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ON THREE-DIMENSIONAL DYNAMIC PROBLEMS OF GENERALIZED THERMOELASTICITY

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Abstract. Lord–Shulman's system of partial differential equations of generalized thermoelasticity [1] is considered, in which the finite velocity of heat propagation is taken into account by introducing a relaxation time constant. General aspects of the theory of boundary value and initial-boundary value problems and representation of solutions by series and quadratures are considered using the method of a potential.

As one knows, in the classical theory of thermoelasticity the velocity of heat propagation is assumed to be infinitely large. However, in studying dynamic thermal stresses in deformable solid bodies, when the inertia terms in the equations of motion cannot be neglected, one must take into account that heat propagates not with an infinite but with a finite velocity; a heat flow arises in the body not instantly but is characterized by a finite relaxation time. Presently, there are at least two different generalizations of the classical theory of thermoelasticity: the first of them, Green–Lindsay's generalization [1] is based on using two heat relaxation time constants; the other one, Lord–Shulman's generalization [2] admits only one relaxation time constant. Both generalizations were developed as an attempt at explaining the paradox of the classical case that the heat propagation velocity is an infinite value.

In this paper, based on [3], we develop a general theory of solvability, as well as of construction of approximate and effective solutions of dynamic problems for the conjugate system of differential equations of thermoelasticity proposed by Lord and Shulman ($L-S$ theory). Green–Lindsay's theory ($G-L$-theory) is developed in [4, 5].

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Note that the use of relaxation time constants much complicates the basic systems of differential equations and essentially changes the type of equations.

Interesting historical information and bibliography on these issues can be found in [6], [7–11], [12].

1. Notation and Definitions

The following notation is introduced:

$\mathbb{R}^n$ is an n-dimensional Euclidean space;

$Ox_1x_2x_3$ is the Cartesian system in $\mathbb{R}^3$;

$x = (x_k), y = (y_k), \dots$ $(k = \overline{1,3})$ are the points of R^3;

$|x - y| = [\sum_{k=1}^{3}(x_k - y_k)^2]^{1/2}$ denotes the distance between x and y;

$D \subset \mathbb{R}^3$ is a finite domain, $S \equiv \partial D \in \mathcal{L}_1(\alpha_0)$,

$\alpha_0 > 0$ (Lyapunov surface) [13];

$\overline{D} \equiv D \cup S$;

$D^- \equiv \mathbb{R}^3 \backslash \overline{D}$ is an infinite domain;

$Z_\infty \equiv \{(x,t) : x \in D, t \in [0, \infty[\}$ is a cylinder in $\mathbb{R}^4$, t is time, $S_\infty \equiv \{(x,t) : x \in S, t \in [0,\infty[\}$ is the lateral surface of Z_∞;

$n(y) = (n_k(y))$, $k = \overline{1,3}$, is the unit normal vector at the point $y \in S$ directed outside D;

$dx = dx_1dx_2dx_3$ is an element of volume, d_yS is an element of the area of S at the point $y \in S$;

δ_{jk} is Kronecker's symbol;

$i = \sqrt{-1}$ is the imaginary unit;

$\Delta \equiv \Delta(\frac{\partial}{\partial x})$ is a three-dimensional Laplace operator;

$v(x,t) = (v_1, v_2, v_3) = \|v_k\|_{3\times 1}$ is a displacement vector (a one-column matrix);

$v_4(x,t)$ is a temperature variation;

T as a superscript denotes transposition;

$\sigma_{jk}(x,t)$ are components of elastic stresses;

$\varepsilon_{jk}(x,t)$ are strain components;

$A(\frac{\partial}{\partial x}) \equiv \|\mu\delta_{jk}\Delta\,(\frac{\partial}{\partial x}) + (\lambda+\mu)\,\frac{\partial^2}{\partial x_j \partial x_k}\|_{3\times 3}$ is the matrix differential Lamé operator;

$T(\frac{\partial}{\partial x}, n) \equiv \|\mu\delta_{jk}\,\frac{\partial}{\partial n(x)} + \lambda n_j(x)\,\frac{\partial}{\partial x_k} + \mu n_k(x)\,\frac{\partial}{\partial x_j}\|_{3\times 3}$ is the matrix operator of elastic stresses;

an m-dimensional vector $f = (f_1, \dots, f_m) = \|f_k\|_{m\times 1}$ (real- or complex-valued) with norm $|f| = [\sum_{j=1}^{m}|f_j|^2]^{1/2}$ is treated as a $m \times 1$ one-column matrix;

the matrix product is obtained by multiplying a row by a column;

if $A = \|A_{kj}\|_{m\times m}$ is a $m \times m$ matrix, then $\overset{k}{A} = \|A_{jk}\|_{m\times 1}$ is the k-th vector-column of the matrix A and we will occasionally write $A =$

$\|\overset{1}{A},\dots,\overset{m}{A}\| = \{\overset{1}{A},\dots,\overset{m}{A}\}$; obvioulsly, if A and B are $m\times m$ matrices and φ is a $m\times 1$ matrix (vector), then

$$AB = \Big\{A\overset{1}{B},\dots,A\overset{m}{B}\Big\},\quad A\varphi = \sum_{k=1}^{m}\overset{k}{A}\varphi_k,$$

$$AB\varphi = \Big\|\sum_{(r,j)} A_{kr}B_{rj}\varphi_j\Big\|_{m\times 1},\quad (AB^T)^T\varphi = \Big\|\sum_{(r,j)} A_{jr}B_{kr}\varphi_j\Big\|_{m\times 1};$$

$$(f\cdot\varphi) = f^T\varphi = \sum_{k=1}^{m} f_k\varphi_k;\quad I = \|\delta_{jk}\|_{m\times m}\ \text{ is the unit matrix.}$$

Lord–Shulman's fundamental system of equations and relations of a field for homogeneous isotropic linear thermoelasticity proposed is written as follows [1], [12], [15]:

1. A strain and displacement relation

$$\varepsilon_{jk}(x,t) = \frac{1}{2}\Big(\frac{\partial v_j(x,t)}{\partial x_k} + \frac{\partial v_k(x,t)}{\partial x_j}\Big),\quad j,k=\overline{1,3},$$

2. An equation of motion

$$\sum_{j=1}^{3}\frac{\partial\sigma_{jk}(x,t)}{\partial x_j} + X_k(x,t) = \rho\frac{\partial^2 v_k(x,t)}{\partial t^2}\ ,\quad k=\overline{1,3},$$

3. An energy equation

$$-\operatorname{div} q(x,t) + r(x,t) = C_\varepsilon\,\frac{\partial v_4}{\partial t} + (3\lambda+2\mu)\alpha\Theta_0\,\frac{\partial}{\partial t}\operatorname{div} v,$$

4. Duhamel–Neuman's law

$$\sigma_{jk}(x,t) = 2\mu\varepsilon_{jk}(x,t) + \big[\lambda\operatorname{div} v - (3\lambda+2\mu)\alpha v_4\big]\delta_{jk},\quad j,k=\overline{1,3},$$

5. A generalized heat conductivity equation

$$q(x,t) + \tau_t\frac{\partial q(x,t)}{\partial t} = -k\operatorname{grad} v_4(x,t),$$

where $\tau_t > 0$ is the relaxation time constant.

In the above formulas $q(x,t) = (q_1,q_2,q_3)$ is the heat flow vector, $r(x,t)$ is the heat source; X_1, X_1, X_3 are the given functions; ρ, λ and μ, α, k and C_ε denote respectively the density, Lamé moduli, thermal expansion coefficient, conductivity and specific heat capacity for zero deformation; Θ_0 is the fixed uniform reference temperature (natural state temperature). These constants satisfy the natural restrictions [1], [12], [15]:

$$\rho>0,\quad k>0,\quad C_\varepsilon>0,\quad \alpha>0,\quad 3\lambda+2\mu>0,\quad \Theta_0>0,\quad \tau_t>0 \quad (1)$$

In the classical case $\tau_t = 0$.

For v, v_4 the above defining relations 1–5 give the basic dynamic system of partial differential equations of generalized thermoelasticity (L–S theory)

$$\begin{aligned} A\Big(\frac{\partial}{\partial x}\Big)v(x,t) - \gamma \operatorname{grad} v_4(x,t) = \rho \frac{\partial^2 v}{\partial t^2} - X(x,t), \\ \Delta v_4(x,t) = \frac{1}{\varkappa}\frac{\partial v_4}{\partial t} + \frac{\tau_t}{\varkappa}\frac{\partial^2 v_4}{\partial t^2} + \eta \frac{\partial}{\partial t} \operatorname{div} v + \\ + \eta\tau_t \frac{\partial^2}{\partial t^2} \operatorname{div} v - X_4(x,t), \end{aligned} \tag{2}$$

where

$$\begin{aligned} \gamma = (3\lambda + 2\mu)\alpha, \quad \eta = \frac{\gamma\Theta_0}{k}\,, \quad \frac{1}{\varkappa} = \frac{C_\varepsilon}{k};, \\ X = (X_1, X_2, X_3), \quad X_4 = \frac{r}{k}\,. \end{aligned}$$

For the stress components we have

$$\sigma_{jk}(x,t) = (\lambda \operatorname{div} v - \gamma v_4)\delta_{jk} + \mu\Big(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j}\Big), \quad j,k = \overline{1,3}. \tag{3}$$

We will consider two possible cases of dependence of $v_k(x,t)$, $k = \overline{1,4}$, on the time t:

I. $v_k(x,t) = \operatorname{Re}[e^{-ipt}u_k(x,p)]$ are stationary oscillations with frequency $p > 0$;

II. $v_k(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\zeta t}u_k(x,\zeta)d\zeta$, $\zeta = \sigma + iq$, $\sigma > 0$, is the representation by the Laplace–Mellin's integral (the general classical case).

One can easily see that in both cases system $(2)_0$ (homogeneous) can be reduced to the form (with respect to $u_k(x,\omega)$)

$$\begin{aligned} A\Big(\frac{\partial}{\partial x}\Big)u(x,\omega) - \gamma \operatorname{grad} u_4(x,\omega) + \rho\omega^2 u(x,\omega) = 0, \\ \Delta u_4(x,\omega) + \frac{i\omega}{\varkappa_t}u_4(x,\omega) + i\omega\eta_t \operatorname{div} u(x,\omega) = 0, \end{aligned} \tag{4}$$

where

$$\begin{aligned} u = (u_1, u_2, u_3) = \|u_k\|_{3\times 1}, \quad \frac{1}{\varkappa_t} = \frac{1}{\varkappa}(1 - i\omega\tau_t), \\ \eta_t = \eta(1 - i\omega\tau_t). \end{aligned} \tag{5}$$

Note that $\omega = p > 0$ in case I and $\omega = i\zeta = -q + i\sigma$ in case II.

2. Energy Identities and Green Formulae

Let $V = (v, v_4) = \|v_k\|_{4\times 1}$ be an arbitrary regular solution of the homogeneous dynamic system $(2)_0$:

$$V, \frac{\partial V}{\partial t} \in C^1(\overline{Z}_\infty) \cap C^2(Z_\infty).$$

We introduce the notation

$$l_t \equiv 1 + \tau_t \frac{\partial}{\partial t}\,, \quad v_t \equiv l_t v, \quad v_{4t} \equiv l_t v_4. \tag{6}$$

Rewrite system $(2)_0$ as

$$\begin{aligned} A\Big(\frac{\partial}{\partial x}\Big) v_t - \gamma \operatorname{grad} v_{4t} &= \rho \frac{\partial^2 v_t}{\partial t^2}\,, \\ \Delta v_4 &= \frac{1}{\varkappa}\, \frac{\partial v_{4t}}{\partial t} + \eta\, \frac{\partial}{\partial t}\, \operatorname{div} v_t. \end{aligned} \tag{7}$$

Multiplying $(7)_1$ by $\frac{\partial v_t^T}{\partial t}$ gives

$$\frac{\partial v_t^T}{\partial t}\Big[A\Big(\frac{\partial}{\partial x}\Big) v_t - \gamma \operatorname{grad} v_{4t}\Big] = \rho\, \frac{\partial v_t^T}{\partial t}\, \frac{\partial^2 v_t}{\partial t^2} = \frac{\rho}{2} \frac{\partial}{\partial t} \Big|\frac{\partial v_t}{\partial t}\Big|^2. \tag{8}$$

The following identity holds [13], [14]:

$$\begin{aligned} \int\limits_D \Big\{ \frac{\partial v_t^T}{\partial t}\, [A v_t - \gamma \operatorname{grad} v_{4t}] + E\Big(\frac{\partial v_t}{\partial t}, v_t\Big) - \gamma v_{4t} \operatorname{div}\, \frac{\partial v_t}{\partial t} \Big\}\, dx = \\ = \int\limits_S \frac{\partial v_t^T}{\partial t} (T v_t - \gamma n v_{4t})\, dS, \end{aligned} \tag{9}$$

where $E(v, u)$ the well-known bilinear form of the theory of elasticity,

$$\begin{aligned} E(v,u) = \frac{3\lambda + 2\mu}{3}\, \operatorname{div} v \operatorname{div} u + \frac{\mu}{2} \sum_{(p\neq q)} \Big(\frac{\partial v_p}{\partial x_q} + \frac{\partial v_q}{\partial x_p}\Big)\Big(\frac{\partial u_p}{\partial x_q} + \frac{\partial u_q}{\partial x_p}\Big) + \\ + \frac{\mu}{3} \sum_{(p,q)} \Big(\frac{\partial v_p}{\partial x_p} - \frac{\partial v_q}{\partial x_q}\Big)\Big(\frac{\partial u_p}{\partial x_p} - \frac{\partial u_q}{\partial x_q}\Big). \end{aligned}$$

It is easy to verify that $E(\frac{\partial v_t}{\partial t}, v_t) = \frac{1}{2}\frac{\partial}{\partial t} E(v_t, v_t)$ and by virtue of (8) formula (9) takes the form

$$\frac{d}{dt} \int\limits_D \Big[\frac{\rho}{2}\Big|\frac{\partial v_t}{\partial t}\Big|^2 + \frac{1}{2} E(v_t, v_t)\Big]\, dx - \gamma \int\limits_D v_{4t} \operatorname{div}\, \frac{\partial v_t}{\partial t}\, dx =$$

$$= \int\limits_S \frac{\partial v_t^T}{\partial t}\, HV_t\, dS, \tag{10}$$

where

$$HV_t = Tv_t - \gamma n v_{4t}.$$

After multiplying $(7)_2$ by v_{4t}, integrating with respect to D and applying the Green formula to the Laplace operator we obtain

$$-\gamma \int\limits_D v_{4t}\, \frac{\partial}{\partial t}\, \operatorname{div} v_t\, dx = \frac{\gamma}{\eta\varkappa} \int\limits_D v_{4t}\, \frac{\partial v_{4t}}{\partial t}\, dt - \frac{\gamma}{\eta} \int\limits_D v_{4t} \Delta v_4\, dx =$$
$$= \frac{\gamma}{2\varkappa\eta}\, \frac{d}{dt} \int\limits_D (v_{4t})^2\, dx + \frac{\gamma}{\eta} \int\limits_D \big[\operatorname{grad} v_{4t}\big]^T \operatorname{grad} v_4\, dx -$$
$$-\frac{\gamma}{\eta} \int\limits_S v_{4t}\, \frac{\partial v_4}{\partial n}\, dS. \tag{11}$$

By taking the identity

$$\big[\operatorname{grad} v_{4t}\big]^T \operatorname{grad} v_4 = |\operatorname{grad} v_4|^2 + \frac{\tau_t}{2}\, \frac{\partial}{\partial t}\, |\operatorname{grad} v_4|^2$$

into account and substituting (11) into (10) we finally have

$$\frac{d}{dt} \int\limits_D \bigg[\frac{\rho}{2}\, \Big|\frac{\partial v_t}{\partial t}\Big|^2 + \frac{1}{2}\, E(v_t, v_t) + \frac{\gamma}{2\varkappa\eta}\, v_{4t}^2 +$$
$$+\frac{\gamma\tau_t}{2\eta}\, |\operatorname{grad} v_4|^2\bigg]\, dx + \frac{\gamma}{\eta} \int\limits_D |\operatorname{grad} v_4|^2\, dx =$$
$$= \int\limits_S \bigg[\frac{\partial v_t^T}{\partial t}\, HV_t + \frac{\gamma}{\eta}\, v_{4t}\, \frac{\partial v_4}{\partial n}\bigg]\, dS \equiv M_S(V). \tag{12}$$

Introducing the notation

$$J(t) = \frac{1}{2} \int\limits_D \bigg[\rho\Big|\frac{\partial v_t}{\partial t}\Big|^2 + E(v_t, v_t) + \frac{\gamma}{\varkappa\eta}\, v_{4t}^2 + \frac{\gamma\tau_t}{\eta}\, |\operatorname{grad} v_4|^2\bigg]\, dx. \tag{13}$$

we rewrite (12) as

$$\frac{dJ(t)}{dt} + \frac{\gamma}{\eta} \int\limits_D |\operatorname{grad} v_4|^2\, dx = M_S(V). \tag{14}$$

Note that formula (14) holds for the infinite domain $D^- \equiv \mathbb{R}^3\backslash\overline{D}$ as well provided that the vector $V(x,t)$ satisfies the following condition at infinity:

$$\left|\frac{\partial^{|\alpha|}V}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}\partial t^{\alpha_4}}\right| \leq \frac{\text{const}\cdot e^{\sigma_0 t}}{1+|x|^{1+|\alpha|}}\ , \quad |\alpha| = \overline{0,3}, \quad \sigma_0 \geq 0,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is the multi-index, $|\alpha| = \sum_{k=1}^{4}\alpha_k$.

Let $u \in C^1(\overline{D}) \cap C^2(D)$, $v \in C^1(\overline{D})$, $u_4 \in C^1(\overline{D})$, $A(\frac{\partial}{\partial x})u \in L_1(D)$, $S \in \mathcal{L}_1(\alpha_0)$, $\alpha_0 > 0$. Then the identity [14]

$$\int\limits_D \left[v^T(Au - \gamma\operatorname{grad} u_4) + E(v,u) - \gamma u_4 \operatorname{div} v\right]dx = \int\limits_S v^T HU\, dS \tag{15}$$

holds. In particular, if $U = (u, u_4)$ is a regular solution of system (4) and $\overline{U} = (\overline{u}, \overline{u}_4)$ is the comple-conjugate vector, then by simple transformations, from (15) and $(14)_0$ we obtain

$$\int\limits_D \left[-\rho\omega^2|u|^2 + E(\overline{u}, u) + \frac{\gamma}{i\overline{\omega}\,\overline{\eta}_t}\,|\operatorname{grad} u_4|^2 + \frac{\gamma}{\overline{\varkappa}_t\overline{\eta}_t}\,|u_4|^2\right]dx =$$
$$= \int\limits_S \left[\overline{u}^T HU + \frac{\gamma}{i\overline{\omega}\,\overline{\eta}_t}\,u_4\,\frac{\partial\overline{u}_4}{\partial n}\right]dS. \tag{16}$$

Let $\omega = p > 0$ be the real parameter. In (16) we pass over to the complex-conjugate expression and subtract the result from (16); taking into account that $\operatorname{Im} E(\overline{u}, u) = 0$,

$$\frac{\gamma}{i\omega\overline{\eta}_t} - \frac{\gamma}{-i\omega\eta_t} = \frac{\gamma}{i\omega|\eta_t|^2}\,2\operatorname{Re}\eta_t = \frac{2\gamma\eta}{i\omega|\eta_t|^2}\ ,$$
$$\frac{\gamma}{\overline{\varkappa}_t\overline{\eta}_t} - \frac{\gamma}{\varkappa_t\eta_t} = \frac{\gamma}{|\varkappa_t|^2|\eta_t|^2}\,2i\operatorname{Im}(\varkappa_t\eta_t) = 0,$$

we obtain

$$\frac{2\gamma\eta}{i\omega|\eta_t|^2}\int\limits_D |\operatorname{grad} u_4|^2\,dx = \int\limits_S \Big\{\left[\overline{u}^T HU - u^T H\overline{U}\right] +$$
$$+ \frac{\gamma}{i\omega\overline{\eta}_t}\,u_4\,\frac{\partial\overline{u}_4}{\partial n} + \frac{\gamma}{i\omega\eta_t}\,\overline{u}_4\,\frac{\partial u_4}{\partial n}\Big\}\,dS. \tag{17}$$

Let now $\omega = i\zeta = -q + i\sigma$, $\sigma > 0$, which means that we are considering the case of pseudo-oscillations. This case is an auxiliary means in studying the general nonstationary problem. Formula (16) takes the form

$$\int\limits_D \left[-\rho\zeta^2|u|^2 + E(\overline{u}, u) + \frac{\gamma}{\overline{\zeta}\,\overline{\eta}_t}\,|\operatorname{grad} u_4|^2 + \frac{\gamma}{\overline{\varkappa}_t\overline{\eta}_t}\,|u_4|^2\right]dx = N_S(U) \equiv$$

$$\equiv \int\limits_S \left[\overline{u}^T HU + \frac{\gamma}{\overline{\zeta}\,\overline{\eta}_t}\, u_4\, \frac{\partial \overline{u}_4}{\partial n} \right] dS, \tag{18}$$

where

$$\frac{1}{\varkappa_t} = \frac{1}{\varkappa}(1+\tau_t\zeta), \quad \eta_t = \eta(1+\tau_t\zeta).$$

Denote by $(\overline{18})$ the complex-conjugate expression and obtain the sum $\overline{\zeta} N_S(U) + \zeta \overline{N_S(U)}$; since $\overline{\zeta}\zeta^2 + \zeta\overline{\zeta}^2 = 2\sigma|\zeta|^2$, $\overline{\zeta}+\zeta = 2\sigma$, $\frac{\gamma}{\overline{\eta}_t} + \frac{\gamma}{\eta_t} = \frac{2\gamma\eta(1+\sigma\tau_t)}{|\eta_t|^2}$, $\frac{\overline{\zeta}\gamma}{\overline{\varkappa}_t\overline{\eta}_t} + \frac{\zeta\gamma}{\varkappa_t\eta_t} = \frac{2\gamma\sigma}{\varkappa\eta}$, we finally have

$$\int\limits_D \left[2\sigma|\zeta|^2\rho|u|^2 + 2\sigma E(\overline{u}, u) + \frac{2\gamma\eta(1+\sigma\tau_t)}{|\eta_t|^2}\,|\operatorname{grad} u_4|^2 + \frac{2\gamma\sigma}{\varkappa\eta}\,|u_4|^2 \right] dx =$$

$$= \overline{\zeta} N_S(U) + \zeta\overline{N_S(U)}. \tag{19}$$

Another generalized Green formula is helpful in investigating boundary value and initial-boundary problems by the method of a potential. To this end, we introduce the differential (matrix) operators

$$L\Big(\frac{\partial}{\partial x}\Big) \equiv \left\| \begin{matrix} \boxed{A\Big(\frac{\partial}{\partial x}\Big)}_{3\times 3}, & \boxed{-\gamma \operatorname{grad}_x}_{3\times 1} \\ \boxed{0}_{1\times 3}, & \Delta \end{matrix} \right\|_{4\times 4},$$

$$L^0\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) \equiv \left\| \begin{matrix} \boxed{\rho\delta_{jk}\,\frac{\partial^2}{\partial t^2}}_{3\times 3}, & \boxed{0}_{3\times 1} \\ \boxed{\eta\,\frac{\partial}{\partial t}(l_t \operatorname{grad}_x)}_{1\times 3}, & \frac{1}{\varkappa}\, l_t\, \frac{\partial}{\partial t} \end{matrix} \right\|_{4\times 4},$$

$$L\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) \equiv L\Big(\frac{\partial}{\partial x}\Big) - L^0\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big).$$

Obviously, the basic nonstationary homogeneous system $(2)_0$ can be rewritten as $(V = (v, v_4))$

$$L\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big) V(x,t) = 0,$$

while system (4) as $(U = (u, u_4))$

$$L\Big(\frac{\partial}{\partial x}, -i\omega\Big) U = 0.$$

Clearly,

$$L^0\Big(\frac{\partial}{\partial x}, -i\omega\Big) \equiv \left\| \begin{matrix} \boxed{-\rho\delta_{jk}\omega^2}_{3\times 3}, & \boxed{0}_{3\times 1} \\ \boxed{-i\omega\eta_t \operatorname{grad}_x}_{1\times 3}, & \frac{i\omega}{\varkappa_t} \end{matrix} \right\|_{4\times 4},$$

$$L^0\Big(\frac{\partial}{\partial x}, \zeta\Big) \equiv \left\| \begin{matrix} \boxed{\rho\delta_{jk}\zeta^2}_{3\times 3}, & \boxed{0}_{3\times 1} \\ \boxed{\eta\zeta l_\zeta \operatorname{grad}_x}_{1\times 3}, & \frac{\zeta}{\varkappa} l_\zeta \end{matrix} \right\|_{4\times 4}, \quad l_\zeta = 1 + \tau_t\zeta.$$

Let $\widetilde{L}(\frac{\partial}{\partial x}, -i\omega)$ be the (Lagrange-)conjugate operator of $L(\frac{\partial}{\partial x}, -i\omega)$,

$$\widetilde{L}\Big(\frac{\partial}{\partial x}, -i\omega\Big) \equiv \left\| \begin{matrix} \boxed{A\Big(\frac{\partial}{\partial x}\Big) + \rho\delta_{jk}\omega^2}_{3\times 3}, & \boxed{-i\omega\eta_t \operatorname{grad}_x}_{3\times 1} \\ \boxed{\gamma \operatorname{grad}_x}_{1\times 3}, & \Delta + \frac{i\omega}{\varkappa_t} \end{matrix} \right\|_{4\times 4},$$

For the regular in D vectors $U = (u, u_4)$ and $\widetilde{U} = (\widetilde{u}, \widetilde{u}_4)$ simple transformations give

$$\begin{gathered} U^T \widetilde{L}\Big(\frac{\partial}{\partial x}, -i\omega\Big)\widetilde{U} - \widetilde{U}^T L\Big(\frac{\partial}{\partial x}, -\omega\Big)U = \\ = \Big[u^T A\Big(\frac{\partial}{\partial x}\Big)\widetilde{u} - \widetilde{u}^T A\Big(\frac{\partial}{\partial x}\Big)u\Big] + \\ + \big[u_4\Delta\widetilde{u}_4 - \widetilde{u}_4\Delta u_4\big] + \gamma \operatorname{div}(u_4\widetilde{u}^T) - i\omega\eta_t \operatorname{div}(\widetilde{u}_4 u^T). \end{gathered}$$

Therefore, by the (Gauss–Ostrogradski's) divergence formula we finally obtain

$$\begin{aligned} \int\limits_D \Big[U^T \widetilde{L}\Big(\frac{\partial}{\partial x}, -i\omega\Big)\widetilde{U} - \widetilde{U}^T L\Big(\frac{\partial}{\partial x}, -i\omega\Big)U\Big]\,dx &= \\ = \int\limits_S \big[U^T \widetilde{R}\widetilde{U} - \widetilde{U}^T R U\big]\,dS, \end{aligned} \tag{20}$$

where

$$\begin{gathered} RU \equiv \Big(HU, \frac{\partial u_4}{\partial n}\Big), \quad \widetilde{R}U \equiv \Big(\widetilde{H}U, \frac{\partial u_4}{\partial n}\Big), \\ HU \equiv Tu - \gamma n u_4, \quad \widetilde{H}U \equiv Tu - i\omega\eta_t n u_4. \end{gathered}$$

Let further

$$\mathcal{P}U \equiv (HU, -u_4), \quad \widetilde{\mathcal{P}}U \equiv (\widetilde{H}U, -u_4), \quad QU \equiv \Big(u, \frac{\partial u_4}{\partial n}\Big).$$

Now formula (20) can be represented as

$$\int\limits_D \Big[U^T \widetilde{L}\Big(\frac{\partial}{\partial x}, -i\omega\Big)\widetilde{U} - \widetilde{U}^T L\Big(\frac{\partial}{\partial x}, -i\omega\Big)U\Big]\,dx = \\ = \int\limits_S \big[(QU)^T \widetilde{\mathcal{P}}\widetilde{U} - (Q\widetilde{U})^T \mathcal{P}U\big]\,dS. \tag{21}$$

3. Decomposition of a Regular Solution. The Properties of Characteristic λ-parameters

In this section we will prove several theorems and lemmas to be essentially used in constructing the general theory of solvability of boundary value problems.

Theorem 1. *In the domain $D \subset \mathbb{R}^3$ any solution $U = (u, u_4)$ of the homogeneous system (4) of the class $U \in \mathbb{C}^2(D)$ belongs to the class $U \in \mathbb{C}^\infty(D)$ and admits a representation*

$$\begin{gathered} U = (u, u_4) = \big(u^{(1)} + u^{(2)}, u_4\big), \quad \operatorname{rot} u^{(1)} = 0, \quad \operatorname{div} u^{(2)} = 0, \\ (\Delta + \lambda_1^2)(\Delta + \lambda_2^2)\begin{pmatrix} u^{(1)} \\ u_4 \end{pmatrix} = 0, \quad (\Delta + \lambda_3^2)u^{(2)} = 0, \quad \lambda_3^2 = \frac{\rho\omega^2}{\mu}\,, \end{gathered} \tag{22}$$

while the characteristic constants $\lambda_k^2(\omega)$, $k = 1, 2$, are the roots of the quadratic equation

$$z^2 - \Big[\rho_0\omega^2(1+\varepsilon_t) + \frac{i\omega}{\varkappa}(1+\varepsilon)\Big]z + \Big[\omega^4\,\frac{\rho_0}{\varkappa}\,\tau_t + \frac{i\omega}{\varkappa}\,\rho_0\omega^2\Big] = 0, \tag{23}$$

where

$$\rho_0 = \rho(\lambda+2\mu)^{-1}, \quad \varepsilon_t = \frac{\tau_t}{\varkappa\rho_0}(1+\varepsilon), \quad \varepsilon = \frac{\varkappa\gamma\eta}{\lambda+2\mu} \ \ (\varepsilon \ll 1).$$

Proof. The fact that the vector U belongs to the class $\mathbb{C}^\infty(D)$ is established by means of the formula for a general representation of a regular solution of system (4) using the matrix of fundamental solutions of the considered differential operator of generalized elastothermoelasticity (see [14], [15]).

Since $\Delta \equiv \operatorname{grad}\operatorname{div} - \operatorname{rot}\operatorname{rot}$, formula (4) implies

$$\begin{gathered} u = u^{(1)} + u^{(2)}, \\ u^{(1)} = \operatorname{grad}\Big(-\frac{\lambda+2\mu}{\rho\omega^2}\,\operatorname{div} u + \frac{\gamma}{\rho\omega^2}\,u_4\Big), \quad u^{(2)} = \operatorname{rot}\Big(\frac{\mu}{\rho\omega^2}\,\operatorname{rot} u\Big). \end{gathered} \tag{24}$$

Obviously, $\operatorname{rot} u^{(1)} = 0$, $\operatorname{div} u^{(2)} = 0$.

Taking into account the value λ_3^2, we find from (24) that

$$(\Delta + \lambda_3^2) u^{(2)} = 0.$$

We have

$$(\lambda + 2\mu)\Delta \operatorname{div} u^{(1)} + \rho\omega^2 \operatorname{div} u^{(1)} - \gamma \Delta u_4 = 0. \tag{25}$$

By putting the value

$$\operatorname{div} u \equiv \operatorname{div} u^{(1)} = -\frac{1}{i\omega\eta_t}\,\Delta u_4 - \frac{1}{\varkappa_t \eta_t}\,u_4$$

from eq. $(4)_2$ into (25) we obtain

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2) u_4 = 0. \tag{26}$$

Hence it easily follows that $(\Delta + \lambda_1^2)(\Delta + \lambda_2^2) \operatorname{div} u^{(1)} = 0$ and by (24) we finally obtain

$$(\Delta + \lambda_1^2)(\Delta + \lambda_2^2) u^{(1)} = 0. \qquad \square$$

Theorem 2. *If the vector-functions* $v^{(k)}(x) = \|v_j^{(k)}\|_{3\times 1}$, $u^{(2)}(x) = \|u_j^{(2)}\|_{3\times 1}$ *and the scalar functions* $v_k(x)$, $k = 1, 2$, *are regular solutions of the equations*

$$(\Delta + \lambda_k^2)\begin{pmatrix} v^{(k)} \\ v_k \end{pmatrix} = 0, \quad k = 1, 2, \quad (\Delta + \lambda_3^2) u^{(2)} = 0, \quad \operatorname{div} u^{(2)} = 0,$$

and are related through the formula

$$v^{(k)}(x) = \gamma\big[\rho\omega^2 - \lambda_k^2(\lambda + 2\mu)\big]^{-1} \operatorname{grad} v_k, \quad k = 1, 2,$$

then the vector

$$U(x) = \big(u^{(1)}(x) + u^{(2)}(x), u_4(x)\big),$$
$$u^{(1)}(x) = v^{(1)}(x) + v^{(2)}(x), \quad u_4(x) = v_1(x) + v_2(x),$$

is a regular solution of system (4).

Proof. By using the conditions of the theorem and performing some simple calculations we obtain

$$A\Big(\frac{\partial}{\partial x}\Big) u - \gamma \operatorname{grad} u_4 + \rho\omega^2 u =$$
$$= \mu\Delta u + (\lambda + \mu) \operatorname{grad} \operatorname{div} u - \gamma \operatorname{grad} u_4 + \rho\omega^2 u = \big[\mu\Delta u^{(2)} + \rho\omega^2 u^{(2)}\big] +$$
$$+ \big[(\lambda + 2\mu)\Delta(v^1 + v^{(2)}) + \rho\omega^2(v^{(1)} + v^{(2)}) - \gamma \operatorname{grad}(v_1 + v_2)\big] =$$
$$= \Big\{\big[\rho\omega^2 - (\lambda + 2\mu)\lambda_1^2\big] v^{(1)} - \gamma \operatorname{grad} v_1\Big\} +$$

$$+\Big\{\big[\rho\omega^2-(\lambda+2\mu)\lambda_2^2\big]v^{(2)}-\gamma\operatorname{grad}v_2\Big\}=0.$$

Similarly, we have

$$\Delta u_4+\frac{i\omega}{\varkappa_t}\,u_4+i\omega\eta_t\operatorname{div}u=$$
$$=\Delta(v_1+v_2)+\frac{i\omega}{\varkappa_t}(v_1+v_2)+i\omega\eta_t\operatorname{div}(v^{(1)}+v^{(2)})=$$
$$=\bigg[\Big(\frac{i\omega}{\varkappa_t}-\lambda_1^2\Big)v_1+i\omega\eta_t\operatorname{div}v^{(1)}\bigg]+\bigg[\Big(\frac{i\omega}{\varkappa_t}-\lambda_2^2\Big)v_2+i\omega\eta_t\operatorname{div}v^{(2)}\bigg]=$$
$$=\Big[\frac{i\omega}{\varkappa_t}-\lambda_1^2-\frac{\gamma\lambda_1^2 i\omega\eta_t}{\rho\omega^2-\lambda_1^2(\lambda+2\mu)}\Big]v_1+\Big[\frac{i\omega}{\varkappa_t}-\lambda_2^2-\frac{\gamma\lambda_2^2 i\omega\eta_t}{\rho\omega^2-\lambda_2^2(\lambda+2\mu)}\Big]v_2=$$
$$=\sum_{k=1}^{2}\frac{v_k}{\rho\omega^2-\lambda_k^2(\lambda+2\mu)}\,\bigg[\frac{i\omega}{\varkappa_t}\,\rho\omega^2-\frac{i\omega}{\varkappa_t}(\lambda+2\mu)\lambda_k^2-$$
$$-\lambda_k^2\rho\omega^2+\lambda_k^4(\lambda+2\mu)-\gamma\lambda_k^2 i\omega\eta_t\bigg]=$$
$$=\sum_{k=1}^{2}\frac{v_k}{\rho\omega^2-\lambda_k^2(\lambda+2\mu)}\,\bigg\{\frac{i\omega}{\varkappa_t}\,\rho\omega^2+\lambda_k^4(\lambda+2\mu)-$$
$$-\lambda_k^2\Big[\frac{i\omega}{\varkappa_t}(\lambda+2\mu)+\rho\omega^2+i\omega\gamma\eta_t\Big]\bigg\}=$$
$$=\sum_{k=1}^{2}\frac{v_k}{\rho\omega^2-\lambda_k^2(\lambda+2\mu)}\,\bigg[\frac{i\omega}{\varkappa_t}\,\rho\omega^2+\lambda_k^4(\lambda+2\mu)-\lambda_k^2(\lambda+2\mu)(\lambda_1^2+\lambda_2^2)\bigg]=$$
$$=\sum_{k=1}^{2}\frac{v_k}{\rho\omega^2-\lambda_k^2(\lambda+2\mu)}\,\Big[\frac{i\omega}{\varkappa_t}\,\rho\omega^2-(\lambda+2\mu)\lambda_1^2\lambda_2^2\Big]\equiv 0.\qquad\square$$

Note that the parameter $\varepsilon=\frac{\varkappa\gamma\eta}{\lambda+2\mu}$ is a physical constant and for most real bodies $\varepsilon\ll 1$ [16]. When $\varepsilon=0$, the deformation and temperature fields get completely separated and we then have a separate (not conjugate) theory by virtue of which (23) implies

$$\lambda_1^2=\frac{i\omega}{\varkappa}+\frac{\omega^2\tau_t}{\varkappa}\,,\quad \lambda_2^2=\frac{\rho\omega^2}{\lambda+2\mu}\,,\quad \lambda_3^2=\frac{\rho\omega^2}{\mu}\,.$$

Clearly, this case is of no particular interest for us as the investigation of boundary value problems actually reduces to problems of the classical theory of elasticity and thermal conductivity taken separately. In what follows it will therefore be assumed that $\varepsilon>0$.

In investigating boundary value problems for system (2) and (4) much importance is attached to the parameters $\lambda_k(\omega)$ depending on the coefficients of the initial systems of differential equations by formula (23). These properties are described by the following lemmas.

Lemma 1. *If $\omega = p > 0$, $\varepsilon > 0$ (stationary oscillation), then λ_1^2 and λ_2^2 are complex numbers.*

Proof. If we assume that the opposite is true, then eq. (23) should have the real root $z = \alpha$, which in turn gives

$$\alpha = \frac{\rho_0\omega^2}{1+\varepsilon}\,, \quad \alpha^2 - \rho_0\omega^2(1+\varepsilon_t)\alpha + \omega^4\,\frac{\rho_0}{\varkappa}\,\tau_t = 0.$$

Hence we readily obtain the contradiction $\varepsilon = 0$. Note that if the relations

$$\tau_t = \frac{\varkappa\rho}{\lambda+2\mu}\cdot\frac{1-\varepsilon}{(1+\varepsilon)^2}\,, \quad \omega = \frac{\frac{1}{\varkappa}(1+\varepsilon)}{\sqrt{\rho_0^2(1+\varepsilon_t)^2 - \frac{4\rho_0\tau_t}{\varkappa}}}\,,$$

are not fulfilled simultaneously (which will be assumed in what follows), then $\lambda_1^2 \neq \lambda_2^2$.

Thus, for stationary oscillations, λ_1 and λ_2 are the complex numbers admitting two signs, and therefore without loss of generality it can always be assumed that $\operatorname{Im}\lambda_k > 0$, $k = 1, 2$. □

Lemma 2. *If $\varepsilon > 0$, $\omega = i\zeta = -q + i\sigma$, $\sigma > 0$, then in the complex half-plane*

$$\Pi_{\sigma_0^*} = \big\{\zeta :\ \operatorname{Re}\zeta > \sigma_0^*\big\}$$

where

$$\sigma_0^* = \max\{\sigma_\varepsilon, 0\}, \quad \sigma_\varepsilon = \frac{\rho_0\varkappa[2-(1+\varepsilon_t)(1+\varepsilon)]}{[\rho_0\varkappa + \tau_t(\varepsilon-1)]^2 + 4\varepsilon\tau_t^2}$$

the parameters λ_k, $k = \overline{1,3}$, possess the properties:

1. $\operatorname{Im}\lambda_k > 0$*;*
2. $\lambda_1 \neq \lambda_2 \neq \lambda_3$*;*
3. $\lambda_k = \lambda_k(\zeta)$ *are the analytic functions ζ which admit estimates $\lambda_k(\zeta) = O(|\zeta|)$ as $\zeta \to \infty$.*

Proof. For $\lambda_3(\zeta) = \sqrt{\frac{\rho}{\mu}}\cdot i\zeta$ these properties are obvious in the half-plane $\operatorname{Re}\zeta > 0$. We will show that they are valid for $\lambda_1(\zeta)$ and $\lambda_2(\zeta)$ as well.

By (23) we have

$$z_{1,2} = -\frac{\rho_0(1+\varepsilon_t)\zeta + (1+\varepsilon)/\varkappa}{2} \pm \sqrt{d}, \tag{$*$}$$

where

$$z_k = \frac{\lambda_k^2}{\zeta}\,, \quad d = \frac{1}{4}\big[\rho_0(1+\varepsilon_t)\zeta + (1+\varepsilon)/\varkappa\big]^2 - \frac{\rho_0}{\varkappa}(1+\tau_t\zeta)\zeta.$$

One can readily see that the zeros of the discriminant $d(\zeta)$ are

$$\zeta_{1,2} = \sigma_\varepsilon \pm i \, \frac{\sqrt{|d_1|}}{\frac{1}{\varkappa^2}\,[\rho_0\varkappa + \tau_t(\varepsilon-1)]^2 + 4\varepsilon\tau_t^2} \,,$$

where

$$d_1 = \Big[\frac{\rho_0}{\varkappa}(1+\varepsilon_t)(1+\varepsilon) - \frac{2\rho_0}{\varkappa}\Big] - \frac{(1+\varepsilon)^2}{\varkappa^2}\,\Big[\rho_0^2(1+\varepsilon_t)^2 - \frac{4\rho_0}{\varkappa}\,\tau_t\Big] = \\ = -\frac{4\rho_0^2}{\varkappa^2}\,\varepsilon < 0.$$

Hence it is clear that if $\operatorname{Re}\zeta > \sigma_0^*$, then the radical in $(*)$ cannot have branching points $(d \neq 0)$ and, after making an appropriate choice of branches, one can regard $z_k(\zeta)$, $k=1,2$, as analytic functions of ζ, $z_1 \neq z_2$, $z_k(\zeta) = O(|\zeta|)$. Therefore $\lambda_k^2(\zeta) = z_k(\zeta)\zeta$ are the analytic functions of ζ and $\lambda_k^2(\zeta) = O(|\zeta|^2)$, and, since $z_k \neq 0$, $\zeta \neq 0$, it can be assumed that $\lambda_k(\zeta)$ are the analytic functions of ζ and $\lambda_k(\zeta) = O(|\zeta|)$, $k=1,2$. Finally, $z_k(\zeta)$, $k=1,2$, as analytic functions of ζ in the half-plane $\Pi_{\sigma_0^*}$, cannot coincide with expressions of form $|a_k|^2\overline{\zeta}$, which are not analytic functions of ζ. Therefore in the half-plane $\Pi_{\sigma_0^*}$ $\lambda_k^2(\zeta)$ are either complex- or real-valued negative numbers [4] and it can always be assumed that $\operatorname{Im}_k(\zeta) > 0$, $k=1,2$. $\square$

The investigation of boundary value problems rests essentially on the above theorems and lemmas.

4. Uniqueness Theorems. Fundamental Solutions and Principal Potentials

Like in the classical case [13], [14], the constructed mathematical methods enable one to perform a complete mathematical analysis of boundary value problems in all cases: stationary oscillations, pseudo-oscillations and general dynamics.

We will briefly discuss the solution uniqueness. By virtue of Theorem 1 and Lemma 1 we introduce

Definition 1. A regular in the infinite domain $D^- \equiv \mathbb{R}^3\backslash\overline{D}$ solution $U = (u^{(1)}+u^{(2)}, u_4)$ of the homogeneous oscillation equation (4) $(\omega = p > 0)$ satisfies the thermoelastic radiation condition at infinity if the following asymptotic estimates hold:

$$\begin{gathered} \begin{pmatrix} u^{(1)}(x) \\ u_4(x) \end{pmatrix} = O(|x|^{-1}), \quad \frac{\partial}{\partial x_k}\begin{pmatrix} u^{(1)}(x) \\ u_4(x) \end{pmatrix} = O(|x|^{-2}), \quad k=\overline{1,3}, \\ u^{(2)}(x) = O(|x|^{-1}), \quad \frac{\partial u^{(2)}(x)}{\partial |x|} - i\lambda_3 u^{(2)}(x) = o(|x|^{-1}), \end{gathered} \tag{27}$$

where $\frac{\partial}{\partial|x|}$ is the derivative along the radius-vector of the point x.

Theorem 3. *A regular in D^- vector $U = (u, u_4) \in C^1(\overline{D}^{\,-}) \cap C^2(D^-)$, which is a solution of the homogeneous oscillation system* (4) ($\omega = p > 0$) *satisfying at infinity the radiation condition and admissible homogeneous conditions on S* [14], *is identically zero in D^-.*

The *proof* immediately follows from identity (17) written for the domain $D_R \equiv D^- \cap \{x : \mathbb{R}^3 : |x| < R\}$, where R is an arbitrary sufficiently large real number (see [14], pp. 242–243). The uniqueness theorems do not hold for the internal oscillation problems (for the finite domain D) because of the emergence of a discrete eigenfrequency spectrum in some boundary value problems of a metaharmonic equation. Such an investigation is carried out similarly to the case of classical thermoelasticity assuming that in all homogeneous internal problems $u_4(x) \equiv \text{const}$ [13], [14].

Theorem 4. *A regular in D vector $U \in C^1(\overline{D}) \cap C^2(D)$, which is a solution of the homogeneous pseudo-oscillation system* (4) ($\omega = i\zeta$, $\operatorname{Re}\zeta > 0$) *satisfying one of the admissible homogeneous boundary conditions on S, is identically zero.*

The proof immediately follows from identity (19) (see [14], p.245). Note that Theorem 4 remains valid for the infinite domain D^- as well (external problems) if the solution $U = (u, u_4)$ satisfies at infinity the asymptotic conditions

$$U = O(|x|^{-1}), \quad \frac{\partial U}{\partial x_k} = O(|x|^{-2}), \quad k = \overline{1,3}. \tag{28}$$

Theorem 5. *A regular in the cylinder Z_∞ vector $V = (v, v_4) \in C^1(\overline{Z}_\infty) \cap C^2(Z_\infty)$, which is a solution of the homogeneous nonstationary system (2)$_0$ satisfying the homogeneous initial conditions $V|_{t=0} = \frac{\partial V}{\partial t}|_{t=0} = 0$ and the corresponding admissible homogeneous boundary conditions on S_∞, is identically zero.*

Proof. (14) implies $M_S(V) \equiv 0$ by virtute of the boundary condition $J'(t) \leq 0$. Therefore $J(t)$ is a nonincreasing function $t \geq 0$. On the other hand, one easily obtains $J(0) = 0$ and therefore $J(t) \leq 0$ for $t \geq 0$. But by (13) $J(t) \geq 0$, i.e., $J(t) = 0$, which in turn gives $v_t(x,t) = 0$, $v_{4t}(x,t) = 0$, $t \geq 0$, $x \in D$. Thus we finally have $\forall x \in D$, $t \geq 0$: $\tau_t \frac{\partial V}{\partial t} + V = 0$, i.e., $V(x,t) = C(x)e^{-\frac{1}{\tau_t}t}$, and by virtue of the initial condition $V|_{t=0} = 0$ we will have $v = 0$, $v_4 = 0$. The theorem also holds for the external (for the infinite domain D^-) initial-boundary problems.

As is well-known, for problems (both stationary and general dynamic) of the conjugate theory of classical thermoelasticity there presently exists a thoroughly developed mathematical theory of solvability which is as general as the available theory for problems of classical elasticity. The final results obtained in this direction are presented in [13], [14], [15].

Further investigations based on the above-presented results showed that the same methods (the method of a potential, theory of multidimensional singular integral equations, Fourier and Laplace–Mellin integral transformation and others) enable one to construct a general mathematical theory of solvability of stationary and nonstationary problems also for problems of the conjugate theory of the nonclassical theory of thermoelasticity by Green–Lindsay (G–L theory) and Lord–Shulman (L–S theory) [3], [4], [5], [15],

In our studies we mostly used the matrix of fundamental solutions of the considered differential operators $L(\frac{\partial}{\partial x},-i\omega)$ and other singular solutions. These solutions are constructed explicitly in terms of elementary functions, which is especially important for obtaining effective and approximate solutions.

The matrix of fundamental solutions has the form [14, 17]:

$$\Phi(x,-i\omega)=\big\|\Phi_{jk}(x,-\omega)\big\|_{4\times 4},$$

$$\Phi_{kj}(x,-i\omega)=\sum_{l=1}^{3}\bigg\{(1-\delta_{k4})(1-\delta_{j4})\Big(\frac{\delta_{kj}}{2\pi\mu}\,\delta_{3l}-\alpha_l\,\frac{\partial^2}{\partial x_k\partial x_l}\Big)+$$

$$+\beta_l\Big[i\omega\eta_t\delta_{k4}(1-\delta_{j4})\,\frac{\partial}{\partial x_j}-\gamma\delta_{j4}(1-\delta_{k4})\,\frac{\partial}{\partial x_k}\Big]+\delta_{k4}\delta_{j4}\gamma_l\bigg\}\,\frac{e^{i\lambda_l|x|}}{|x|}\ ,$$

$$k_1^2=\frac{\rho\omega^2}{\lambda+2\mu}\ ,\quad \beta_l=\frac{(-1)^l(\delta_{1l}+\delta_{2l})}{2\pi(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)}\ ,\quad \sum_{l=1}^{3}\beta_l=0,$$

$$\gamma_l=\frac{(-1)^l(\lambda_l^2-k_1^2)(\delta_{1l}+\delta_{2l})}{2\pi(\lambda_2^2-\lambda_1^2)}\ ,\quad \sum_{l=1}^{3}\gamma_l=0,$$

$$\alpha_l=\frac{(-1)^l(1-\frac{i\omega}{\varkappa_t}\,\lambda_l^{-2})(\delta_{1l}+\delta_{2l})}{2\pi(\lambda+2\mu)(\lambda_2^2-\lambda_1^2)}-\frac{\delta_{3l}}{2\pi\rho\omega^2}\ ,\quad \sum_{l=1}^{3}\alpha_l=0.$$

The behavior of the fundamental matrix largely depends on the property of the characteristic parameters λ_k, $k=\overline{1,3}$.

The relation $\widetilde{\Phi}(x,-i\omega)=\Phi^T(-x,-i\omega)$ holds, where $\widetilde{\Phi}(x,-i\omega)$ is the matrix of fundamental solutions of the conjugated operator $\widetilde{L}(\frac{\partial}{\partial x},-i\omega)$.

Note that the matrix of fundamental solutions of the static operator $L(\frac{\partial}{\partial x})$ $(\omega=0)$ has the form:

$$\Phi(x)=\frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}\times$$

$$\times \left\| \begin{matrix} \boxed{\dfrac{2(\lambda+2\mu)}{\lambda+\mu}\,\dfrac{\delta_{jk}}{|x|} - \dfrac{\partial^2|x|}{\partial x_j \partial x_k}}_{3\times 3}, & \boxed{\dfrac{\mu\gamma}{\lambda+\mu}\,\mathrm{grad}\,|x|}_{3\times 1} \\ \boxed{0}_{1\times 3}, & \frac{2(\lambda+2\mu)\mu}{\lambda+\mu}\cdot\frac{1}{|x|} \end{matrix} \right\|.$$

We have

$$\Phi(x,-i\omega) - \Phi(x) = \mathrm{const} + O(|x|),$$
$$\frac{\partial}{\partial x_p}\big[\Phi(x,-i\omega) - \Phi(x)\big] = O(1), \quad p = \overline{1,3},$$
$$\frac{\partial^2}{\partial x_p \partial x_q}\big[\Phi_{kj}(x,-i\omega) - \Phi_{kj}(x)\big] = O\big(|x|^{-1}\big), \quad k,j = \overline{1,3},$$
$$k = j = 4, \;\; p,q = \overline{1,3},$$
$$\frac{\partial^2}{\partial x_p \partial x_q}\big[\Phi_{kj}(x,-i\omega) - \Phi_{kj}(x)\big] = O(1), \quad k = \overline{1,3}, \;\; j = 4,$$
$$\text{and} \;\; k = 4, \;\; j = \overline{1,3}.$$

Fundamental solutions are used in a usual manner to construct new singular solutions of the equations which serve as measures of the generalized potentials of the considered boundary value problems.

Principal potentials for system (4) are written as:

$$V(x,\varphi) = \int_S \Phi(x-y,-i\omega)\varphi(y)\,d_yS \quad \text{(the simple-layer potential)};$$
$$W(x,\varphi) = \int_S \Big[\widetilde{R}\Big(\frac{\partial}{\partial y}, n\Big)\widetilde{\Phi}(y-x,-i\omega)\Big]^T \varphi(y)\,d_yS$$
$$\text{(the double-layer potential)};$$
$$M^{(1)}(x,\varphi) = \int_S \Big[\widetilde{\mathcal{P}}\Big(\frac{\partial}{\partial y}, n\Big)\widetilde{\Phi}(y-x,-i\omega)\Big]^T \varphi(y)\,d_yS,$$
$$M^{(2)}(x,\varphi) = \int_S \Big[Q\Big(\frac{\partial}{\partial y}, n\Big)\widetilde{\Phi}(y-x,-i\omega)\Big]^T \varphi(y)\,d_yS$$
$$\text{the mixed-type potentials};$$
$$U(\psi)(x) \equiv U(x,\psi) = \int_D \Phi(x-y,-i\omega)\psi(y)\,dy \quad \text{(the volume potential)}.$$

By virtue of the above constructed mathematical tool the investigation of these potentials, as well as their application in the theory of boundary value problems are accomplished in the same way as in classical thermoelasticity [13], [14]. Therefore all the results obtained in [14] on classical thermoelasticity for oscillations and pseudo-oscillations remain valid in the considered case as well. As for nonstationary problems of general dynamics, they demand a separate investigate, since, for instance, the fourth equation of system (2) is not a parabolic but a hyperbolic one with respect to $v_4(x,t)$. The "compatibility" and other conditions will be changed accordingly [14]. However we do not intend to dwell on this topic here. $\square$

Remark. By Theorem 1 and Lemma 1 an explicit expression of the fundamental matrix $\Phi(x-y,-i\omega)$ containing expressions of the form $\exp(i\lambda_k|x-y|)$, $k=\overline{1,3}$, implies that the radiation conditions (27) are satisfied by each column- vector of this matrix, while for pseudo-oscillations condition (28) is satisfied due to Lemma 2. Thus the constructed potentials possess the required properties at infinity. We note further that using (20) and (21) one can derive formulas for general integral representations of an arbitrary regular vector, which can in turn be used as a basis for investigating boundary value problems and as a source of new estimates.

5. Completeness Theorem

We will prove here some completeness theorems for the definite sets of vector-functions generated by the matrix of fundamental solutions $\Phi(x,-i\omega)$. In doing so, a sufficiently general situation will be considered. These theorems underlie the construction of approximate solutions of the corresponding boundary value problems by the Riesz–Fisher–Kupradze method (the method of discrete singularities).

Let: $D_k\subset\mathbb{R}^3$ be a finite domain with the boundary $S_k\in\mathcal{L}_2(\alpha_0)$, $\alpha_0>0$, ($S_k\cap S_j=\varnothing$, $k\neq j=\overline{0,m}$), S_0 cover all the other domains but the latter S_j not cover one another; $\overline{D}_k\equiv D_k\cup S_k$, $S\equiv S^+\equiv\cup_{k=0}^m S_k$, $D^+\equiv D_0\setminus\cup_{k=1}^m\overline{D}_k$, $D^-\equiv\mathbb{R}^3\setminus\cup_{k=1}^m\overline{D}_k$ be an infinite connected domain with the boundary $S^-\equiv\cup_{k=1}^m S_k$; $\widetilde{D}_k$ be a domain entirely lying strictly within D_k: $\overline{\widetilde{D}}_k\Subset\widetilde{D}_0$, $k=\overline{1,m}$; $\widetilde{D}_0$ be a domain covering D_0; $\overline{D}_0\Subset\widetilde{D}_0$; $\widetilde{S}_k=\partial\widetilde{D}_k$, $k=\overline{0,m}$; $\widetilde{S}\equiv\widetilde{S}^+\equiv\cup_{k=0}^m\widetilde{S}_k$, $\widetilde{S}^-\equiv\cup_{k=0}^m\widetilde{S}_k$. Clearly, $S\cap\widetilde{S}=\varnothing$, $S^-\cap\widetilde{S}\equiv\varnothing$.

Denote by $\{x^k\}_{k=1}^\infty$ a dense everywhere countable set of points on the auxiliary surface $\widetilde{S}$. We introduce the matrix

$$M(y-x,-i\omega)=\big\|\overset{1}{M},\overset{2}{M},\overset{3}{M},\overset{4}{M}\big\|_{4\times4},$$

where

$$M(y-x,-i\omega)=$$
$$=\begin{cases}\Phi(y-x,-i\omega), & y\in\overset{m_1}{\underset{k=0}{\cup}}S_k,\quad x\in\mathbb{R}^3,\\ R\Big(\dfrac{\partial}{\partial y},n\Big)\Phi(y-x,-i\omega), & y\in\overset{m_2}{\underset{k=m_1+1}{\cup}}S_k,\quad x\in\mathbb{R}^3,\\ Q\Big(\dfrac{\partial}{\partial y},n\Big)\Phi(y-x,-i\omega), & y\in\overset{m_3}{\underset{k=m_2+1}{\cup}}S_k,\quad x\in\mathbb{R}^3,\\ \mathcal{P}\Big(\dfrac{\partial}{\partial y},n\Big)\Phi(y-x,-i\omega), & y\in\overset{m}{\underset{k=m_3+1}{\cup}}S_k,\quad x\in\mathbb{R}^3,\end{cases}\tag{29}$$

m_j, $j=\overline{1,3}$, m are arbitrary natural numbers; $0\le m_1\le m_2\le m_3\le m$.

Theorem 6. *The countable set of vectors*

$$\big\{\overset{j}{M}(y-x^k,\zeta)\big\}_{k=1,\;j=1}^{\infty,\;4},\quad y\in S'\equiv\overset{m}{\underset{k=0}{\cup}}S_k \tag{30}$$

is linearly independent and complete in the (vector) Hilbert space $L_2(S)$.

Proof. a) Linear Independence. By assuming that the opposite is true we have

$$\sum_{j=1}^{4}\sum_{k=1}^{N}c_{jk}\overset{j}{M}(y-x^k,\zeta)\equiv 0,\quad y\in S, \tag{31}$$

where at least one $c_{jk}\neq 0$, N is an arbitrary natural number. Consider the vector

$$V(x)=\sum_{j=1}^{4}\sum_{k=1}^{N}c_{jk}\overset{j}{\Phi}(y-x^k,\zeta).$$

By (31) $V(x)$ is a regular in D^+ solution of the homogeneous problem $M_0^+(\zeta)$: $\forall x\in D^+$: $L(\frac{\partial}{\partial x},\zeta)V=0$; $\forall z\in S_k$, $k=\overline{0,m_1}$: $V^+(z)=0$; $\forall z\in S_k$, $k=m_1+1,\dots,m_2$: $[RV(z)]^+=0$; $\forall z\in S_k$, $k=m_2+1,\dots,m_3$: $[QV(z)]^+=0$; $\forall z\in S_k$, $k=m_3+1,\dots,m$: $[\mathcal{P}V(z)]^+=0$. Therefore by virtue of Theorem 4 $V(x)\equiv 0$, $x\in D^+$. Hence, since V is analytic, we have $V(x)\equiv 0$, $x\subset\widetilde{D}_0\setminus\cup_{k=1}^{m}\overline{\widetilde{D}}_k$. Further, by making the point x approach x^k, $k=\overline{1,N}$, and repeating the standard reasoning from [13] we obtain the contradiction $c_{jk}=0$, $j=1,4$, $k=\overline{1,N}$.

b) Completeness. Let $\varphi\in L_2(S)$ be an arbitrary four-dimensional vector. Since $L_2(S)$ is a complete Hilbert space, the necessary and sufficient condition for set (29) to be complete consists in that the orthogonality condition

$$\int_S\overset{j}{M}{}^T(y-x^k,\zeta)\overline{\varphi}(y)\,dS=0,\quad j=\overline{1,4},\quad k=\overline{1,\infty}, \tag{32}$$

would imply the equality $\varphi(y)=0$ almost everywhere on S.

Consider the mixed-type potential

$$\begin{aligned}\widetilde{M}(\varphi)(x)\equiv\widetilde{M}(x,\varphi)=\sum_{k=0}^{m_1}\int\limits_{S_k}\widetilde{\Phi}(x-y,\zeta)\overline{\varphi}(y)\,d_yS+\\+\sum_{k=m_1+1}^{m_2}\int\limits_{S_k}\Big[R\Big(\frac{\partial}{\partial y},n\Big)\Phi(y-x,\zeta)\Big]^T\overline{\varphi}y\,dS+\\+\sum_{k=m_2+1}^{m_3}\int\limits_{S_k}\Big[Q\Big(\frac{\partial}{\partial y},n\Big)\Phi(y-x,\zeta)\Big]^T\overline{\varphi}y\,d_yS+\\+\sum_{k=m_3+1}^{m}\int\limits_{S_k}\Big[\mathcal{P}\Big(\frac{\partial}{\partial y},n\Big)\Phi(y-x,\zeta)\Big]^T\overline{\varphi}y\,d_yS.\end{aligned}\tag{33}$$

Using (29), (30), (32) and assuming that points x^k, $k=\overline{1,\infty}$, are dense everywhere on $\widetilde{S}$, we have $\widetilde{M}(x,\varphi)\equiv 0$, $x\in\widetilde{S}$. Hence, by the uniqueness theorem for the conjugate problem, we readily obtain

$$\widetilde{M}(x,\varphi)=0,\quad x\in\bigcup_{k=1}^{m}\widetilde{D}_k\cup\big(\mathbb{R}^3\backslash\overline{\widetilde{D}}_0\big).\tag{34}$$

Since $\widetilde{M}(x,\varphi)$ is analytic in the domain $\mathbb{R}^3\backslash\cup_{k=0}^{m}S_k$, (34) implies [13]

$$\widetilde{M}(x,\varphi)=0,\quad x\in\bigcup_{k=1}^{m}D_k\cup\big(\mathbb{R}^3\backslash\overline{D}_0\big),\quad\overline{\varphi}\in C^{(1,\alpha)}(S).\tag{35}$$

To solve the homogeneous singular integral equations [13] we have used here some theorems of the embedding type. Applying familiar theorems of the Liapunov-Tauber type (see [13], [14], [15]) to (35), we conclude that $\widetilde{M}(x,\varphi)$ is a regular solution of the homogeneous boundary value problem $\widetilde{M}_0^+(\zeta)$:

$$\begin{gathered}\forall x\in D^+:\ \widetilde{L}\Big(\frac{\partial}{\partial x},\zeta\Big)\widetilde{M}(x,\varphi)=0,\\ \forall z\in\bigcup_{k=0}^{m_1}S_k:\ \widetilde{M}^+(z,\varphi)=0,\quad\forall z\in\bigcup_{k=m_1}^{m_2}S_k:\ \big[\widetilde{R}\widetilde{M}(z,\varphi)\big]^+=0,\\ \forall z\in\bigcup_{k=m_2}^{m_3}S_k:\ \big[\widetilde{Q}\widetilde{M}(z,\varphi)\big]^+=0,\quad\forall z\in\bigcup_{k=m_3}^{m}S_k:\ \big[\widetilde{\mathcal{P}}\widetilde{M}(z,\varphi)\big]^+=0,\end{gathered}$$

and therefore, by the uniqueness theorem,

$$\widetilde{M}(x,\varphi)=0,\quad\forall x\in D^+.\tag{36}$$

Finally, relations (35) and (36) give in turn $\overline{\varphi}(y)=0$, $y\in S$, i.e., $\varphi(y)=0$, which completes the proof of Theorem 6.

Replace, in (29), $y \in \cup_{k=0}^{m_1} S$ by $y \in \cup_{k=1}^{m_1} S_k$ and denote by $M'(y-x,-i\omega)$ the matrix $M(y-x,-i\omega)$ (this means that the surface S_0 "expands" to infinity). □

By a reasoning similar to the above one can prove the following theorems:

Theorem 7. *A countable set of vectors*

$$\big\{\overset{j}{M}(y-x^k,-ip)\big\}_{k=1,\,j=1}^{\infty,\,4}, \quad p>0, \quad y \in S \equiv \overset{m}{\underset{k=0}{\cup}} S_k,$$

is linearly independent and complete in the space $L_2(S)$ if p^2 is different from the eigenvalues of the homogeneous problem $M_0^+(-ip)$.

Theorem 8. *A countable set of vectors*

$$\big\{\overset{j}{M}{}'(y-x^k,\zeta)\big\}_{k=1,\,j=1}^{\infty,\,4}, \quad y \in S^- \equiv \overset{m}{\underset{k=1}{\cup}} S_k,$$

is linearly independent and complete in $L_2(S^-)$.

Theorem 9. *A countable set of vectors*

$$\big\{\overset{j}{M}{}'(y-x^k,-ip)\big\}_{k=1,\,j=1}^{\infty,\,4}, \quad p>0, \quad y \in S^-,$$

is linearly independent and complete in $L_2(S^-)$.

(In Theorems 8 and 9 $\{x^k\} \in \widetilde{S}^- \equiv \cup_{k=1}^{m} \widetilde{S}_k$ is a dense everywhere countable set of points).

6. Solution Continuation Theorems

As is well known, in the three-dimensional domain with the boundary whose some part is plane, a regular solution of the Laplace (also Helmholtz) equation analytically continues across this plane part into an additional domain which is the mirror reflection of the initial domain. In that case the continuable solutions must satisfy additional homogeneous conditions imposed on the plane part of the boundary. This property known by the name of the symmetry principle is important in solving effectively (in quadratures) the boundary value problems for some classes of infinite domains. This principle was generalized by E. I. Obolashvili for equations of classical elasticity [18], [19], by V. D. Kupradze and T. V. Burchuladze for equations of conjugate thermoelasticity, and by T. V. Burchuladze for systems of equations of elastothermodiffusion [15].

Let: Σ be the plane part of the boundary of the domain D given by the equation $\sum_{k=1}^{3} b_k x_k + b_4 = 0$; $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3)$ be the mirror reflection of the point $x \in D$ in Σ (it is assumed that D lies on one side of Σ),

$$\widehat{U}(x) = (\widehat{u}, \widehat{u}_4) = \big\|\widehat{u}_k(x)\big\|_{4\times 1},$$

$$\widehat{u}_4(x) = b^{-2}\sum_{l=1}^{3}\big(2b_k b_l - \delta_{lk}b^2\big)u(\widehat{x}), \quad k=\overline{1,3},$$

$$\widehat{u}_4(x) = -u_4(\widehat{x}), \quad b^2 = \sum_{k=1}^{3} b_k^2.$$

Theorem 10. *If $U(x)=(u,u_4)=\|u_k(x)\|_{4\times 1}$ is a regular in D solution of the homogeneous system* (4)*, satisfying, on Σ, the boundary conditions $u-n(u\cdot n)=0$, $(HU\cdot n)=0$, $u_4=0$, then the vector $V(x)=\|v_k(x)\|_{4\times 1}$,*

$$V(x)=\begin{cases} U(x), & x\in D,\\ \widehat{U}(x), & x\in \widehat{D}\end{cases}$$

is a regular solution of system (4) *in the domain $D\cup\Sigma\cup\widehat{D}$.*

Theorem 11. *If the vector $U=\|u_k(x)\|_{4\times 1}$ is a regular in D solution of the homogeneous system* (4)*, satisfying, on Σ, the boundary conditions $(u\cdot n)=0$, $HU-n(HU\cdot n)=0$, $\frac{\partial u_4}{\partial n}=0$, then the vector $V(x)=\|v_k(x)\|_{4\times 1}$,*

$$V(x)=\begin{cases} U(x), & x\in D,\\ -\widehat{U}(x), & x\in \widehat{D}\end{cases}$$

is a regular solution of system (4) *in the domain $D\cup\Sigma\cup\widehat{D}$.*

The proof of these theorems rests on the equalities

$$\Delta\Big(\frac{\partial}{\partial x}\Big)=\Delta\Big(\frac{\partial}{\partial \widehat{x}}\Big), \quad \operatorname{div}\widehat{U}(x) = -\operatorname{div}U(\widehat{x})$$

which are easy to verify, and on general integral representations of the solution of system (4) (see [13], [14]).

Like in [20], [21], [22], [23], one can use these theorems and Theorem 2 to solve effectively (in quadratures) the boundary value problems for system (4) in the case of infinite domains such as a half-space; right, dihedral and trihedral angles (a quarter and eighth part of the space); dihedral and trihedral angles with opening $\pi/2^m$ (m is a natural number), an infinite layer, a half-layer, a nonhomogeneous (piecewise homogeneous) space, a half-space and so on.

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