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COMMUTATIVITY FOR A CERTAIN CLASS OF RINGS

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Abstract. We first establish the commutativity for the semiprime ring satisfying $[x^n, y]x^r = \pm y^s[x, y^m]y^t$ for all x, y in R, where m, n, r, s and t are fixed non-negative integers, and further, we investigate the commutativity of rings with unity under some additional hypotheses. Moreover, it is also shown that the above result is true for s-unital rings. Also, we provide some counterexamples which show that the hypotheses of our theorems are not altogether superfluous. The results of this paper generalize some of the well-known commutativity theorems for rings which are right s-unital.

1. Introduction

Let R be an associative ring with $N(R)$, $Z(R)$, $C(R)$, $N'(R)$, and R^+ denoting the set of nilpotent elements, the center, the commutator ideal, the set of all zero divisors, and the additive group of R, respectively. For any x, y in R, $[x, y] = xy - yx$. By $GF(q)$ we mean the Galois field (finite field) with q elements, and by $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$. We set

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in $(GF(p))_2$ for prime p.

There are several results dealing with the conditions under which R is commutative. Generally, such conditions are imposed either on the ring itself or on its commutators. Very recently, Abujabal and Perić [1] remarked that if a ring R satisfies either $[x^n, y]y^t = \pm y^s[x, y^m]$ or $[x^n, y]y^t = \pm[x, y^m]y^s$ for all x, y in R, where $m > 1$, $n \geq 1$, and R has the property $Q(m)$ (see below) for $n > 1$, then R is commutative. Also, under different and appropriate constraints, the commutativity of R has been studied for other values of m, n, s and t (see [1]).

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The objective of this paper is to generalize the above-mentioned commutativity results to a certain class of rings satisfying the property

$$[x^n, y]x^r = \pm y^s[x, y^m]y^t \quad \text{for all} \quad x, y \in R \tag{$*$}$$

for any fixed non-negative integers m, n, r, s, and t (see [2], [3], and [4]).

In Section 2, we shall prove the commutativity of semiprime rings satisfying $(*)$ and, in Section 3, study the commutativity of rings with unity satisfying $(*)$. However, in Section 4, we extend these results to the wider class of rings that are called right s-unital.

2. Commutativity Theorem for Semiprime Rings

Theorem 1. *Suppose that $n > 0, m, r, s,$ and t are fixed non-negative integers such that $(n, r, s, m, t) \neq (1, 0, 0, 1, 0)$. Let R be a semiprime ring satisfying $(*)$. Then R is commutative.*

Proof. Let R be a semiprime ring satisfying the polynomial identity

$$h(x, y) = [x^n, y]x^r \mp y^s[x, y^m]y^t = 0. \tag{1}$$

Then R is isomorphic to a subdirect sum of prime rings R_i, $i \in I$ (the index set), each of which as a homomorphic image of R satisfies the hypothesis placed on R. Thus we can assume that R is a prime ring satisfying (1). By Posner's theorem [5, Sec. 12.6, Theorem 8], the central quotient of R is a central simple algebra over a field. If the ground field is finite, then the center of R is a finite integral domain, and so R is equal to its central quotient and is a matrix ring $M_\alpha(S)$ for some $\alpha \geq 1$ and some finite field S. Further, we prove that $\alpha = 1$.

If the ground field is infinite and $h(x, y) = 0$ is the polynomial identity for R, we write $h = h_0 + h_1 + h_2 + h_3 + \cdots + h_{m-1} + h_m$ where h_j, $j = 0, 1, 2, \ldots, m$, is a homogeneous polynomial in x, y. Then $g_0(x, y) = g_1(x, y) = \cdots = g_m(x, y) = 0$ for every x, y in R, since the center of R is infinite. Hence $h_0 = h_1 = h_2 = \cdots = h_m = 0$ is also valid in the central quotient of R. Thus $h = h_0 + h_1 + h_2 + \cdots + h_m = 0$ is satisfied by elements in the central quotient of R. Moreover, $h = 0$ is satisfied by elements in $A \otimes_S B$, where A is the central quotient of R, S the center of A, and B any field extension of S [5, Sec. 12.5, Proposition 3]. As a special case, choosing B to be a splitting field of A, we have $A \otimes_S B \simeq M_\alpha(S)$. Now $f = 0$ is satisfied by the elements in $M_\alpha(S)$. So it is enough to prove $\alpha = 1$. Let e_{ij}, $1 \leq i, j \leq \alpha$, be the matrix units in the ring of $\alpha \times \alpha$ matrices. Suppose that $\alpha \geq 2$. Then (1) can be rewritten as $h(x, y) = [x^n, y]x^r \mp y^s[x, y^m]y^t = 0$, which does not hold in $M_\alpha(S)$ because $h(e_{11} + e_{21}, e_{12}) \neq 0$. Hence we get a contradiction, i.e., $\alpha = 1$. So the central quotient of R is contained in the respective ground field. Hence this proves that R itself is commutative. $\square$

Remark 1. In 1989 W. Streb [6] gave some classification of minimal commutative factors of non-commutative rings. This classification is very useful in proving commutativity theorems for rings satisfying conditions that are not necessarily identities (see details in [7]).

If we take $(n, r, s, m, t) = (1, 0, 0, 1, 0)$ in Theorem 1, then $(*)$ becomes an identity.

The following example demonstrates that we cannot extend the above theorem to arbitrary rings.

Example 1. Let $R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix} \middle| \alpha, \beta \text{ and } \gamma \text{ are integers} \right\}$. Then it can be easily verified that R satisfies (1). However, R is not commutative.

One might ask a natural question: "What additional conditions are needed to force the commutativity for arbitrary rings which satisfies $(*)$?". To investigate the commutativity of the ring R with the property $(*)$, we need some extra conditions on R such as the property:

$Q(m)$: for all x, y in $R, m[x, y] = 0$ implies $[x, y] = 0$, where m is some positive integer.

The property $Q(m)$ is an H-property in the sense of [8]. It is easy to check that every m-torsion free ring R has the property $Q(m)$, and every ring has the property $Q(1)$. Further, it is clear that if the ring R has the property $Q(m)$, then R has the property $Q(n)$ for every divisor n of m. In this direction we prove the following theorems.

3. Some Commutativity Theorems for Rings with Unity

The following theorem is a generalization of H. E. Bell and A. Yaquab obtained in the 80s.

Theorem 2. *Suppose that $n > 1, m, r, s,$ and t are fixed non-negative integers, and let R be a ring with unity 1 satisfying $(*)$. Further, if R has property $Q(n)$, then R is commutative.*

We begin with the following well-known result [9, p. 221].

Lemma 1. *Let x, y be elements in a ring R such that $[x, [x, y]] = 0$. Then for any positive integer h, $[x^h, y] = hx^{h-1}[x, y]$.*

The next four lemmas are essentially proved in [3], [10], [11], and [12], respectively.

Lemma 2. *Let R be a ring with unity 1 and x, y in R. If $k[x, y]x^m = 0$ and $k[x, y](x+1)^m = 0$ for some integers $m \geq 1$ and $k \geq 1$, then necessarily $k[x, y] = 0$.*

Lemma 3. *Let R be a ring with unity 1 and let there exist relatively prime positive integers m and n such that $m[x,y]=0$ and $n[x,y]=0$. Then $[x,y]=0$ for all x,y in R.*

Lemma 4. *Let R be a ring with unity 1. If for each x in R there exists a pair m and n of relative prime positive integers for which $x^m \in Z(R)$ and $x^n \in Z(R)$, then R is commutative.*

Lemma 5. *Let R be a ring with unity 1. If $(1-y^k)=0$, then $(1-y^{km})x=0$ for any positive integers m and k.*

Further, the following results play a key role in proving the main results of this paper. The first and the second are due to Herstein [13] and [14], and the third is due to Kezlan [15].

Theorem A. *Suppose that R is a ring and $n>1$ is an integer. If x^n-x in $Z(R)$ for all x in R, then R is commutative.*

Theorem B. *If for every x and y in a ring R we can find a polynomial $p_{x,y}(z)$ with integral coefficients which depends on x and y such that $[x^2p_{x,y}(x)-x,y]=0$, then R is commutative.*

Theorem C. *Let f be a polynomial in non-commuting indeterminates $x_1,x_2,\ldots,x_n$ with integer coefficients. Then the following statements are equivalent:*
(i) *For any ring R satisfying $f=0$, $C(R)$ is a nil ideal.*
(ii) *For every prime p, $(GF(p))_2$ fails to satisfy $f=0$.*

Now we shall prove the following lemmas:

Lemma 6. *Let $n>1$, m,r,s, and t be fixed non-negative integers, and let R be a ring with unity 1 which satisfies the property $(*)$. Further, if R has $Q(n)$, then $N(R)\subseteq Z(R)$.*

Proof. Let $a\in N(R)$. Then there exists a positive integer p such that

$$a^h \in Z(R) \tag{2}$$

for all $h\geq p$, where p is minimal.

If $p=1$, then $a\in Z(R)$. Now we assume that $p>1$ and $b=a^{p-1}$. Replacing x by b in $(*)$ we get $[b^n,y]y^r=\pm y^s[b,y^m]y^t$. Using (2) and the fact that $(p-1)n\geq p$ for $n>1$, we obtain

$$\pm y^s[b,y^m]y^t=0 \quad \text{for all} \quad y\in R. \tag{3}$$

Replacing $1+b$ for x in $(*)$, we get $[(1+b)^n,y](1+b)^r=\pm y^s[b,y^m]y^t$.

Since $(1+b)$ is invertible (3) leads to

$$[(1+b)^n,y]=0 \quad \text{for all} \quad y\in R. \tag{4}$$

Combining (2) with (4) yields $0 = [(1+b)^n, y] = [1+nb, y] = n[b, y]$. Using the property $Q(n)$ gives $[b, y] = 0$. Thus $a^{p-1} \in Z(R)$, which is a contradiction for the minimality of p. Hence $p = 1$ and $a \in Z(R)$, which implies $N(R) \subseteq Z(R)$. □

Lemma 7. *Let $n > 0, m, r, s,$ and t be fixed non-negative integers, and let R be a ring with unity 1 satisfying $(*)$. Then $C(R) \subseteq Z(R)$.*

Proof. Let $x = e_{11} + e_{21}$ and $y = e_{12}$. Then x and y fail to satisfy $(*)$ whenever $n > 0$ except for $r = 0$, $s = 0$, and $m = 1$. For other cases we can also choose $x = e_{12}$ and $y = e_{22}$. Thus by Theorem C, $C(R)$ is nil and hence by Lemma 6 we get $C(R) \subseteq Z(R)$. □

Remark 2. In view of the above lemma it is guaranteed that Lemma 1 holds for each pair of elements x and y in the ring R which satisfies $(*)$.

Proof of Theorem 2. To get $[x^n, y]x^r = 0$ let $m = 0$ in $(*)$. By Lemmas 1 and 7 this becomes $n[x, y]x^{r+n-1} = 0$. By Lemma 2 and the property of $Q(n)$ this yields the commutativity of R. Let $m \geq 1$ and $k = (\lambda^{n+r} - \lambda)$, where λ is a prime. Then by $(*)$ we have

$$k[x^n, y]x^r = (\lambda^{n+r} - \lambda)[x^n, y]x^r, \quad [(\lambda x)^n, y](\lambda x)^r \mp y^s[(\lambda x), y]y^t = 0.$$

Again by Lemmas 7 and 2 one gets $0 = kn[x, y] = kn[x, y]x^{r+n-1}$. Suppose that $h = kn$; this gives $h[x, y] = 0$. So $[x^h, y] = hx^{h-1}[x, y] = 0$, whence

$$x^h \in Z(R) \quad \text{for all} \quad x \in R. \tag{5}$$

Now, we consider two cases:

Case (1). If $m = 1$ in $(*)$ we get $[x^n, y]x^r = \pm y^s[x, y]y^t$ for all x, y in R. By Lemmas 1 and 7

$$n[x, y]x^{r+n-1} = \pm y^{s+t}[x, y]. \tag{6}$$

Replacing x by x^n in (6), we have

$$n[x, y]x^{n(r+n-1)} = \pm y^{s+t}[x^n, y] \quad \text{for all} \quad x, y \in R. \tag{7}$$

Using Lemmas 1 and 7 together with $(*)$ (7) yields

$$n[x^n, y]x^{n(r+n-1)} = \pm nx^{n-1}[x, y]y^{s+t} = n[x^n, y]x^{r+n-1} \quad \text{for all} \quad x, y \in R.$$

This implies that

$$n[x^n, y]x^{r+n-1}(1 - x^{(n-1)(r+n-1)}) = 0 \quad \text{for all} \quad x, y \in R.$$

By Lemma 5 one can write

$$n[x^n, y]x^{r+n-1}(1 - x^{h(n-1)(r+n-1)}) = 0 \quad \text{for all} \quad x, y \in R. \tag{8}$$

Since R is isomorphic to a subdirect sum of subdirectly irreducible rings Ri $(i \in I)$, each Ri satisfies $(*)$, Lemma 7, and (8). Now we take the ring Ri,

$i \in I$, and assume H is the heart of Ri (i.e., the intersection of all non-zero ideals of Ri). Then $H \neq (0)$ and $Hd = 0$ for any central zero divisor d.

Let $a \in N'(R_i)$. Then by (8) we have

$$n[a^n, y]a^{r+n-1}(1 - a^{h(n-1)(r+n-1)}) = 0 \quad \text{for all} \quad y \in R_i.$$

If $n[a^n, y]a^{r+n-1} \neq 0$, then $a^{h(n-1)(r+n-1)}$ and $1 - a^{h(n-1)(r+n-1)}$ are central zero divisors. Hence $(0) = H(1 - a^{h(n-1)(r+n-1)}) = H$. But $H \neq (0)$, which leads to a contradiction. Hence $n[a^n, y]a^{r+n-1} = 0$ for all $y \in Ri$, $i \in I$. From (7) and the above condition we get

$$0 = \pm y^{s+t}[a^n, y] = n[a^n, y]a^{n(r+n-1)}.$$

Again by Lemma 2, we get $[a^n, y] = 0$ for all $y \in R_i$. Hence

$$\pm[a, y]y^{s+t} = [a^n, y]a^r = 0 \quad \text{and} \quad [a, y] = 0.$$

Let $z \in N(R_i)$. Then by $(*)$ we have

$$(z^{r+n} - z)[x^n, y]x^r = [(zx)^n, y](zx)^r \mp y^s[(zx), y^m]y^t = 0.$$

Lemmas 1, 2, and 7 together with $Q(n)$ give

$$(z^{r+n} - z)[x, y] = 0 \quad \text{for all} \quad x, y \in R_i. \tag{9}$$

Now, as a special case , using (3) we get $(x^{h(r+n)} - x^h)[x, y] = 0$. If $[x, y] = 0$ for all x, y in R_i, then R satisfies $[x, y] = 0$ for all x, y in R and R is commutative. Further, if $[x, y] \neq 0$ for each x, y in R_i, then $x^{h(r+n-1)+1} - x \in N'(R_i)$ and so $x^{h(r+n-1)+1} - x \in N'(R)$. But $[a, y] = 0$ is satisfied by R. So $[x^{h(r+n-1)+1} - x, y] = 0$ for each x, y in R. Hence R is commutative by Theorem A.

Case (2). Let $m > 1$. Then by $(*)$ and together with Lemma 7, we get

$$[x^n, y]x^r = \pm m[x, y]y^{s+t+m-1} \quad \text{for all} \quad x, y \in R. \tag{10}$$

Replacing y by y^m in (10), we get

$$[x^n, y^m]x^r = \pm[x, y^m]y^{m(s+t+m-1)} \quad \text{for all} \quad x, y \in R.$$

Thus by Lemma 1 we obtain $my^{m-1}[x^n, y]x^r = \pm m[x, y^m]y^{m(s+t+m-1)}$.

Applying $(*)$ and Lemma 3, this becomes

$$m[x, y^m]y^{s+t+m-s}(1 - y^{h(m-1)(s+t+m-1)}) = 0 \quad \text{for all} \quad x, y \in R. \tag{11}$$

By Lemmas 7 and 1, $(*)$ becomes

$$n[x, y]x^{r+n-1} = \pm[x, y]y^{s+t+m-1} \quad \text{for all} \quad x, y \in R. \tag{12}$$

Suppose that $a \in N'(R_i)$. Then using (11) and the same argument as in case (1), we write $m[x, a^m]a^{s+t+m-1}(1 - a^{h(m-1)(s+t+m-1)}) = 0$. We can prove that

$$m[x, a^m]a^{s+t+m-1} = 0 \quad \text{for all} \quad x \in R_i. \tag{13}$$

Combining (12) and (13) we get $n[x, a^m]x^{r+n-1} = \pm m[x, a^m]a^{m(s+t+m-1)} = 0$ for all $x \in R_i$. Again using Lemma 1, this yields $n[x, a^m] = 0$. So $nm[x, a]a^{m-1} = 0$. So we shall show that

$$n^2[x, a]x^{r+n-1} = n(n[x, a]x^{r+n-1}) = n([x^n, a]x^r)$$

By (∗) and Lemma 7 we get $n^2[x, a]x^{r+n-1} = \pm n(m[x, a]a^{s+t+m-1}) = 0$; replacing x by $x + 1$ and applying Lemma 2 we have $n^2[x, a] = 0$ for all $x \in R_i$, so that $[x^{n^2}, a] = n^2 x^{n^2-1}[x, a] = 0$. This implies that

$$[x^{n^2}, a] = 0 \quad \text{for all} \quad x \in R_i. \tag{14}$$

Next, let $z \in Z(R_i)$. By arguments similar to those we used in case (1) we have $(z^{r+n} - z)[x^n, y] = 0$ for all $x, y \in R_i$. Using (2), we get

$$(y^{h(r+n)} - y^h)[x^n, y] = 0 \quad \text{for all} \quad x, y \in R_i. \tag{15}$$

Let $y \in R_i$ and let $[x^n, y] = 0$. Then $[x^{n^2}, y^j - y] = 0$ for all positive integers j and $x \in R_i$. If $[x^n, y] = 0$, then $[x^{n^2}, y] = 0$. If $[x^n, y] \neq 0$, then (15) implies that $y^{h(r+n)} - y^h$ is a zero divisor. Hence $y^{h(r+n-1)+1} - y$ is also a zero divisor. But $[x^{n^2}, a] = 0$. Therefore

$$[x^{n^2}, y^{h(r+n-1)+1} - y] = 0 \quad \text{for all} \quad x, y \in R_i. \tag{16}$$

Since each R satisfies (16), the original ring R also satisfies (16). But R possesses $Q(n)$. So by Lemma 1 (16) gives $[x, y^{(r+n-1)+1} - y] = 0$. Hence R is commutative by Theorem A. □

Remark 3. The property $Q(n)$ is essential in Theorem 2. To show this, we consider

Example 2. Let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over Z_2, the ring of integers mod 2. If R is the ring generated by the matrices A, B, and S, then by the Dorroh construction with Z_2, we get a ring with unity. But R is noncommutative and satisfies $[x^2, y] = [x, y^2]$ for all x, y in R.

Remark 4. Also, if we neglect the restriction of unity in the hypothesis, R may be badly noncommutative. Indeed,

Example 3. Let D_k be the ring of $k \times k$ matrices over a division ring D and $A_k = \{a_{ij} \in D_k / a_{ij} = 0,\ i \leq j\}$. A_k is necessarily noncommutative for any positive integer $k > 2$. Now A_3 satisfies $(*)$ for all positive integers m, n.

Example 4. Let

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

be elements of the ring of all 3×3 matrices over Z_2. If R is the ring generated by the elements A_1, B_1, and S_1, then for each integer $m \geq 1$, the ring R satisfies the identity $[x^m, y] = [x, y^m]$ for all x, y in R, but R is noncommutative.

The following results are direct consequences of Theorem 2.

Corollary 1. *Let $n > 1$ and m be positive integers, and let r and t be any non-negative integers. Suppose that R is a ring with unity satisfying the polynomial identity $[x^n, y]x^r = [x, y^m]y^t$ for all $x, y \in R$. Further, if R has property $Q(n)$, then R is commutative.*

Corollary 2 ([3, Theorem 6]). *Let R be a ring with unity 1, and $n > 1$ be a fixed integer. If R^+ is n-torsion free and R satisfies the identity $x^n y - y x^n = x y^n - y^n x$ for all x, y in R, then R is commutative.*

Corollary 3 ([7, Theorem 2]). *Let $n \geq m \geq 1$ be fixed integers such that $mn > 1$, and let R be a ring with unity 1. Suppose that every commutator in R is m-torsion free. Further, if R satisfies the polynomial identity $[x^n, y] = [x, y^m]$ for all x, y in R. Then R is commutative.*

Now, the following theorem shows that the conclusion of Theorem 2 is still valid if the property $Q(n)$ is replaced by requiring m and n to be relatively prime positive integers.

Theorem 3. *Suppose that $m > 1$, and $n > 1$ are relatively prime positive integers, and r, s, and t are nonnegative integers. Let R be a ring with unity satisfying $(*)$. Then R is commutative.*

Proof. Without loss of generality we may assume that R is a subdirectly irreducible ring. Suppose that $a \in N(R)$ and choose p and b as in Lemma 7. By the arguments of the proof of Lemma 7 we get $n[b, y] = 0$ and $m[b, y] = 0$, whence by Lemma 3 $[b, y] = 0$ for all y in R. But $a^{p-1} \in Z(R)$, i.e., $N(R) \subseteq Z(R)$, and by Lemma 7 we get $C(R) \subseteq N(R) \subseteq Z(R)$. The proof of (2) also works in the present situation. So there exists an integer h such that

$$x^h \in Z(R) \quad \text{for all} \quad x \in R. \tag{17}$$

Suppose that $c \in N'(R)$. Using the same steps as in the proof of Theorem 2 (see (14)), we obtain $[x^{n^2}, c] = 0$ and $[x^{m^2}, c] = 0$. Hence by Lemma 3 we get

$$[x, c] = 0 \quad \text{for all} \quad x \in R \quad \text{and} \quad c \in N'(R). \tag{18}$$

As is observed in the proof followed by (9) one can see that

$$n(z^{r+n} - z)[x, y] = 0 \quad \text{and} \quad m(z^{r+n} - z)[x, y] = 0, \quad \text{where} \quad z \in Z(R).$$

Again using Lemma 3, we get

$$(z^{r+n} - z)[x, y] = 0 \quad \text{for all} \quad x, y \in R \quad \text{and} \quad z \in Z(R). \tag{19}$$

Since $y^h \in Z(R)$, (17) yields $(y^{h(r+n)} - y^h)[x, y] = 0$ for all $x, y \in R$. Using the same arguments as in the proof of Theorem 2, we finally get $y^{h(r+n-1)+1} - y \in N'(R)$ so that (18) gives $y^{h(r+n-1)+1} - y \in Z(R)$ for all $y \in R$. Hence by Theorem A, R is commutative. □

As a consequence of Theorem 3 we obtain

Corollary 4. *Suppose that m and n are relatively prime positive integers, and let r and t be any nonnegative integers. Let R be a ring with unity satisfying $[x^n, y]x^r = [x, y^m]y^t$. Then R is commutative.*

Further, the following result deals with the commutativity of R for the case where $(*)$ is satisfied with $n = 1$. Thus we prove:

Theorem 4. *Suppose that R is a ring with unity, and $m, r, s,$ and t are fixed nonnegative integers such that $(m, r, s, t) \neq (1, 0, 0, 0)$. If R satisfies*

$$[x, y]x^r = \pm y^s[x, y^m]y^t \quad \textit{for all} \quad x, y \in R, \tag{**}$$

then R is commutative.

Proof. First, we consider the following cases:

Case 1. Let $m = 0$ in $(**)$. Then $[x, y]x^r = 0$. Replacing x by $x + 1$ and using Lemma 2, we obtain the commutativity of R.

Case 2. Let $m > 1$ in $(**)$. Then we choose the matrix for $x = e_{22}$ and $y = e_{12}$ fail to satisfy $(**)$. Thus $C, C(R) \subseteq Z(R)$ by Theorem.

Let $a \in N(R)$. Then there exists a positive integer p such that

$$a^h \in Z(R) \quad \text{for all} \quad h \geq p \quad \text{and} \quad p \quad \text{is minimal.} \tag{20}$$

If $p = 1$, then $a \in Z(R)$. Suppose that $p > 1$ and let $b = a^{p-1}$. Replacing b by y in $(**)$ we get $[x, b]x^r = \pm b^s[x, b^m]b^t$. Using (20) we have $[x, b]x^r = 0$ which by Lemma 2 becomes $[x, b] = 0$. Thus $a^{p-1} \in Z(R)$, which contradicts the minimality of p. Hence $p = 1$ and $N(R) \subseteq Z(R)$. Thus $C(R) \subseteq N(R) \subseteq Z(R)$ and the proof of Theorem 2 enables us to establish the commutativity of R.

Case 3. If $m = 1$ in $(**)$, we have

$$[x, y]x^r = \pm y^s[x, y]y^t \quad \text{for all} \quad x, y \in R. \tag{21}$$

Step (i). Assuming $r = 0$ in (21), we get

$$[x, y] = \pm y^s[x, y]y^t \quad \text{for all} \quad x, y \in R. \tag{22}$$

Then either $s > 0$ or $t > 0$. Trivially, we can see that $x = e_{22}$ and $y = e_{12}$ fail to satisfy (22). Hence $C(R) \subseteq N(R)$. Suppose that p and b are defined as in Case (2). Then (22) holds and becomes $[x, b] = \pm b^s[x, b]b^t = 0$ for all x in R, which is a contradiction. Hence $a \in Z(R)$ so that $N(R) \subseteq Z(R)$. Therefore

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{23}$$

Using (23) and Lemma 1 we get $[x, y] = \pm[x, y]y^{r+s}$ for x, y in R. Hence R is commutative by Kezlan [16].

Step (ii). If $s = 0$ in (21), we get

$$[x, y]x^r = \pm[x, y]y^t \quad \text{for all} \quad x, y \in R. \tag{24}$$

Let $t = 0$. Then $r > 0$ and (24) become $[x, y]x^r = \pm[x, y]$. Hence R is commutative [12]. Now, let $r = 0$ and $t > 0$. Then (24) gives $[x, y] = \pm[x, y]y^t$ for all $x, y \in R$. Again R is commutative by Kezlan [16].

Finally, if $r > 0$, $t > 0$, then $x = e_{22}$ and $y = e_{12}$ fail to satisfy (24). Hence by Theorem C, $C(R) \subseteq N(R)$. For any positive integer k we have

$$[x, y]x^{kr} = \pm[x, y]y^{kt} \quad \text{for all} \quad x, y \in R. \tag{25}$$

Let $a \subseteq N(R)$. Then for sufficiently large k, we get $[x, a]x^{kr} = 0$. Using Lemma 2 this gives $a \in Z(R)$ and thus $C(R) \subseteq N(R) \subseteq Z(R)$. Further, we choose $q = (p^{t+1} - p) > 0$ for $t > 0$, p is a prime. We can prove that

$$x^q \in Z(R) \quad \text{for all} \quad x \in R. \tag{26}$$

Using (25) and (26), we get $[x^{qr+1}, y] = \pm[x, y^{qt+1}]$. In view of Proposition 3 (ii) of [10], there exists a positive integer l such that $[x, y^{(qt+1)l}] = 0$ for each x, y in R. But $(qt + l)^l = gh + 1$. So (25) becomes $[x, y]y^{hq} = 0$ and by Lemma 2 R is commutative.

Step (iii). Setting $t = 0$ in (21), we get

$$[x, y] = \pm y^s[x, y] \quad \text{for all} \quad x, y \in R. \tag{27}$$

Now we have $r > 0$ and $s > 0$. Without loss of generality we may assume that $s > 0$. So, trivially, we can see that $x = e_{22}$ and $y = e_{12}$ fail to satisfy (27). Hence by Theorem C, $C(R) \subseteq N(R)$. By the same arguments as in Step (ii) we can show the commutativity of R.

Step (iv). Suppose that $r > 0$, $s > 0$, and $t > 0$ in (21), and suppose that $x = e_{22}$, $y = e_{12}$ fail to satisfy (21). So $C(R) \subseteq N(R)$. p and b are defined in the same manner as in Case (2). So $[x, b]x^r = \pm b^s[x, b]b^t = 0$. Using Lemma 2, we get $[x, b] = 0$, which contradicts the minimality of p. Hence $N(R) \subseteq Z(R)$ so that $C(R) \subseteq N(R) \subseteq Z(R)$.

Since $C(R) \subseteq Z(R)$, we can write $[x, b]x^r = \pm[x, b]y^{s+t}$.

Using the same argument as in step (ii), we can get the commutativity of R. □

Theorem 5. *Suppose that $n > 0$ and m (resp. $m > 0$ and n) are two fixed non-negative integers. Suppose that a ring with unity satisfies the polynomial identity $[x^n \pm y^m, yx] = 0$ for all x, y in R. Further, if R has the property $Q(n)$, then R is commutative.*

Proof. By hypothesis, we $[x^n, y]x = \pm y[x, y^r]$. Hence R is commutative by Theorem 2. □

Corollary 5. *t Let $m > 1$ and $n > 1$ be relatively prime integers and R be a ring with unity satisfying $[x^n \pm y^m, yx] = 0$ for all x, y in R. Then R is commutative.*

Recently, Harmanci [4] proved that if $n > 1$ is a fixed integer and R is a ring with unity 1 which satisfies the identities $[x^n, y] = [x, y^n]$ and $[x^{n+1}, y] = [x, y^{n+1}]$ for each $x, y \in R$, then R must be commutative. In [17], Bell generalized this result. The following theorem further extends the result of Bell.

Theorem 6. *Suppose that $m > 1$ and $n > 1$ are fixed relatively prime integers, and let $r, s,$ and t be fixed non-negative integers: R is a ring with unity satisfying both identities*

$$[x^n, y]x^r = \pm y^s[x, y^n]y^t \quad \textit{and} \quad [x^m, y]x^r = \pm y^s[x, y^m]y^t. \tag{$***$}$$

Then R is commutative.

Proof. Suppose that b is as in the proof of Lemma 6. Using the proof of Theorem 1 and Theorem 2 of [4], we can show that $n[b, y] = 0$ and $m[b, y] = 0$ for all y in R. Applying Lemma 3, we get $[b, y] = 0$. By the same argument as in the proof of Lemma 6, we get $N(R) \subseteq Z(R)$. The matrices $x = e_{22}$ and $y = e_{12}$ fail to satisfy $(***)$. Thus by Theorem C, $C(R) \subseteq N(R)$.

And thus $C(R) \subseteq N(R) \subseteq C(R)$. Carrying out the argument of subdirectly irreducible rings for n and m, we obtain integers $\alpha > 1$ and $h > 1$ such that $[x^\alpha - x, y^{n^2}] = 0$ and $[x^h - x, y^{n^2}] = 0$ for all $x, y \in R$. Suppose that $g(x) = (x^\alpha - x)^h - (x^\alpha - x)$. Then $0 = [g(x), y^{n^2}] = n^2[g(x), y]y^{n^2-1}$ and $0 = [g(x), y^{m^2}] = m^2[g(x), y]y^{m^2-1}$. By Lemma 3 and Lemma 4 we get $[g(x), y]y^s = 0$ for all $x, y \in R$ and $s = \max\{m^2-1, n^2-1\}$. So $g(x) \in Z(R)$. But $g(x) = x^2h(x) - x$ with $h(x)$ having integral coefficients. Hence R is commutative by Theorem B. $\square$

As a consequence of Theorem 6 we get the result which is proved in [3].

Corollary 6. *Let $m > 1$ and $n > 1$ be relatively prime positive integers. If R is any ring with unity satisfying both identities $[x^m, y] = [x, y^m]$ and $[x^n, y] = [x, y^n]$ for all $x, y \in R$, then R is commutative.*

4. Extension for s-Unital Rings

We pause to recall a few preliminaries in order to make our paper self-contained as far as possible. A ring R is called right (resp. left) s-unital if $x \in xR$ (resp. $x \in Rx$) for all x in R, and R is called s-unital if for any finite subset F of R there exists an element e in R such that $xe = ex = x$(resp. $ex = x$ or $xe = x$) for all $x \in F$. The element is called a pseudo-identity of F (see [18]). The results proved in the earlier sections can be extended further to right s-unital rings.

The following Lemma is proved in [19].

Lemma 8. *Let R be a right (resp. left) s-unital ring. If for each pair of elements x, y of R there exist a positive integer $k = k(x, y)$ and an element $e^1 = e^1(x, y)$ of R such that $e'x^k = x^k$ and $e'y^k = y^k$ (resp $x^ke' = x^k$ and $y^ke' = y^k$), then R is s-unital.*

Theorem 7. *Suppose that $n > 1, m, r, s,$ and t are fixed non-negative integers, and let R be a right s-unital ring satisfying $(*)$. Further, if R has property $Q(n)$, then R is commutative.*

Proof. Let x and y be arbitrary elements of R. Suppose that R is a right s-unital ring. Then there exists an element $e \in R$ such that $xe = x$ and $ye = y$. Replacing x by e in $(*)$ we get $[e^n, y]e^r = \pm y^s[e, y^m]y^t$ for all $y \in R$. This implies that $y = e^ny$ for all $y \in R$. So $y \in Ry$. Hence in view of Lemma 8 R is an s-unital ring and by Proposition 1 of [8], we may assume that R has unity 1. Thus R is commutative by Theorem 2. $\square$

Corollary 7. *Let r and m be two fixed nonnegative integers. Suppose that R satisfies the polynomial identity $[x, y]x^r = [x, y^m]$ for all $x, y \in R$. Further,*

(i) *if R is a right s-unital ring, then R is commutative except* $(m, r) = (1, 0)$*;*

(ii) *if R is a left s-unital ring, then R is commutative except when* $(m, r) = (0, 1)$ *and* $r > 0$ *and* $m = 1$.

Remark 5. Let $t = 0$ in $(*)$. Then Theorem 7 and Corollary 7 are special cases of [20, Corollary 3] and [20, Theorem 5].

Remark 6. In Corollary 7, for $m > 1, R$ is commutative by Theorem 6. However, for $m = 0$ (resp. $m = 1$ and $r > 0$) it is trivial to prove the commutativity of R.

Theorems such as Theorem 3, Theorem 4, Theorem 5, and Theorem 6 can also be proved for right s-unital rings by the same lines as above employing the necessary variations.

Remark 7. If we take $m = 0$ and $n \geq 1$ in $(*)$, then Theorem 7 need not be true for left s-unital rings. Also, when $m = 0$ and $t = 1$, Corollary 4 is not valid for s-unital rings. Indeed,

Example 5. Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ be a subring of all 2×2 matrices over $GF(2)$ which is a non-commutative left s-unital ring satisfying $(*)$.

Remark 8. If $m = 0$ and $n > 0$ in $(*)$, then Theorem 7 need not be true for left s-unital rings. Owing to this fact, Example 5 disproves Theorems 3, 4, 5, 6, and 7 for left s-unital case whenever both r and s are positive.

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