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MULTIPLICATIVE B-PRODUCT AND ITS PROPERTIES

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Abstract. The author considers a special kind of B-product which is called the multiplicative B-product. The most important property of multiplicative B-product is that it preserves multiplicativity of functions.

In the previous work [1], the author showed a new kind of convolution product called the B-product defined as follows. For every natural number n let B_n be the set of some pairs (r, s) of divisors of n. For the arithmetical functions f and g their B-product $f *_B g$ is as follows:

$$(f *_B g)(n) = \sum_{(r,s)\in B_n} f(r)g(s), \quad \text{for} \quad n = 1, 2, \dots .$$

This B-product generalizes simultaneously the A-product of W. Narkiewicz [2] and the l.c.m. product and has a nonempty intersection with the ψ-product of D. H. Lehmer [3]. The τ-product of H. Scheid [4] is also a particular case of the B-product.

In this paper the author considers a special kind of B-product which is called the multiplicative B-product. It is defined that a B-product is multiplicative if for every pair (m, n) of relatively prime natural numbers

$$B_{mn} = \big\{(r_1 r_2, s_1 s_2) : (r_1, s_1) \in B_m, (r_2, s_2) \in B_n\big\}.$$

This definition can also be formulated as follows. For every k and n denote by $k^{(n)}$ the g.c.d. of k and n. Then the B-product is multiplicative iff for every pair (m, n) of relatively prime natural numbers we have

$$(r, s) \in B_{mn} \quad \text{iff} \quad (r^{(m)}, s^{(m)}) \in B_m \quad \text{and} \quad (r^{(n)}, s^{(n)}) \in B_n. \tag{1}$$

We say that an arithmetical function f is multiplicative iff $f(mn) = f(m)f(n)$ for every pair (m, n) of relatively prime natural numbers.

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A multiplicative B-product is determined uniquely by the sets B_{p^n}, where p is a prime and $n \geq 1$. These sets can be given arbitrarily. Moreover, for every multiplicative B-product we have $B_1 = \{(1,1)\}$ provided that $B_n \neq \varnothing$ for some n. As for multiplicative function, it is determined uniquely by its values on arguments of the form p^n, where p is a prime and $n \geq 1$. These values can be given arbitrarily. Moreover, for every multiplicative function f we have $f(1) = 1$ provided that $f(n) \neq 0$ for some n.

In this paper we discuss some properties of this multiplicative B-product. The most important property is that this product preserves multiplicativity of functions. More precisely, the following theorem holds.

Theorem 1.1. *The B-product of multiplicative functions is a multiplicative function iff the B-product is multiplicative.*

Proof. $\Leftarrow$ Suppose that the B-product is multiplicative. Let f, g be multiplicative functions and let (m,n) be a pair of relatively prime natural numbers. Then

$$(f *_B g)(mn) = \sum_{(r,s)\in B_{mn}} f(r)g(s) = \sum_{(r^{(m)},s^{(m)})\in B_m} \sum_{(r^{(n)},s^{(n)})\in B_n} f(r^{(m)}, r^{(n)}) \times$$
$$\times g(s^{(m)}, s^{(n)}) = \sum_{(r^{(m)},s^{(m)})\in B_m} f(r^{(m)})g(s^{(m)}) \sum_{(r^{(n)},s^{(n)})\in B_n} f(r^{(n)})g(s^{(n)}) =$$
$$= (f *_B g)(m) \cdot (f *_B g)(n).$$

$\Rightarrow$ Let (m,n) be a pair of relatively prime natural numbers and let $r|mn$, $s|mn$. We shall prove by induction on rs that (1) holds:

Let $rs = 1$, i.e., $r = s = 1$. For every d we have

$$(e_1 *_B e_1)(d) = \sum_{(r,y)\in B_d} e_1(r)\, e_1(y) = \begin{cases} 1 & \text{if } \ (1,1) \in B_d, \\ 0 & \text{otherwise.} \end{cases}$$

By assumption, the function $e_1 *_B e_1$ is multiplicative. Therefore $(e_1 *_B e_1)(mn) = (e_1 *_B e_1)(m) \cdot (e_1 *_B e_1)(n)$. Consequently, $(e_1 *_B e_1)(mn) = 1$ iff $(e_1 *_B e_1)(m) = (e_1 *_B e_1)(n) = 1$. Thus $(1,1) \in B_{mn}$ iff $(1,1) \in B_m$ and $(1,1) \in B_n$.

Now let $r_0|mn$, $s_0|mn$, $r_0 s_0 > 1$ and assume that for every $r|mn$, $s|mn$ with $rs < r_0 s_0$ (1) holds. Put

$$f(d) = \begin{cases} 1 & \text{if } \ d|r_0, \\ 0 & \text{otherwise,} \end{cases} \qquad g(d) = \begin{cases} 1 & \text{if } \ d|s_0, \\ 0 & \text{otherwise.} \end{cases}$$

The functions f and g are multiplicative. Therefore, by assumption, the function $f *_B g$ is multiplicative. In particular, we have

$$(f *_B g)(mn) = (f *_B g)(m) \cdot (f *_B g)(n). \tag{2}$$

For every k we have

$$(f *_B g)(k) = \sum_{(r,y)! \in B_k} f(r)g(y) = \sum_{\substack{(r,y)\in B_k \\ r|r_0^{(k)}, y|s_0^{(k)}}} 1 = \sum_{(r_0^{(k)}, s_0^{(k)}) \in B_k} 1 + \sum_{\substack{(r,y)\in B_k r|r_0^{(k)}, s|s_0^{(k)} \\ rs < r_0^{(k)} s_0^{(k)}}} 1.$$

Substituting this into (2), for $k = mn$, m, and n, and using the multiplicativity of the B-product, we get

$$\sum_{(r_0,s_0)\in B_{mn}} 1 + \sum_{\substack{(r_0,s)\in B_{mn} \\ r|r_0, s|s_0 \\ rs<r_0 s_0}} 1 = \sum_{\substack{(r_0^{(m)}, s_0^{(m)})\in B_m \\ (r_0^{(n)}, s_0^{(n)})\in B_n}} 1 + \sum_{\substack{(r,s)\in B_{mn} \\ r|r_0, s|s_0 \\ rs<r_0 s_0}} 1.$$

Thus it follows that $(r_0, s_0) \in B_{mn}$ iff $(r_0^{(m)}, s_0^{(m)}) \in B_m$ and $(r_0^{(n)}, s_0^{(n)}) \in B_n$. □

Now we give conditions for the commutativity and associativity of the multiplicative B-product analogous to those given in [1] for an arbitrary B-product. We omit the proofs since they are straightforward.

Theorem 1.2. *If a B-product is multiplicative, then it is commutative iff the following condition holds:*

For every nonnegative integer k, m, n and every prime number p we have $(p^k, p^m) \in B_{p^n}$ iff $(p^m, p^k) \in B_{p^n}$.

Theorem 1.3. *If a B-product is multiplicative, then it is associative iff the following condition is satisfied.*

For every nonnegative integer j, k, m, n and a prime number p we have

$$\sum_{\substack{b \\ (p^b,p^j)\in B_{p^n}, (p^k,p^m)\in B_{p^b}}} 1 = \sum_{\substack{c \\ (p^k,p^c)\in B_{p^n}, (p^m,p^j)\in B_{p^c}}} 1.$$

Theorem 1.4. *If e is the unit in the system $R_B = \langle \mathbb{C}^{\mathbb{N}}, +, *_B \rangle$, where $\mathbb{C}^{\mathbb{N}}$ stands for the set of all arithmetical functions, i.e., the set of all complex valued functions defined on the set of natural numbers and $*_B$ is a multiplicative B-product, then the function e is multiplicative.*

Proof. Let m and n be relatively prime natural numbers. We shall prove that $e(mn) = e(m)\, e(n)$ proceeding by induction on mn. For $mn = 1$, this equality follows from Corollary 2.5 of [1]. Let $mn > 1$. In view of Theorem 2.4 of [1], for every $r|mn$ and $k|mn$ we have

$$e_{k^{(m)}}(m) = \sum_{\substack{r^{(m)} \\ (r^{(m)},k^{(m)})\in B_m}} e(r^{(m)}), \quad e_{k^{(n)}}(n) = \sum_{\substack{r^{(n)} \\ (r^{(n)},k^{(n)})\in B_n}} e(r^{(n)}).$$

Therefore from the multiplicativity of the B-product we get

$$e_k(mn) = e_{k^{(m)}}(m) \cdot e_{k^{(n)}}(n) = \sum_{\substack{r \\ (r,k)\in B_{mn}}} e(r^{(m)}) \cdot e(r^{(n)}). \qquad (3)$$

In view of Corollary 2 of Theorem 2.5 of [1] there exists k such that $(mn, k) \in B_{mn}$. For this k, in view of (3) and the inductive assumption, we have $e_k(mn) = \sum_{\substack{r \\ (r,k)\in B_{mn}, r<mn}} e(r) + e(m) \cdot e(n)$. On the other hand, from Theorem 2.4 of [1] it follows that

$$e_k(mn) = \sum_{\substack{r \\ (r,k)\in B_{mn}}} e(r) = \sum_{\substack{r \\ (r,k)\in B_{mn}, r<mn}} e(r) + e(mn).$$

The two latter relations lead to $e(mn) = e(m)\, e(n)$. □

Theorem 1.5. *If a B-product is multiplicative, then a multiplicative function e is the unit of the system R_B iff, for every nonnegative integer m and t and a prime number p, it satisfies the condition*

$$\sum_{\substack{s \\ (p^t,p^s)\in B_{p^m}}} e(p^s) = \sum_{\substack{s \\ (p^s,p^t)\in B_{p^m}}} e(p^s) = e_t(m).$$

Corollary 1.6. *If the B-product is multiplicative, then the function e_1 is the unit of the system R_B iff $(p^k, 1) \in B_{p^n} \Leftrightarrow k = n \Leftrightarrow (1, p^k) \in B_{p^n}$ for every nonnegative integer k and n and a prime number p.*

Proof. $\Rightarrow$ The result follows immediately from Corollary 2.6 of [1]. We have

$$\sum_{\substack{s \\ (p^k,p^s)\in B_{p^n}}} e_1(p^s) = \begin{cases} 1 & \text{if} \quad (p^k, 1) \in B_{p^n} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if} \quad k = n \\ 0 & \text{otherwise} \end{cases} = e_{p^k}(p^n).$$

Similarly, one can prove that $\sum_{\substack{s \\ (p^s,p^k)\in B_{p^n}}} e_1(p^s) = e_{p^k}(p^n)$. Therefore from Theorem 1.5 it follows that e_1 is the unit of the system R_B. □

By extending Corollary 2.5 of [1] we shall give below some necessary conditions for a function e to be the unit of the system R_B.

Proposition 1.7. *If e is the unit of the system R_B, then*

(i) *$e(1) = 1$ and $B_1 = \{(1, 1)\}$.*

(ii) *For every prime number p we have $e(p) = 1$ and $B_p = \{(p, p)\}$ or $e(p) = -1$ and $B_p = \{(1, 1), (1, p), (p, 1)\}$ or*

$$e(p) = 0 \;\; \text{and} \;\; B_p = \{(1, p), (p, 1)\} \;\; \text{or} \;\; \{(1, p), (p, 1), (p, p)\}.$$

(iii) *For every prime number p, $e(p^2) = 0, 1,$ or -1.*

Proof. To prove (i) see Corollary 2.5 of [1]. As for (ii), denote for fixed n,

$$x_{rs} = x_{rs}^{(n)} = \begin{cases} 1 & \text{if } \ (r,s) \in B_n, \\ 0 & \text{otherwise.} \end{cases}$$

For $n = p$ Theorem 1.5 gives

$$x_{11} + x_{1p}e(p) = x_{11} + x_{p1}e(p) = 0, \quad x_{p1} + x_{pp}e(p) = x_{1p} + x_{pp}e(p) = 1.$$

From these equalities it easily follows that:

If $x_{11} = 1$, then $x_{1p} = x_{p1} = 1$, $e(p) = -1$,

If $x_{11} = 0$ and $x_{1p} = 0$, then $x_{p1} = 0$, $x_{pp} = 1$, $e(p) = 1$,

If $x_{11} = 0$ and $x_{1p} = 1$, then $x_{p1} = 1$, $e(p) = 0$ and $x_{pp} = 0$ or 1.

To prove (iii) a reasoning similar to the above leads for $n = p^2$ to the equalities

$$\begin{aligned} &x_{11} + x_{1p}\, e(p) + x_{1p^2}\, e(p^2) = x_{11} + x_{p1}\, e(p) + x_{p^21}e(p^2) = 0, \\ &x_{p1} + x_{pp}\, e(p) + x_{pp^2}\, e(p^2) = x_{1p} + x_{pp}\, e(p) + x_{p^2p}\, e(p^2) = 0, \\ &x_{p^21} + x_{p^2p}\, e(p) + x_{p^2p^2}\, e(p^2) = x_{1p^2} + x_{pp^2}\, e(p) + x_{p^2p^2}\, e(p^2) = 1. \end{aligned} \tag{4}$$

Suppose that $e(p^2) \neq 0, 1, -1$. Then from the last equality it follows that $x_{p^21} = 1$ or $x_{p^2p} = 1$. Therefore from the first two equalities of (4) we deduce respectively that $e(p^2) = -x_{11} - x_{p1}e(p)$ or $e(p^2) = -x_{1p} - x_{pp}e(p)$.

Since in view of (ii) we have $e(p) = 0, 1$, or -1, we conclude that in both cases $e(p^2) = -2$, $e(p) = 1$ and respectively $x_{11} = x_{p1} = 1$ or $x_{1p} = x_{pp} = 1$.

In the first case from the first equality of (4) we deduce that $x_{1p^2} = 1$ and from the second one that $x_{pp} = x_{pp^2} = 1$. Hence the last part of the third equality of (4) takes the form $1 + 1 - 2x_{p^2p^2} = 1$. This is a contradiction. In the second case we obtain a contradiction in a similar way. □

Theorem 1.8. *If e is the unit in the system R_B, where $*_B$ is a multiplicative B-product, then $|e(n)| < n$ for every n.*

Proof. In view of Theorem 1.4 the function e is multiplicative. Therefore it is sufficient to prove that $|e(p^n)| < p^n$ for every prime number p and every natural number n.

From Theorem 1.5 we have

$$\sum_{\substack{r \\ (p^r, p^k) \in B_{p^n}}} e(p^r) = e_{p^k}(p^n) = \sum_{\substack{r \\ (p^k, p^r) \in B_{p^n}}} e(p^r). \tag{5}$$

From (10), for $k = n$, it follows that $(p^t, p^n) \in B_{p^n}$ for some t. Take the least value of t. If $t = n$, then from (5), for $k = n$, it follows that $e(p^n) = 1$.

If $t < n$, then from (5), for $k = t$ (in the second equality) we get

$$0 = \sum_{\substack{r \\ (p^t, p^r) \in B_{p^n}}} e(p^r) = \sum_{\substack{r<n \\ (p^t, p^r) \in B_{p^n}}} e(p^r) + e(p^n).$$

Therefore, in view of the inductive assumption, we get

$$|e(p^n)| \le \sum_{r=0}^{n-1} |e(p^r)| \le \sum_{r=0}^{n-1} p^r < p^n. \quad \square$$

Remark. In fact, the above proof jointly with the Proposition 1.7 gives a stronger inequality $|e(p^n)| \le 2^{n-1}$ for $n \ge 1$, and therefore $|e(n)| \le 2^{\Omega(n)-\omega(n)}$ for every n, where $\omega(n)$ is the number of different prime factors of n, and $\Omega(n)$ is the total number of prime factors of n.

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