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SOLUTION OF SOME WEIGHT PROBLEMS FOR THE RIEMANN–LIOUVILLE AND WEYL OPERATORS

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Abstract. The necessary and sufficient conditions are found for the weight function v, which provide the boundedness and compactness of the Riemann–Liouville operator R_α from L^p to L^q_v. The criteria are also established for the weight function w, which guarantee the boundedness and compactness of the Weyl operator W_α from L^p_w to L^q.

In this paper, the necessary and sufficient conditions are found for the weight function v (w), which provide the boundedness and compactness of the Riemann-Liouville transform $R_\alpha f(x) = \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt$ (of the Weyl transform $W_\alpha f(x) = \int_x^\infty \frac{f(t)}{(t-x)^{1-\alpha}} dt$) from L^p to L^q_v (from L^p_w to L^q) when $1 < p,\ q < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$ ($\frac{q-1}{q} < \alpha < 1$ or $\alpha > 1$).

A complete description of the weight pairs (v, w) providing the boundedness of the operators R_α and W_α from L^p_w to L^q_v when $1 < p < q < \infty$ and $0 < \alpha < 1$ is given in [1]. For $1 < p \le q < \infty$ and $\alpha > 1$ a similar problem has been solved by many authors (see, e.g., [2, 3]).

The necessary and sufficient conditions for pairs of weights, which provide the boundedness of the above-mentioned operators when $1 < q < p < \infty$ and $\alpha > 1$, are obtained in [4].

For $1 < q \le p < \infty$ and $0 < \alpha < 1$, the two-weight problem for the operators R_α and W_α remains unsolved and in this context the results presented here are interesting.

Let v and w be positive almost everywhere, locally integrable functions defined on $\mathbb{R}_+$.

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Denote by L^p_v $(1 < p < \infty)$ a class of all Lebesgue-measurable functions defined on $\mathbb{R}_+$ for which

$$\|f\|_{L^p_v} = \Big(\int\limits_0^\infty |f(x)|^p v(x)\,dx\Big)^{\frac{1}{p}} < \infty.$$

First, let us recall some familiar results.

Theorem A ([5–10]). *Let $1 \le p \le q < \infty$. The inequality*

$$\Big(\int\limits_0^\infty \Big|\int\limits_0^x f(t)dt\Big|^q v(x)dx\Big)^{\frac{1}{q}} \le c\Big(\int\limits_0^\infty |f(x)|^p w(x)dx\Big)^{\frac{1}{p}}, \tag{1}$$

where the positive constant c does not depend on f, is fulfilled iff

$$D = \sup_{t>0}\Big(\int\limits_t^\infty v(x)dx\Big)^{\frac{1}{q}}\Big(\int\limits_0^t w^{1-p'}(x)dx\Big)^{\frac{1}{p'}} < \infty \quad \Big(p' = \frac{p}{p-1}\Big).$$

Moreover, if c is the best constant in (1), then $c \approx D$ (the symbol $\approx$ here denotes a two-sided inequality).

Theorem B ([10]). *Let $1 \le q < p < \infty$. Then inequality (1) holds iff*

$$D_1 = \Bigg(\int\limits_0^\infty \Big[\Big(\int\limits_t^\infty v(x)dx\Big)\Big(\int\limits_0^t w^{1-p'}(x)dx\Big)^{q-1}\Big]^{\frac{p}{p-q}} w^{1-p'}(t)dt\Bigg)^{\frac{p-q}{pq}} < \infty.$$

Moreover, if c is the best constant in (1), then $c \approx D_1$.

We also need Kolmogorov's theorem formulated as follows (see, e.g., [11]):

Theorem C. *Let $1 < p, q < \infty$ and $K : L^p \to L^q_v$ be an integral operator of the form $Kf(x) = \int_0^\infty k(x,y)f(y)dy$. If*

$$\|\ \|k(x,\cdot)\|_{L^{p'}}\|_{L^q_v} < \infty,$$

then the operator K is compact.

Theorem 1. *Let $1 < p \le q < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$. The inequality*

$$\|R_\alpha f\|_{L^q_v} \le A\|f\|_{L^p}, \tag{2}$$

where the positive constant A does not depend on f, is fulfilled iff

$$B = \sup_{t>0} B(t) = \sup_{t>0}\Big(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)^{\frac{1}{q}} t^{\frac{1}{p'}} < \infty. \tag{3}$$

Moreover, if A is the best constant in (2), then $A \approx B$.

Proof. Sufficiency. Denoting $I_{1\alpha}f(x) = \int_0^{\frac{x}{2}} \frac{f(t)}{(x-t)^{1-\alpha}}dt$ and $I_{2\alpha}f(x) = \int_{\frac{x}{2}}^{x} \frac{f(t)}{(x-t)^{1-\alpha}}dt$ for $f \in L^p$ we write $R_\alpha f$ as $R_\alpha f(x) = I_{1\alpha}f(x) + I_{2\alpha}f(x)$. We obtain

$$\|R_\alpha f\|^q_{L^q_v} \leq c_1 \int_0^\infty |I_{1\alpha}f(x)|^q v(x)dx + c_1 \int_0^\infty |I_{2\alpha}f(x)|^q v(x)dx = S_1 + S_2.$$

If $0 < t < \frac{x}{2}$, then $(x-t)^{\alpha-1} \leq bx^{\alpha-1}$, where the positive constant b depends only on α. Consequently, using Theorem A with $w \equiv 1$, we have

$$S_1 \leq c_2 \int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}} \Big(\int_0^x |f(t)|dt \Big)^q dx \leq c_3 B^q \|f\|^q_{L^p}.$$

Now we shall estimate S_2. Using the Hölder inequality and the condition $\frac{1}{p} < \alpha$, we obtain

$$S_2 = c_1 \int_0^\infty v(x) \Big| \int_{\frac{x}{2}}^x \frac{f(t)}{(x-t)^{1-\alpha}}dt \Big|^q dx \leq$$

$$\leq c_1 \int_0^\infty v(x) \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt \Big)^{\frac{q}{p}} \Big(\int_{\frac{x}{2}}^x \frac{dt}{(x-t)^{(1-\alpha)p'}} \Big)^{\frac{q}{p'}} dx =$$

$$= c_4 \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q + \frac{q}{p'}} \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt \Big)^{\frac{q}{p}} dx \leq$$

$$\leq c_4 \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{\frac{q}{p}} \Big(\int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q + \frac{q}{p'}} dx \Big) \leq$$

$$\leq c_5 \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{\frac{q}{p}} \Big(\int_{2^k}^{2^{k+1}} v(x) \cdot x^{(\alpha-1)q} dx \Big) \cdot 2^{\frac{kq}{p'}} \leq$$

$$\leq c_5 B^q \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt \Big)^{\frac{q}{p}} \leq c_6 B^q \|f\|^q_{L^p}$$

which proves the sufficiency.

Necessity. Let $f(x) = \chi_{(0,\frac{t}{2})}(x)$. Note that if $0 < y < \frac{t}{2}$ and $x > t$, then $(x-y)^{\alpha-1} \geq b_1 x^{\alpha-1}$, where the positive constant b_1 depends only on α.

We have

$$\|R_\alpha f\|_{L^q_v} \geq \bigg(\int\limits_t^\infty v(x)\Big(\int\limits_0^{\frac{t}{2}} \frac{dy}{(x-y)^{1-\alpha}}\Big)^q dx\bigg)^{\frac{1}{q}} \geq c_7\Big(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)^{\frac{1}{q}} \cdot t.$$

On the other hand, $\|f\|_{L^p} = c_8 t^{\frac{1}{p}}$ and by virtue of inequality (2) we find that $B(t) \leq c_9 A$ for all $t > 0$. $\square$

A most complicated proof of a similar theorem is given in [12] for the case $p = q = 2$.

Remark 1. Condition (3) is equivalent to the condition

$$\widetilde{B} = \sup_{k\in\mathbb{Z}}\Big(\int\limits_{2^k}^{2^{k+1}} \frac{v(x)}{x^{(1-\alpha)q-\frac{q}{p'}}}dx\Big)^{\frac{1}{q}} < \infty. \tag{4}$$

Moreover, $B \approx \widetilde{B}$.

Indeed, the fact that (3) implies (4) follows from the proof of Theorem 1. Now let condition (4) be satisfied and $t \in (0, \infty)$. Then $t \in (2^m, 2^{m+1}]$ for some $m \in \mathbb{Z}$. We have

$$B(t)^q = \Big(\int\limits_t^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)t^{\frac{q}{p'}} \leq \Big(\int\limits_{2^m}^\infty \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)2^{\frac{(m+1)q}{p'}} =$$

$$= c_1 2^{\frac{mq}{p'}} \sum_{k=m}^\infty \Big(\int\limits_{2^k}^{2^{k+1}} \frac{v(x)}{x^{(1-\alpha)q}}dx\Big) \leq c_2 2^{\frac{mq}{p'}} \sum_{k=m}^\infty 2^{-\frac{kq}{p'}} \int\limits_{2^k}^{2^{k+1}} \frac{v(x)x^{\frac{q}{p'}}}{x^{(1-\alpha)q}}dx \leq$$

$$\leq c_2 \widetilde{B}^q 2^{\frac{mq}{p'}} \sum_{k=m}^\infty 2^{-\frac{kq}{p'}} \leq c_3 \widetilde{B}^q$$

and therefore $B \leq c_4 \widetilde{B} < \infty$.

By the duality argument and Theorem 1 we obtain

Theorem 2. *Let $1 < p \leq q < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. For the inequality*

$$\|W_\alpha f\|_{L^q} \leq \overline{A}\|f\|_{L^p_w}, \tag{5}$$

where the positive constant $\overline{A}$ does not depend on f, to be valid it is necessary and sufficient that

$$\overline{B} = \sup_{t>0} \overline{B}(t) = \sup_{t>0}\Big(\int\limits_t^\infty \frac{w^{1-p'}(x)}{x^{(1-\alpha)p'}}dx\Big)^{\frac{1}{p'}} t^{\frac{1}{q}} < \infty. \tag{6}$$

Moreover, if $\overline{A}$ is the best constant in inequality (5), *then* $\overline{A} \approx \overline{B}$.

We shall now consider the case $1 < q < p < \infty$. Applying the integration by parts, we obtain

Lemma 1. *Let $1 < q < p < \infty$ and u be a locally integrable function on $\mathbb{R}_+$. Then the equality*

$$\Big(\int_a^b u(x)dx\Big)^{\frac{p}{p-q}} = \frac{p}{p-q}\int_a^b \Big(\int_x^b u(t)dt\Big)^{\frac{q}{p-q}} u(x)dx$$

holds, where $0 \le a < b < \infty$.

Theorem 3. *Let $1 < q < p < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$. The inequality*

$$\|R_\alpha f\|_{L^q_v} \le A_1 \|f\|_{L^p} \tag{7}$$

is fulfilled iff

$$B_1 = \Big(\int_0^\infty \Big(\int_x^\infty \frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}} x^{\frac{(q-1)p}{p-q}}dx\Big)^{\frac{p-q}{pq}} < \infty. \tag{8}$$

Moreover, if A_1 is the best constant in inequality (7), *then* $A_1 \approx B_1$.

Proof. Sufficiency. In the notation introduced in the proof of Theorem 1 we have

$$\|R_\alpha f\|^q_{L^q_v} \le S_1 + S_2.$$

Using Theorem B with $w \equiv 1$ and the argument from the proof of Theorem 1, we obtain

$$S_1 \le c_2 B_1^q \|f\|^q_{L^p}.$$

Applying the Hölder inequality twice and the fact that $\frac{1}{p} < \alpha$, we have

$$S_2 \le c_1 \int_0^\infty \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt\Big)^{\frac{q}{p}} \Big(\int_{\frac{x}{2}}^x \frac{dt}{(x-t)^{(1-\alpha)p'}}\Big)^{\frac{q}{p'}} v(x)dx =$$

$$= c_3 \sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \Big(\int_{\frac{x}{2}}^x |f(t)|^p dt\Big)^{\frac{q}{p}} v(x) x^{(\alpha-1)q+\frac{q}{p'}} dx \le$$

$$\le c_3 \sum_{k\in\mathbb{Z}} \Big(\int_{2^{k-1}}^{2^{k+1}} |f(t)|^p dt\Big)^{\frac{q}{p}} \Big(\int_{2^k}^{2^{k+1}} v(x) x^{(\alpha-1)q+\frac{q}{p'}} dx\Big) \le$$

$$\le c_3\Big(\sum_{k\in\mathbb{Z}}\int\limits_{2^k}^{2^{k+1}}|f(t)|^p dt\Big)^{\frac{q}{p}}\Big(\sum_{k\in\mathbb{Z}}\Big(\int\limits_{2^k}^{2^{k+1}}v(x)x^{(\alpha-1)q+\frac{q}{p'}}dx\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}\le$$

$$\le c_4\|f\|^q_{L^p}\Big(\sum_{k\in\mathbb{Z}}\Big(\int\limits_{2^k}^{2^{k+1}}v(x)x^{(\alpha-1)q+\frac{q}{p'}}dx\Big)^{\frac{p}{p-q}}\Big)^{\frac{p-q}{p}}=c_4\|f\|^q_{L^p}\Big(\sum_{k\in\mathbb{Z}}S_{2k}\Big)^{\frac{p-q}{p}}.$$

By Lemma 1, we find for S_{2k} that

$$S_{2k}\le 2^{\frac{(k+1)qp}{p'(p-q)}}\Big(\int\limits_{2^k}^{2^{k+1}}v(x)x^{(\alpha-1)q}dx\Big)^{\frac{p}{p-q}}\le$$

$$\le c_5 2^{\frac{kqp}{p'(p-q)}}\int\limits_{2^k}^{2^{k+1}}\Big(\int\limits_{x}^{2^{k+1}}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{q}{p-q}}\frac{v(x)}{x^{(1-\alpha)q}}dx\le$$

$$\le c_5\int\limits_{2^k}^{2^{k+1}}\Big(\int\limits_{x}^{\infty}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{q}{p-q}}\frac{v(x)}{x^{(1-\alpha)q}}\cdot x^{\frac{q(p-1)}{p-q}}dx.$$

Using integration by parts we get

$$S_2\le c_6\|f\|^q_{L^p}\Big(\int\limits_0^{\infty}\Big(\int\limits_x^{\infty}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{q}{p-q}}\frac{v(x)}{x^{(1-\alpha)q}}\cdot x^{\frac{q(p-1)}{p-q}}dx\Big)^{\frac{p-q}{p}}=$$

$$=c_7\|f\|^q_{L^p}\Big(\int\limits_0^{\infty}\Big(\int\limits_x^{\infty}\frac{v(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}}x^{\frac{p(q-1)}{p-q}}dx\Big)^{\frac{p-q}{p}}=c_7\|f\|^q_{L^p}B_1^q$$

and finally we obtain inequality (7).

Necessity. Let $\frac{1}{p}<\alpha<1$ and $v_0(t)=v(t)\cdot\chi_{(a,b)}(t)$, $w_0(t)=\chi_{(a,b)}(t)$, where $0<a<b<\infty$, and let

$$f(x)=\Big(\int\limits_x^{\infty}\frac{v_0(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{1}{p-q}}\Big(\int\limits_0^x w_0(t)dt\Big)^{\frac{q-1}{p-q}}w_0(x).$$

Then we have

$$\|f\|_{L^p}=\Big(\int\limits_a^b\Big(\int\limits_x^{\infty}\frac{v_0(t)}{t^{(1-\alpha)q}}dt\Big)^{\frac{p}{p-q}}\Big(\int\limits_0^x w_0(t)dt\Big)^{\frac{(q-1)p}{p-q}}dx\Big)^{\frac{1}{p}}<\infty.$$

On the other hand,

$$\|R_\alpha f\|_{L^q_v} = \Big(\int_0^\infty v(x) \Big(\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \Big)^q dx \Big)^{\frac{1}{q}} \geq$$

$$\geq c_8 \Big(\int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}} \Big(\int_0^x f(t) dt \Big)^q dx \Big)^{\frac{1}{q}} \geq c_9 \Big(\int_0^\infty \frac{v(x)}{x^{(1-\alpha)q}} \times$$

$$\times \Big(\int_x^\infty \frac{v_0(y)}{y^{(1-\alpha)q}} dy \Big)^{\frac{q}{p-q}} \Big(\int_0^x \Big(\int_0^t w_0(y) dy \Big)^{\frac{q-1}{p-q}} w_0(t) dt \Big)^q dx \Big)^{\frac{1}{q}} \geq$$

$$\geq c_{10} \Big(\int_0^\infty \frac{v_0(x)}{x^{(1-\alpha)q}} \Big(\int_x^\infty \frac{v_0(y)}{y^{(1-\alpha)q}} dy \Big)^{\frac{q}{p-q}} \Big(\int_0^x w_0(y) dy \Big)^{\frac{(p-1)q}{p-q}} dx \Big)^{\frac{1}{q}} =$$

$$= c_{11} \Big(\int_0^\infty \Big(\int_x^\infty \frac{v_0(t)}{t^{(1-\alpha)q}} dt \Big)^{\frac{p}{p-q}} \Big(\int_0^x w_0(t) dt \Big)^{\frac{(q-1)p}{p-q}} w_0(x) dx \Big)^{\frac{1}{q}} =$$

$$= c_{11} \Big(\int_a^b \Big(\int_x^\infty \frac{v_0(t)}{t^{(1-\alpha)q}} dt \Big)^{\frac{p}{p-q}} \Big(\int_0^x w_0(t) dt \Big)^{\frac{(q-1)p}{p-q}} dx \Big)^{\frac{1}{q}}.$$

From inequality (7) we have

$$\Big(\int_a^b \Big(\int_x^\infty \frac{v_0(t)}{t^{(1-\alpha)q}} dt \Big)^{\frac{p}{p-q}} \Big(\int_0^x w_0(t) dt \Big)^{\frac{(q-1)p}{p-q}} dx \Big)^{\frac{q-p}{pq}} \leq c_{12} A_1,$$

where c_{12} does not depend on a and b. By Fatou's lemma we finally obtain condition (8). The case $\alpha > 1$ is proved similarly. □

By the duality argument and Theorem 3 we have

Theorem 4. *Let $1 < q < p < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. The inequality*

$$\|W_\alpha f\|_{L^q} \leq \overline{A}_1 \|f\|_{L^p_w}, \tag{9}$$

where the positive constanr $\overline{A}_1$ does not depend f, holds iff

$$\overline{B}_1 = \Big(\int_0^\infty \Big(\int_x^\infty \frac{w^{1-p'}(t)}{t^{(1-\alpha)p'}} dt \Big)^{\frac{q(p-1)}{p-q}} x^{\frac{q}{p-q}} dx \Big)^{\frac{p-q}{pq}} < \infty. \tag{10}$$

Moreover, if $\overline{A}_1$ is the best constant in inequality (9)*, then $\overline{A}_1 \approx \overline{B}_1$.*

Let us now investigate the compactness of the operators R_α and W_α.

Theorem 5. *Let* $1 < p \le q < \infty$, $\frac{1}{p} < \alpha < 1$ *or* $\alpha > 1$. *The operator* R_α *is compact from* L^p *to* L^q_v *iff condition* (3) *and the condition*

$$\lim_{t\to 0} B(t) = \lim_{t\to\infty} B(t) = 0$$

is satisfied.

Proof. Sufficiency. Let $0 < a < b < \infty$. We write $R_\alpha f$ as

$$R_\alpha f = \chi_{[0,a]} R_\alpha(f\cdot\chi_{(0,a)}) + \chi_{(a,b)} R_\alpha(f\cdot\chi_{(0,b)}) + \chi_{[b,\infty)} R_\alpha(f\cdot\chi_{(0,\frac{b}{2})}) + \\ +\chi_{[b,\infty)} R_\alpha(f\cdot\chi_{(\frac{b}{2},\infty)}) = P_{1\alpha}f + P_{2\alpha}f + P_{2\alpha}f + P_{4\alpha}f.$$

For $P_{2\alpha}f$ we have $P_{2\alpha}f(x) = \chi_{(a,b)}(x)\int_0^\infty k_1(x,y)f(y)dy$, with $k_1(x,y) = (x-y)^{\alpha-1}$ for $y < x$ and $k_1(x,y) = 0$ for $y \ge x$. Consequently

$$\int\limits_a^b v(x)\Big(\int\limits_0^\infty (k_1(x,y))^{p'}dy\Big)^{\frac{q}{p'}}dx \le \Big(\int\limits_a^b \frac{v(x)}{x^{(1-\alpha)q}}dx\Big) b^{\frac{q}{p'}} < \infty$$

and by Theorem C we conclude that $P_{2\alpha}$ is compact from L^p to L^q_v.

In a similar manner we show that $P_{3\alpha}$ is compact too.

Using Theorem 1 for the operators $P_{1\alpha}$ and $P_{4\alpha}$, we obtain

$$\|P_{1\alpha}\| \le c_1 \sup_{0<t<a} B(t) \quad \text{and} \quad \|P_{4\alpha}\| \le c_2 \sup_{t>\frac{b}{2}} B(t).$$

Consequently

$$\|R_\alpha - P_{2\alpha} - P_{3\alpha}\| \le \|P_{1\alpha}\| + \|P_{4\alpha}\| \le c_1 \sup_{0<t<a} B(t) + c_2 \sup_{t>\frac{b}{2}} B(t) \to 0$$

as $a \to 0$ and $b \to \infty$.

Thus the operator R_α is compact, since it is a limit of compact operators. The sufficiency is proved.

Necessity. Note that the fact $B < \infty$ follows from Theorem 1. Thus we need to prove the remaining part. Let $f_t(x) = \chi_{(0,t)}(x)t^{-1/p}$. Then the sequence f_t is weakly convergent to 0. Indeed, assuming that $\varphi \in L^{p'}$, we obtain

$$\Big|\int\limits_0^\infty f_t(x)\varphi(x)dx\Big| \le \Big(\int\limits_0^t |\varphi(x)|^{p'}dx\Big)^{\frac{1}{p'}} \to 0 \ \text{ as } \ t \to 0.$$

On the other hand, we have

$$\|R_\alpha f_t\|_{L^q_v} \ge \Big(\int\limits_t^\infty v(x)\Big(\int\limits_0^t \frac{f_t(y)}{(x-y)^{1-\alpha}}dy\Big)^q dx\Big)^{\frac{1}{q}} \ge$$

$$\geq c_3\Big(\int\limits_t^{\infty} \frac{v(x)}{x^{(1-\alpha)q}}dx\Big)^{\frac{1}{q}} t^{\frac{1}{p'}} = c_3 B(t).$$

Using the fact that a compact operator maps a weakly convergent sequence into a strongly convergent form, we find that $B(t) \to 0$ as $t \to 0$.

Keeping in mind that the operator W_α is compact from $L^{q'}_{v^{1-q'}}$ to $L^{p'}$ and arguing as above, we prove the remaining part of the theorem. □

In [12] a similar theorem is proved for the case $p = q = 2$.

Since the operator W_α is compact from L^p_w to L^q iff the operator R_α is compact from $L^{q'}$ to $L^{p'}_{w^{1-p'}}$, by Theorem 5 we obtain

Theorem 6. *Let $1 < q < p < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. The operator W_α is compact from L^p_w to L^q iff condition (6) and the condition*

$$\lim_{t\to 0} \overline{B}(t) = \lim_{t\to\infty} \overline{B}(t) = 0$$

are fulfilled.

Theorem 7. *Let $1 < q < p < \infty$, $\frac{1}{p} < \alpha < 1$ or $\alpha > 1$. The operator R_α is compact from L^p to L^q_v iff condition (8) is satisfied.*

Proof. The sufficiency is proved as in proving Theorem 5 while the necessity follows from Theorem 3. □

By the duality argument we have

Theorem 8. *Let $1 < q < p < \infty$, $\frac{1}{q'} < \alpha < 1$ or $\alpha > 1$. The operator W_α is compact from L^p_w to L^q iff condition (10) is fulfilled.*

In [13, 14] the necessary and sufficient conditions are found for the operators R_α and W_α to be compact when $1 < p \leq q < \infty$ and $\alpha = 1$.

An analogous problem for $\alpha > 1$ was investigated in [15].

Remark 2. In Theorems 1 and 5 it suffices to consider v as a measurable almost everywhere, positive function. The same assumption can be made for w in Theorems 2 and 6.

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