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ON THE CORRECTNESS OF THE DIRICHLET PROBLEM IN A CHARACTERISTIC RECTANGLE FOR FOURTH ORDER LINEAR HYPERBOLIC EQUATIONS

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Abstract. It is proved that the Dirichlet problem is correct in the characteristic rectangle $D_{ab} = [0,a] \times [0,b]$ for the linear hyperbolic equation

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = p_0(x,y)u + p_1(x,y)\frac{\partial u}{\partial x} + p_2(x,y)\frac{\partial u}{\partial y} + p_3(x,y)\frac{\partial^2 u}{\partial x \partial y} + q(x,y)$$

with the summable in D_{ab} coefficients p_0, p_1, p_2, p_3 and q if and only if the corresponding homogeneous problem has only the trivial solution. The effective and optimal in some sense restrictions on p_0, p_1, p_2 and p_3 guaranteeing the correctness of the Dirichlet problem are established.

§ 1. Formulation of the Problem and Main Results

The Dirichlet problem for second order hyperbolic equations and some higher order linear hyperbolic equations with constant coefficients has long been attracting the attention of mathematicians. The problem has been the subject of numerous studies (see [1–20] and the references therein) but still remains investigated very little for a wide class of hyperbolic equations. This class includes the fourth order hyperbolic equation

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}=p_0(x,y)u+p_1(x,y)\frac{\partial u}{\partial x}+p_2(x,y)\frac{\partial u}{\partial y}+p_3(x,y)\frac{\partial^2 u}{\partial x \partial y}+q(x,y) \quad (1.1)$$

for which the Dirichlet problem is considered here in the characteristic rectangle $D_{ab} = [0,a] \times [0,b]$.

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Throughout the paper it will be assumed that $0 < a < +\infty$, $0 < b < +\infty$, and $p_k : D_{ab} \to \mathbb{R}$ $(k = 0, 1, 2, 3)$ and $q : D_{ab} \to \mathbb{R}$ are Lebesgue summable functions.

A function $u : D_{ab} \to \mathbb{R}$ will be called a solution of system (1.1) if it is absolutely continuous together with $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial^2 u}{\partial x \partial y}$ and satisfies (1.1) almost everywhere in D_{ab}.

Let Γ_{ab} be the boundary of the rectangle D_{ab}, and $\varphi : \Gamma_{ab} \to \mathbb{R}$ be a continuous function having the absolutely continuous partial derivatives $\frac{\partial \varphi(x,0)}{\partial x}$, $\frac{\partial \varphi(x,b)}{\partial x}$ and $\frac{\partial \varphi(0,y)}{\partial y}$, $\frac{\partial \varphi(a,y)}{\partial y}$ in $[0, a]$ and $[0, b]$, respectively.

The Dirichlet problem for equation (1.1) in D_{ab} consists in finding a solution $u : D_{ab} \to \mathbb{R}$ of (1.1) satisfying the boundary condition

$$u(x, y) = \varphi(x, y) \quad \text{for} \quad (x, y) \in \Gamma_{ab}. \tag{1.2}$$

Along with (1.1),(1.2), we shall consider the corresponding homogeneous problem

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = p_0(x, y)u + p_1(x, y)\frac{\partial u}{\partial x} + p_2(x, y)\frac{\partial u}{\partial y} + p_3(x, y)\frac{\partial^2 u}{\partial x \partial y}, \tag{1.1$_0$}$$

$$u(x, y) = 0 \quad \text{for} \quad (x, y) \in \Gamma_{ab}, \tag{1.2$_0$}$$

and also the perturbed problem

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \overline{p}_0(x, y)u + \overline{p}_1(x, y)\frac{\partial u}{\partial x} + \overline{p}_2(x, y)\frac{\partial u}{\partial y} + \overline{p}_3(x, y)\frac{\partial^2 u}{\partial x \partial y} + \overline{q}(x, y), \tag{1.3}$$

$$u(x, y) = \overline{\varphi}(x, y) \quad \text{for} \quad (x, y) \in \Gamma_{ab}. \tag{1.4}$$

Before formulating the main results, we introduce some notation and a definition.

$\mathbb{R}$ is the set of real numbers.

$L(D_{ab})$ is the space of Lebesgue summable functions $z : D_{ab} \to \mathbb{R}$.

$\widetilde{C}(D_{ab})$ is the space of absolutely continuous functions $z : D_{ab} \to \mathbb{R}$ (see the definition in [21] or [22]).

$\widetilde{C}^1(D_{ab})$ is the space of absolutely continuous functions $z : D_{ab} \to \mathbb{R}$ having the absolutely continuous partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial x \partial y}$.

$\widetilde{C}^1(\Gamma_{ab})$ is the space of absolutely continuous functions $\zeta : \Gamma_{ab} \to \mathbb{R}$ having the absolutely continuous partial derivatives $\frac{\partial \zeta(x,0)}{\partial x}$, $\frac{\partial \zeta(x,b)}{\partial x}$ in $[0, a]$ and the absolutely continuous partial derivatives $\frac{\partial \zeta(0,y)}{\partial y}$, $\frac{\partial \zeta(a,y)}{\partial y}$ in $[0, b]$.

Definition 1.1. Problem (1.1), (1.2) will be called correct if it has a unique solution u and for an arbitrary positive number ε and nonnegative function $\gamma \in L(D_{ab})$ there exists a positive number δ such that for arbitrary functions $\overline{p}_k \in L(D_{ab})$ $(k = 0, 1, 2, 3)$, $\overline{q} \in L(D_{ab})$ and $\overline{\varphi} \in \widetilde{C}^1(\Gamma_{ab})$

satisfying the conditions

$$|\overline{p}_k(x,y) - p_k(x,y)| \le \gamma(x,y) \quad \text{almost everywhere in } D_{ab} \ \ (k=0,1,2,3),$$

$$\Big|\int\limits_0^x \int\limits_0^y [\overline{p}_k(s,t) - p_k(s,t)]\,ds\,dt\Big| \le \delta \quad \text{for} \quad (x,y)\in D_{ab} \ \ (k=0,1,2,3),$$

$$\Big|\int\limits_0^x \int\limits_0^y [\overline{q}(s,t) - q(s,t)]\,ds\,dt\Big| \le \delta \quad \text{for} \quad (x,y)\in D_{ab},$$

$$|\overline{\varphi}(x,y) - \varphi(x,y)| + \Big|\frac{\partial\overline{\varphi}(x,0)}{\partial x} - \frac{\partial\varphi(x,0)}{\partial x}\Big| + \Big|\frac{\partial\overline{\varphi}(x,b)}{\partial x} - \frac{\partial\varphi(x,b)}{\partial x}\Big| +$$
$$+\Big|\frac{\partial\overline{\varphi}(0,y)}{\partial y} - \frac{\partial\varphi(0,y)}{\partial y}\Big| + \Big|\frac{\partial\overline{\varphi}(a,y)}{\partial y} - \frac{\partial\varphi(a,y)}{\partial y}\Big| < \delta \quad \text{for} \quad (x,y)\in\Gamma_{ab},$$

problem $(1.3), (1.4)$ has a unique solution $\overline{u}$ and

$$|\overline{u}(x,y) - u(x,y)| + \Big|\frac{\partial\overline{u}(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial x}\Big| + \Big|\frac{\partial\overline{u}(x,y)}{\partial y} - \frac{\partial u(x,y)}{\partial y}\Big| +$$
$$+\Big|\frac{\partial^2\overline{u}(x,y)}{\partial x\partial y} - \frac{\partial^2 u(x,y)}{\partial x\partial y}\Big| < \varepsilon \quad \text{for} \quad (x,y)\in D_{ab}.$$

Theorem 1.1. *Problem* $(1.1), (1.2)$ *is correct if and only if the corresponding homogeneous problem* $(1.1_0), (1.2_0)$ *has only the trivial solution.*

Now consider the sequence of boundary value problems

$$\frac{\partial^4 u}{\partial x^2\partial y^2} = p_{0m}(x,y)u + p_{1m}(x,y)\frac{\partial u}{\partial x} + p_{2m}(x,y)\frac{\partial u}{\partial y} +$$
$$+p_{3m}(x,y)\frac{\partial^2 u}{\partial x\partial y} + q_m(x,y), \qquad (1.5_m)$$

$$u(x,y) = \varphi_m(x,y) \quad \text{for} \quad (x,y)\in\Gamma_{ab}, \qquad (1.6_m)$$

under the following assumptions on the functions p_{km}, q_m and φ_m:

(i) $p_{km} \in L(D_{ab})$, $q \in L(D_{ab})$ $(k=0,1,2,3; m=1,2,\dots)$ and the equalities

$$\lim_{m\to\infty}\int\limits_0^x \int\limits_0^y [p_{km}(s,t) - p_k(s,t)]\,ds\,dt = 0 \ \ (k=0,1,2,3),$$

$$\lim_{m\to\infty}\int\limits_0^x \int\limits_0^y [q_m(s,t) - q(s,t)]\,ds\,dt = 0$$

hold uniformly on D_{ab};

(ii) There exists a nonnegative function $p \in L(D_{ab})$ such that the inequalities

$$|p_{km}(x,y)| \le p(x,y) \quad (k=0,1,2,3;\ m=1,2,\dots)$$

hold almost everywhere in D_{ab};

(iii) $\varphi_m \in \widetilde{C}^1(\Gamma_{ab})$ $(m=1,2,\dots)$ and the equalities

$$\lim_{m\to\infty} \frac{\partial^k \varphi_m(x,0)}{\partial x^k} = \frac{\partial^k \varphi(x,0)}{\partial x^k}, \quad \lim_{m\to\infty} \frac{\partial^k \varphi_m(x,b)}{\partial x^k} = \frac{\partial^k \varphi(x,0)}{\partial x^k} \ (k=0,1),$$
$$\lim_{m\to\infty} \frac{\partial^k \varphi_m(0,y)}{\partial y^k} = \frac{\partial^k \varphi(0,y)}{\partial y^k}, \quad \lim_{m\to\infty} \frac{\partial^k \varphi_m(a,y)}{\partial y^k} = \frac{\partial^k \varphi(a,y)}{\partial y^k} \ (k=0,1)$$

hold uniformly on $[0,a]$ and $[0,b]$ respectively.

Theorem 1.1 can be reformulated in the manner as follows.

Theorem 1.1′. *If the homogeneous problem* $(1.1_0),(1.2_0)$ *has only the trivial solution, then problem* $(1.1),(1.2)$ *has the unique solution* u. *Moreover, if the conditions* (i)–(iii) *hold, then beginning from some sufficiently large* m_0 *problem* $(1.5_m),(1.6_m)$ *has the unique solution* u_m *and*

$$\lim_{m\to\infty} \frac{\partial^{j+k} u_m(x,y)}{\partial x^j \partial y^k} = \frac{\partial^{j+k} u(x,y)}{\partial x^j \partial y^k} \quad \textit{uniformly on} \quad D_{ab} \ \ (j,k=0,1).^*$$

Remark 1.1. The conditions (i) and (ii) can be fulfilled even in the case where the differences $p_{km}(x,y)-p_k(x,y)$ $(k=0,1,2,3)$ and $q_m(x,y)-q(x,y)$ have no limits as $m\to\infty$. For example, the functions

$$p_{km}(x,y) = p_k(x,y) + \sin mx \sin my,$$
$$q_m(x,y) = q(x,y) + m \sin mx \sin my \ \ (k=0,1,2,3;\ m=1,2,\dots)$$

satisfy the conditions (i) and (iii).

Finally, we introduce the effective conditions for the correctness of problem (1.1),(1.2).

Theorem 1.2. *Let the inequalities*

$$|p_k(x,y) - p_{k0}(x,y)| \le l_k \ \ (k=1,2), \quad |p_3(x,y)| \le l_3, \tag{1.7}$$
$$p_0(x,y) - \frac{1}{2}\frac{\partial p_{10}(x,y)}{\partial x} - \frac{1}{2}\frac{\partial p_{20}(x,y)}{\partial y} \le l_0 \tag{1.8}$$

hold almost everywhere in D_{ab}, *where* l_k $(k=0,1,2,3)$ *are constants and the functions* $p_{10} \in L(D_{ab})$ *and* $p_{20} \in L(D_{ab})$ *are absolutely continuous in the first and in the second argument, respectively, and have the partial*

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*Here and in what follows it is assumed that $\frac{\partial^0 u(x,y)}{\partial x^0} \equiv \frac{\partial^0 u(x,y)}{\partial y^0} \equiv \frac{\partial^0 u(x,y)}{\partial x^0 \partial y^0} \equiv u(x,y)$.

derivatives $\frac{\partial p_{10}(x,y)}{\partial x}$ *and* $\frac{\partial p_{20}(x,y)}{\partial y}$ *summable in* D_{ab}. *Moreover, let either* $l_0 \geq 0$ *and*

$$\frac{a^2b^2}{\pi^4}l_0 + \frac{ab^2}{\pi^3}l_1 + \frac{a^2b}{\pi^3}l_2 + \frac{ab}{\pi^2}l_3 < 1, \tag{1.9}$$

or $l_0 < 0$ *and*

$$\frac{b}{\pi}l_1 + \frac{a}{\pi}l_2 + l_3 < 2\sqrt{|l_0|}. \tag{1.10}$$

Then problem (1.1), (1.2) *is correct.*

Remark 1.2. The strict inequality (1.9) in Theorem 1.2 cannot be replaced by the nonstrict one

$$\frac{a^2b^2}{\pi^4}l_0 + \frac{ab^2}{\pi^3}l_1 + \frac{a^2b}{\pi^3}l_2 + \frac{ab}{\pi^2}l_3 \leq 1. \tag{1.9$'$}$$

To make sure that is so, let us consider the equation

$$\frac{\partial^4 u}{\partial x^2 \partial y^2} = \frac{\pi^4}{a^2b^2}u \tag{1.11}$$

with the boundary condition (1.2_0). For equation (1.11) conditions (1.7) and (1.8) hold, where $p_0(x,y) \equiv l_0 = \frac{\pi^4}{a^2b^2}$, $p_k(x,y) \equiv 0$, $l_k = 0$ $(k = 1,2,3)$ and $p_{k0}(x,y) \equiv 0$ $(k = 1,2)$, and instead of (1.9) condition (1.9$'$) holds. And yet problem (1.11), (1.2_0) is incorrect, since it has the infinite set of solutions of the form $u(x,y) = c \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$, where c is an arbitrary real constant.

§ 2. Auxiliary Statements

We introduce some additional notation to be used in this section.

If $\varphi \in \widetilde{C}^1(\Gamma_{ab})$, then

$$\begin{aligned}\eta(\varphi)(x,y) = &\Big(1-\frac{y}{b}\Big)\varphi(x,0) + \frac{y}{b}\varphi(x,b) + \Big(1-\frac{x}{a}\Big)\varphi(0,y) + \\ &+\frac{x}{a}\varphi(a,y) - \Big(1-\frac{y}{b}\Big)\Big(1-\frac{x}{a}\Big)\varphi(0,0) - \frac{y}{b}\Big(1-\frac{x}{a}\Big)\varphi(0,b) - \\ &-\Big(1-\frac{y}{b}\Big)\frac{x}{a}\varphi(a,0) - \frac{xy}{ab}\varphi(a,b) \quad \text{for} \quad (x,y) \in D_{ab}.\end{aligned} \tag{2.1}$$

It is obvious that $\eta(\varphi) \in \widetilde{C}^1(D_{ab})$ and

$$\eta(\varphi)(x,y) = \varphi(x,y) \quad \text{for} \quad (x,y) \in \Gamma_{ab}. \tag{2.2}$$

We denote by g_1 and g_2 the Green's functions of the boundary value problems

$$\frac{d^2z}{dx^2} = 0, \;\; z(0) = z(a) = 0 \quad \text{and} \quad \frac{d^2z}{dy^2} = 0, \;\; z(0) = z(b) = 0.$$

Therefore

$$g_1(x,s)=\begin{cases}(x-a)\frac{s}{a} & \text{for} \quad s\le x,\\ (s-a)\frac{x}{a} & \text{for} \quad s>x,\end{cases} \tag{2.3}$$

$$g_2(y,t)=\begin{cases}(y-b)\frac{t}{b} & \text{for} \quad t\le y,\\ (t-b)\frac{y}{b} & \text{for} \quad t>y.\end{cases} \tag{2.4}$$

Under $C^{1,1}(D_{ab})$ we understand the Banach space of continuous functions $z: D_{ab}\to\mathbb{R}$ having the continuous partial derivatives $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$, $\frac{\partial^2 z}{\partial x\partial y}$, with the norm

$$\|z\|_{C^{1,1}}=\max\Big\{\sum_{j,k=0}^{1}\Big|\frac{\partial^{j+k}z(x,y)}{\partial x^j\partial y^k}\Big| : (x,y)\in D_{ab}\Big\}.$$

It is obvious that $\widetilde{C}^1(D_{ab})\subset C^{1,1}(D_{ab})$. On the other hand, Arcela–Ascoli lemma gives rise to

Lemma 2.1. *Let $Z\subset\widetilde{C}^1(D_{ab})$ and there exist summable functions $h_1: [0,a]\to[0,+\infty)$ and $h_2:[0,b]\to[0,+\infty)$ such that every function $z\in Z$ satisfies the inequality*

$$\Big|\frac{\partial^3 z(x,y)}{\partial x^2\partial y}\Big|\le h_1(x)$$

for every $y\in[0,b]$ and almost every $x\in[0,a]$ and the inequality

$$\Big|\frac{\partial^3 z(x,y)}{\partial x\partial y^2}\Big|\le h_2(y)$$

for every $x\in[0,a]$ and almost every $y\in[0,b]$. Moreover, if Z is bounded in the topology of the space $C^{1,1}(D_{ab})$, then it is the relative compact of this space.

In $C^{1,1}(D_{ab})$ consider the operator equation

$$u(x,y)=\eta(\varphi)(x,y)+\int_0^a\int_0^b g_1(x,s)g_2(y,t)[\mathcal{Q}(u)(s,t)+q(s,t)]dsdt, \tag{2.5}$$

where

$$\mathcal{Q}(u)(x,y)=p_0(x,y)u(x,y)+p_1(x,y)\frac{\partial u(x,y)}{\partial x}+ \\ +p_2(x,y)\frac{\partial u(x,y)}{\partial y}+p_3(x,y)\frac{\partial^2 u(x,y)}{\partial x\partial y}. \tag{2.6}$$

Under a solution of equation (2.5) we understand a function $u \in C^{1,1}(D_{ab})$ satisfying (2.5) at every point of D_{ab}.

2.1. Lemma on Equivalence of Problem (1.1),(1.2) and Equation (2.5).

Lemma 2.2. *Problem* (1.1), (1.2) *is equivalent to equation* (2.5)*. More precisely, problem* (1.1), (1.2) *is solvable if and only if equation* (2.5) *is solvable and the sets of their solutions coincide.*

Proof. First we show that if problem (1.1), (1.2) is solvable, then its arbitrary solution u is a solution of equation (2.5). Put

$$v(x,y) = u(x,y) - \eta(\varphi)(x,y). \tag{2.7}$$

Then in view of (2.1) and (2.6)

$$\frac{\partial^2}{\partial x^2}\Big(\frac{\partial^2 v(x,y)}{\partial y^2}\Big) = \mathcal{Q}(u)(x,y) + q(x,y) \text{ almost for every } (x,y) \in D_{ab}, \tag{2.8}$$

since

$$\frac{\partial^4 \eta(\varphi)(x,y)}{\partial x^2 \partial y^2} = 0 \quad \text{for} \quad (x,y) \in D_{ab}.$$

On the other hand, by virtue of (1.2) and (2.2) we have

$$\frac{\partial^2 v(0,y)}{\partial y^2} = \frac{\partial^2 v(a,y)}{\partial y^2} = 0 \quad \text{for} \quad 0 \le y \le b, \tag{2.9}$$

$$v(x,0) = v(x,b) = 0 \quad \text{for} \quad 0 \le x \le a. \tag{2.10}$$

By the condition $\mathcal{Q}(u) \in L(D_{ab})$ it follows from (2.8) and (2.9) that the function v satisfies the equality

$$\frac{\partial^2 v(x,y)}{\partial y^2} = \int_0^a g_1(x,s)[\mathcal{Q}(u)(s,y) + q(s,y)]\,ds$$

for every $x \in [0,a]$ and almost for every $y \in [0,b]$. Hence, taking into the account (2.9) and (2.7), we get

$$u(x,y) - \eta(\varphi)(x,y) = \int_0^a \int_0^b g_1(x,s) g_2(y,t)[\mathcal{Q}(u)(s,t) + q(s,t)]\,ds\,dt$$

for $(x,y) \in D_{ab}$. Thus u is a solution of equation (2.5).

To complete the proof, we have to show that if equation (2.5) is solvable, then its arbitrary solution u is a solution of problem (1.1), (1.2). Indeed, since $\eta(\varphi) \in \widetilde{C}^1(D_{ab})$ and $u \in C^{1.1}(D_{ab})$, it follows from (2.3)–(2.6) that $u \in \widetilde{C}^1(D_{ab})$ and

$$\frac{\partial^2 u(x,y)}{\partial y^2} = \frac{\partial^2 \eta(\varphi)(x,y)}{\partial y^2} + \int\limits_0^a g_1(x,s)[\mathcal{Q}(u)(s,y) + q(s,y)]\,ds,$$

$$\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} = \mathcal{Q}(u)(x,y) + q(x,y) \quad \text{almost for every} \quad (x,y) \in D_{ab},$$

i.e., u is the solution of equation (1.1). On the other hand, by (2.2)–(2.4) it follows from (2.5) that u satisfies the boundary condition (1.2). □

2.2. Lemma on the Fredholm Property of Problem (1.1),(1.2).

Lemma 2.3. *Problem* (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem* $(1.1_0), (1.2_0)$ *has only the trivial solution.*

Proof. For an arbitrary $u \in C^{1,1}(D_{ab})$ put

$$\mathcal{A}(u)(x,y) = \int\limits_0^a \int\limits_0^b g_1(x,s) g_2(y,t) \mathcal{Q}(u)(s,t)\,ds\,dt, \tag{2.11}$$

where $\mathcal{Q} : C^{1,1}(D_{ab}) \to L(D_{ab})$ is the operator given by (2.6). In view of (2.3), (2.4) and (2.6) it is clear that $\mathcal{A} : C^{1,1}(D_{ab}) \to C^{1,1}(D_{ab})$ is a linear bounded operator.

By Lemma 2.1 problem (1.1), (1.2) is equivalent to the equation

$$u(x,y) = u_0(x,y) + \mathcal{A}(u)(x,y), \tag{2.12}$$

where

$$u_0(x,y) = \eta(\varphi)(x,y) + \int\limits_0^a \int\limits_0^b g_1(x,s) g_2(y,t) q(s,t)\,ds\,dt$$

and $u_0 \in \widetilde{C}^1(D_{ab})$. As for problem $(1.1_0), (1.2_0)$, it is equivalent to the equation

$$u(x,y) = \mathcal{A}(u)(x,y). \tag{2.12_0}$$

Now, we have to show that equation (2.12) is uniquely solvable if and only if equation (2.12_0) has only the trivial solution.

By Fredholm's alternative for operator equations ([2.3], Theorem XII. 2.3) we need only to show that $\mathcal{A}$ is a completely continuous operator, which is equivalent to the fact that $\mathcal{A}(B_\rho) = \{v = \mathcal{A}(u) : u \in B_\rho\}$ is a relative compact in $C^{1,1}(D_{ab})$, where ρ is an arbitrary positive number and $B_\rho = \{u \in C^{1,1}(D_{ab}) : \|u\|_{C^{1,1}} \leq \rho\}$.

By condition (2.6) every function $u \in B_\rho$ satisfies the inequality

$$|\mathcal{Q}(u)(x,y)| \leq \rho p(x,y)$$

almost everywhere in D_{ab}, where

$$p(x,y) = \sum_{k=0}^{3} |p_k(x,y)|.$$

With regard to this fact and equalities (2.3), (2.4) and (2.6), we conclude from (2.11) that $\mathcal{A}(B_\rho) \subset \widetilde{C}^1(D_{ab})$. Moreover, for any $v \in \mathcal{A}(B_\rho)$ the inequality

$$\Big|\frac{\partial^3 v(x,y)}{\partial x^2 \partial y}\Big| \leq h_1(x)$$

holds for every $y \in [0,b]$ and almost for every $x \in [0,a]$ and the inequality

$$\Big|\frac{\partial^3 v(x,y)}{\partial x \partial y^2}\Big| \leq h_2(y)$$

holds for every $x \in [0,a]$ and almost for every $y \in [0,b]$, where

$$h_1(x) = \rho \int_0^b p(x,t)\,dt, \quad h_2(y) = \rho \int_0^a p(s,y)\,ds.$$

Moreover, $h_1 : [0,a] \to [0,+\infty)$ and $h_2 : [0,b] \to [0,+\infty)$ are summable functions. Hence Lemma 2.1 implies that $\mathcal{A}(B_\rho)$ is a relative compact of the space $C^{1,1}(D_{ab})$. $\square$

2.3. Krasnosel'skii–Krein Type Lemmas.

Lemma 2.4. *Let* $f \in L(D_{ab})$, $z \in C(D_{ab})$,

$$f_m \in L(D_{ab}), \quad z_m \in \widetilde{C}(D_{ab}) \ \ (m = 1,2,\dots), \tag{2.13}$$

$$\lim_{m\to\infty} \int_0^x \int_0^y [f_m(s,t) - f(s,t)]\,ds\,dt = 0 \ \ \textit{uniformly on} \ \ D_{ab}, \tag{2.14}$$

and

$$\lim_{m\to\infty} z_m(x,y) = z(x,y) \ \ \textit{uniformly on} \ \ D_{ab}. \tag{2.15}$$

Moreover, if

$$\limsup_{m\to\infty}\Big(\int_0^a\Big|\frac{\partial z_m(s,0)}{\partial s}\Big|\,ds+\int_0^b\Big|\frac{\partial z_m(0,t)}{\partial t}\Big|\,dt+ \\ +\int_0^a\int_0^b\Big|\frac{\partial^2 z_m(s,t)}{\partial s\partial t}\Big|\,ds\,dt\Big)<+\infty, \tag{2.16}$$

then

$$\lim_{m\to\infty}\int_0^x\int_0^y[f_m(s,t)z_m(s,t)-f(s,t)z(s,t)]dsdt=0 \text{ uniformly on } D_{ab}{}^*. \tag{2.17}$$

Proof. By virtue of condition (2.16) and the boundedness of the sequence $(z_m)_{m=1}^\infty$ there exists a positive constant ρ such that

$$|z_m(x,y)|+\int_0^a\int_0^b|f(s,t)|\,ds\,dt+\int_0^a\Big|\frac{\partial z_m(s,y)}{\partial s}\Big|\,ds+\int_0^b\Big|\frac{\partial z_m(x,t)}{\partial t}\Big|\,dt+ \\ +\int_0^a\int_0^b\Big|\frac{\partial^2 z_m(s,t)}{\partial s\partial t}\Big|\,ds\,dt<\rho \quad\text{for}\quad (x,y)\in D_{ab}. \tag{2.18}$$

For an arbitrary natural m set

$$\gamma_m(x,y)=\int_0^x\int_0^y[f_m(s,t)-f(s,t)]\,ds\,dt, \tag{2.19}$$

$$\delta_m(x,y)=\int_0^x\int_0^y[f_m(s,t)z_m(s,t)-f(s,t)z(s,t)]\,ds\,dt. \tag{2.20}$$

Then

$$\delta_m(x,y)=\int_0^x\int_0^y\frac{\partial^2\gamma_m(s,t)}{\partial s\partial t}z_m(s,t)dsdt+\int_0^x\int_0^y f(s,t)[z_m(s,t)-z(s,t)]dsdt=$$

$$=\gamma_m(x,y)z_m(x,y)+\int_0^x\int_0^y\frac{\partial^2 z_m(s,t)}{\partial s\partial t}\gamma_m(s,t)dsdt-\int_0^x\frac{\partial z_m(s,y)}{\partial s}\gamma_m(s,y)ds-$$

*An analogous statement for functions of one variable was proved for the first time in [24].

$$-\int_0^y \frac{\partial z_m(x,t)}{\partial t}\gamma_m(x,t)\,dt+\int_0^x\int_0^y f(s,t)[z_m(s,t)-z(s,t)]\,ds\,dt. \quad (2.21)$$

On the other hand, by (2.14) and (2.15) there exists a sequence of positive numbers $(\varepsilon_m)_{m=1}^{\infty}$ such that

$$\gamma_m(x,y)| < \varepsilon_m, \quad |z_m(x,y)-z(x,y)| < \varepsilon_m \quad \text{for} \quad (x,y)\in D_{ab}, \quad (2.22)$$

and

$$\lim_{m\to\infty}\varepsilon_m = 0. \quad (2.23)$$

Taking into account (2.18) and (2.22), from (2.21) we find

$$|\delta_m(x,y)| < \rho\varepsilon_m \quad \text{for} \quad (x,y)\in D_{ab} \ \ (m=1,2,\dots).$$

Hence, by virtue of equalities (2.20) and (2.23), we obtain condition (2.17). □

Lemma 2.5. *Let $f\in L(D_{ab})$, $z\in C(D_{ab})$ and along with* (2.13)–(2.15) *the conditions*

$$z_m(x,y) = z_{0m}(x,y)+\overline{z}_m(x,y), \quad \overline{z}_m\in \widetilde{C}(D_{ab}) \ (m=1,2,\dots), \quad (2.24)$$

$$\limsup_{m\to\infty}\Big(\int_0^a\Big|\frac{\partial \overline{z}_m(s,0)}{\partial s}\Big|\,ds+\int_0^b\Big|\frac{\partial \overline{z}_m(0,t)}{\partial t}\Big|\,dt+ + \int_0^a\int_0^b\Big|\frac{\partial^2 \overline{z}_m(s,t)}{\partial s\partial t}\Big|\,ds\,dt\Big) < +\infty, \quad (2.25)$$

$$\lim_{m\to\infty} z_{0m}(x,y) = z_0(x,y) \quad \textit{uniformly on} \quad D_{ab} \quad (2.26)$$

hold, where

$$z_0\in\widetilde{C}(D_{ab}). \quad (2.27)$$

Moreover, let there exist a nonnegative function $f^*\in L(D_{ab})$ such that the inequality

$$|f_m(x,y)| \le f^*(x,y) \quad (m=1,2,\dots) \quad (2.28)$$

holds almost everywhere in D_{ab}. Then condition (2.17) *is fulfilled.*

Proof. By conditions (2.13), (2.15), (2.24) and (2.26)

$$\lim_{m\to\infty}\overline{z}_m(x,y) = \overline{z}(x,y) \quad \text{uniformly on} \ \ D_{ab},$$

where $\overline{z} \in C(D_{ab})$. But by Lemma 2.4 the latter condition, along with (2.14) and (2.25), yields

$$\lim_{m\to\infty} \int_0^x \int_0^y [f_m(s,t)\overline{z}_m(s,t) - f(s,t)\overline{z}(s,t)]\, ds\, dt = 0 \quad \text{uniformly on } D_{ab}.$$

Thus to complete the proof we have only to show that

$$\lim_{m\to\infty} \delta_{0m}(x,y) = 0 \quad \text{uniformly on } D_{ab}, \tag{2.29}$$

where

$$\delta_{0m}(x,y) = \int_0^x \int_0^y [f_m(s,t)z_{0m}(s,t) - f(s,t)z_0(s,t)]\, ds\, dt.$$

In view of (2.19) and (2.27)

$$\delta_{0m}(x,y) = \int_0^x \int_0^y \frac{\partial^2 \gamma_m(s,t)}{\partial s \partial t} z_0(s,t)\, ds\, dt +$$
$$+ \int_0^x \int_0^y f_m(s,t)[z_{0m}(s,t) - z_0(s,t)]\, ds\, dt = \gamma_m(x,y)z_0(x,y) +$$
$$+ \int_0^x \int_0^y \frac{\partial^2 z_0(s,t)}{\partial s \partial t} \gamma_m(s,t)\, ds\, dt - \int_0^x \frac{\partial z_0(s,y)}{\partial s} \gamma_m(s,y)\, ds -$$
$$- \int_0^y \frac{\partial z_0(x,t)}{\partial t} \gamma_m(x,t)\, dt + \int_0^x \int_0^y f_m(s,t)[z_{0m}(s,t) - z_0(s,t)]\, ds\, dt,$$

which with regard to (2.28) gives

$$|\delta_{0m}(x,y)| \le \rho \varepsilon_m \quad \text{for} \quad (x,y) \in D_{ab} \ \ (m = 1, 2, \dots), \tag{2.30}$$

where

$$\rho = \max\Big\{ |z_0(x,y)| + \int_0^a \Big| \frac{\partial z_0(s,y)}{\partial s} \Big|\, ds + \int_0^b \Big| \frac{\partial z_0(x,t)}{\partial t} \Big|\, dt : (x,y) \in D_{ab} \Big\} +$$
$$+ \int_0^a \int_0^b \Big| \frac{\partial^2 z_0(s,t)}{\partial s \partial t} \Big|\, ds\, dt + \int_0^a \int_0^b f^*(s,t)\, ds\, dt,$$
$$\varepsilon_m = \max\{ |\gamma_m(x,y)| + |z_{0m}(x,y) - z_0(x,y)| : (x,y) \in D_{ab} \}.$$

But by (2.14) and (2.26) equality (2.23) holds. (2.23) and (2.30) yield condition (2.29). □

Lemma 2.6. *Let*

$$v_m(x,y) = \int_0^a \int_0^b g_1(x,s) g_2(y,t) f_m(s,t) z_m(s,t)\, ds\, dt, \tag{2.31}$$

$$v(x,y) = \int_0^a \int_0^b g_1(x,s) g_2(y,t) f(s,t) z(s,t)\, ds\, dt, \tag{2.32}$$

where g_1 and g_2 are the functions given by (2.3) *and* (2.4)*, and f_m, z_m $(m = 1, 2, \dots)$, f and z are the functions satisfying either the conditions of Lemma* 2.4*, or the conditions of Lemma* 2.5*. Then*

$$\lim_{m\to\infty} \|v_m - v\|_{C^{1,1}} = 0. \tag{2.33}$$

Proof. By (2.31) and (2.32)

$$v_m \in \widetilde{C}^1(D_{ab}) \ \ (m = 1, 2, \dots), \quad v \in \widetilde{C}^1(D_{ab})$$

and the equalities

$$\frac{\partial^{j+k} v_m(x,y)}{\partial x^j \partial y^k} = \\ = \int_0^a \int_0^b g_{1j}(x,s) g_{2k}(y,t) f_m(s,t) z_m(s,t)\, ds\, dt \ \ (j,k = 0,1), \tag{2.34}$$

$$\frac{\partial^{j+k} v(x,y)}{\partial x^j \partial y^k} = \int_0^a \int_0^b g_{1j}(x,s) g_{2k}(y,t) f(s,t) z(s,t)\, ds\, dt \ (j,k = 0,1), \tag{2.35}$$

hold, where $g_{10}(x,s) = g_1(x,s)$, $g_{11}(x,s) = \frac{\partial g_1(x,s)}{\partial x}$, $g_{20}(y,t) = g_2(y,t)$ and $g_{21}(y,t) = \frac{\partial g_1(y,t)}{\partial y}$.

Let $\sigma(x,y)$ be an arbitrary polynomial of two variables. If the functions f_m, z_m $(m = 1, 2, \dots)$, f and z satisfy the conditions of Lemma 2.4 (Lemma 2.5), then the functions f_m, σz_m $(m = 1, 2, \dots)$, f and σz satisfy the conditions of this lemma too. Therefore for arbitrary fixed $a_0 \in [0,a]$ and $b_0 \in [0,b]$ we have

$$\lim_{m\to\infty} \int_{a_0}^x \int_{b_0}^y \sigma(s,t)[f_m(s,t) z_m(s,t) - f(s,t) z(s,t)]\, ds\, dt = 0$$

uniformly on D_{ab}. Hence by (2.34) and (2.35) we conclude that

$$\lim_{m\to+\infty}\frac{\partial^{j+k}v_m(x,y)}{\partial x^j\partial y^k}=\lim_{m\to+\infty}\frac{\partial^{j+k}v(x,y)}{\partial x^j\partial y^k}\quad (j,k=0,1)\ \text{ uniformly on }\ D_{ab}.$$

Therefore condition (2.33) holds. □

When $z_m(x,y)\equiv z(x,y)\equiv 1$ $(m=1,2,\dots)$ Lemma 2.6 takes the form of

Lemma 2.7. *Let*

$$v_m(x,y)=\int\limits_0^a\int\limits_0^b g_1(x,s)g_2(y,t)f_m(s,t)\,ds\,dt\quad (m=1,2,\dots),$$

$$v(x,y)=\int\limits_0^a\int\limits_0^b g_1(x,s)g_2(y,t)f(s,t)\,ds\,dt,$$

where $f_m\in L(D_{ab})$ $(m=1,2,\dots)$ and $f\in L(D_{ab})$ are the functions satisfying condition (2.14). *Then condition* (2.33) *holds.*

2.4. Lemma on Sequences of Solutions of Problems $(1.5_{\mathbf{m}})$,$(1.6_{\mathbf{m}})$ ($\mathbf{m}$=1,2, …).

Lemma 2.8. *Let the functions p_{km},q_m and φ_m $(k=0,1,2,3;\ m=1,2,\dots)$ satisfy the conditions* (i)–(iii) *and let, for every natural m, problem* $(1.5_m),(1.6_m)$ *have a solution u_m. Moreover, let*

$$\rho=\sup\{\|u_m\|_{C^{1,1}}:m=1,2,\dots\}<+\infty. \tag{2.36}$$

Then there exist a subsequence $(u_{m_n})_{n=1}^{\infty}$ of the sequence $(u_m)_{m=1}^{\infty}$ and a solution u of problem (1.1), (1.2) *such that*

$$\lim_{n\to\infty}\|u_{m_n}-u\|_{C^{1,1}}=0. \tag{2.37}$$

Proof. By Lemma 2.2 and the conditions (ii) and (2.36) for any natural m the function u_m admits the representation

$$u_m(x,y)=v_m(x,y)+\overline{v}_m(x,y), \tag{2.38}$$

where

$$v_m(x,y)=\eta(\varphi_m)(x,y)+\int\limits_0^a\int\limits_0^b g_1(x,s)g_2(y,t)q_m(s,t)\,ds\,dt, \tag{2.39}$$

$$\overline{v}_m(x,y)=\int\limits_0^a\int\limits_0^b g_1(x,s)g_2(y,t)Q_m(s,t)\,ds\,dt, \tag{2.40}$$

$$Q_m(x,y)=p_{0m}(x,y)u_m(x,y)+p_{1m}(x,y)\frac{\partial u_m(x,y)}{\partial x}+$$
$$+p_{2m}(x,y)\frac{\partial u_m(x,y)}{\partial y}+p_{3m}(x,y)\frac{\partial^2 u_m(x,y)}{\partial x\partial y} \tag{2.41}$$

and

$$|Q_m(x,y)|\le \rho p(x,y) \quad \text{almost everywhere in } \ D_{ab}. \tag{2.42}$$

According to the conditions (i), (iii) and Lemma 2.7

$$\lim_{m\to\infty}\|v_m-v\|_{C^{1,1}}=0, \tag{2.43}$$

where

$$v(x,y)=\eta(\varphi)(x,y)+\int\limits_0^a\int\limits_0^b g_1(x,s)g_2(y,t)q(s,t)\,ds\,dt \tag{2.44}$$

and

$$v\in \widetilde{C}^1(D_{ab}). \tag{2.45}$$

By (2.40) and (2.42) for any natural m we have

$$\overline{v}_m\in \widetilde{C}^1(D_{ab}),\ \|\overline{v}_m\|_{C^{1,1}}\le (ab+a+b+1)\rho\int\limits_0^a\int\limits_0^b p(s,t)ds\,dt, \tag{2.46}$$

$$\Big|\frac{\partial^3\overline{v}_m(x,y)}{\partial x^2\partial y}\Big|\le h_1(x) \quad \text{for} \quad x\in[0,a]\backslash I_{1y},\ y\in[0,b], \tag{2.47}$$

$$\Big|\frac{\partial^3\overline{v}_m(x,y)}{\partial x\partial y^2}\Big|\le h_2(y) \quad \text{for} \quad x\in[0,a],\ y\in[0,b]\backslash I_{2x}, \tag{2.48}$$

and

$$\Big|\frac{\partial^4\overline{v}_m(x,y)}{\partial x^2\partial y^2}\Big|\le \rho p(x,y) \quad \text{almost everywhere in } \ D_{ab}, \tag{2.49}$$

where

$$h_1(x)=\rho\int\limits_0^b p(x,t)\,dt, \quad h_2(y)=\rho\int\limits_0^a p(s,y)\,ds,$$

and $I_{1y}\subset[0,a]$ and $I_{2x}\subset[0,b]$ are some sets of zero measure.

By virtue of Lemma 2.1 conditions (2.46)–(2.49) guarantee that the sequence $(\overline{v}_m)_{m=1}^{\infty}$ is a relative compact in the topology of the space $C^{1,1}(D_{ab})$.

Therefore there exist a function $\overline{v}_0 \in C^{1,1}(D_{ab})$ and a subsequence $(\overline{v}_{m_n})_{n=1}^{\infty}$ of this sequence such that

$$\lim_{m\to\infty} \|\overline{v}_{m_n} - \overline{v}\|_{C^{1,1}} = 0.$$

This and (2.43) imply condition (2.37), where

$$u(x,y) = v(x,y) + \overline{v}(x,y)$$

and $u \in C^{1,1}(D_{ab})$. By Lemma 2.2, to complete the proof, it remains to show that u is a solution of equation (2.5).

Put

$$z_{0n}(x,y) = u_{m_n}(x,y), \quad z_{1n}(x,y) = \frac{\partial z_{0n}(x,y)}{\partial x}, \quad z_{2n}(x,y) = \frac{\partial z_{0n}(x,y)}{\partial y},$$
$$z_{3n}(x,y) = \frac{\partial^2 z_{0n}(x,y)}{\partial x \partial y};$$
$$w_{0n}(x,y) = v_{m_n}(x,y), \quad w_{1n}(x,y) = \frac{\partial w_{0n}(x,y)}{\partial x}, \quad w_{2n}(x,y) = \frac{\partial w_{0n}(x,y)}{\partial y},$$
$$w_{3n}(x,y) = \frac{\partial^2 w_{0n}(x,y)}{\partial x \partial y};$$
$$\overline{w}_{0n}(x,y) = \overline{v}_{m_n}(x,y), \quad \overline{w}_{1n}(x,y) = \frac{\partial \overline{w}_{0n}(x,y)}{\partial x}, \quad \overline{w}_{2n}(x,y) = \frac{\partial \overline{w}_{0n}(x,y)}{\partial y},$$
$$\overline{w}_{3n}(x,y) = \frac{\partial^2 \overline{w}_{0n}(x,y)}{\partial x \partial y};$$
$$w_0(x,y) = v(x,y), \quad w_1(x,y) = \frac{\partial v(x,y)}{\partial x}, \quad w_2(x,y) = \frac{\partial v(x,y)}{\partial y},$$
$$w_3(x,y) = \frac{\partial^2 v(x,y)}{\partial x \partial y}.$$

Then by (2.37), (2.43) and (2.45) we have

$$\begin{gathered}\lim_{n\to\infty} z_{0n}(x,y) = u(x,y), \quad \lim_{n\to\infty} z_{1n}(x,y) = \frac{\partial u(x,y)}{\partial x},\\ \lim_{n\to\infty} z_{2n}(x,y) = \frac{\partial u(x,y)}{\partial y}, \ \lim_{n\to\infty} z_{3n}(x,y) = \frac{\partial^2 u(x,y)}{\partial x \partial y},\\ \lim_{n\to\infty} w_{kn}(x,y) = w_k(x,y)\end{gathered} \tag{2.50}$$

uniformly on D_{ab} and

$$w_k \in \widetilde{C}(D_{ab}) \quad (k = 0,1,2,3). \tag{2.51}$$

On the other hand, it follows from (2.46)–(2.49) that

$$\limsup_{n\to\infty}\Big(\int_0^a\Big|\frac{\partial\overline{w}_{kn}(s,0)}{\partial s}\Big|\,ds+\int_0^b\Big|\frac{\partial\overline{w}_{kn}(0,t)}{\partial t}\Big|\,dt+ \\ +\int_0^a\int_0^b\Big|\frac{\partial^2\overline{w}_{kn}(s,t)}{\partial s\partial t}\Big|\,ds\,dt\Big)<+\infty\quad(k=0,1,2,3). \tag{2.52}$$

According to (2.38), (2.40) and (2.41) we have

$$z_{kn}(x,y)=w_{kn}(x,y)+\overline{w}_{kn}(x,y)\quad(k=0,1,2,3), \tag{2.53}$$

$$z_{0n}(x,y)=w_{0n}(x,y)+\sum_{k=0}^{3}\int_0^a\int_0^b g_1(x,s)g_2(y,t)p_{kn}(s,t)z_{kn}(s,t)dsdt. \tag{2.54}$$

But by Lemma 2.6 and the conditions (i), (ii), and (2.50)–(2.53) from (2.54) we obtain

$$u(x,y)=v(x,y)+\int_0^a\int_0^b g_1(x,s)g_2(y,t)\mathcal{Q}(u)(s,t)\,ds\,dt,$$

where $\mathcal{Q}$ is the operator given by equality (2.6). Taking this and equality (2.44) into account, it becomes clear that u is the solution of equation (2.5). □

2.5. Lemma on the Stability of the Unique Solvability Property of Problem (1.1_0),(1.2_0).

Lemma 2.9. *Let problem* (1.1_0), (1.2_0) *have only the trivial solution. Then for any nonnegative function* $\gamma\in L(D_{ab})$ *there exists a positive number* δ_0 *such that for arbitrary functions* $\overline{p}_k\in L(D_{ab})$ $(k=0,1,2,3)$ *satisfying the inequalities*

$$|\overline{p}_k(x,y)-p_k(x,y)|\le\gamma(x,y)\ \textit{almost for every}\ (x,y)\in D_{ab}\ (k=0,1,2,3),$$

$$\Big|\int_0^x\int_0^y[\overline{p}_k(s,t)-p_k(s,t)]\,ds\,dt\Big|\le\delta\quad\textit{for}\quad(x,y)\in D_{ab}\ \ (k=0,1,2,3),$$

the differential equation

$$\frac{\partial^4u}{\partial x^2\partial y^2}=\overline{p}_0(x,y)u+\overline{p}_1(x,y)\frac{\partial u}{\partial x}+\overline{p}_2(x,y)\frac{\partial u}{\partial y}+\overline{p}_3(x,y)\frac{\partial^2u}{\partial x\partial y}$$

has only the trivial solution satisfying the boundary conditions (1.2_0).

Proof. Assume the contrary, i.e., that the lemma is not true. Then there exist a nonnegative function $\gamma \in L(D_{ab})$ and a sequence of summable in D_{ab} functions $(p_{km})_{m=1}^{\infty}$ $(k=0,1,2,3)$ such that

$$|p_{km}(x,y)-p_k(x,y)| \le \gamma(x,y) \quad \text{almost for every} \quad (x,y)\in D_{ab}$$
$$(k=0,1,2,3;\ m=1,2,\dots),$$
$$\Big|\int_0^x\int_0^y [p_k(s,t)-p_k(s,t)]\,ds\,dt\Big| \le \frac{1}{m} \quad \text{for} \quad (x,y)\in D_{ab}$$
$$(k=0,1,2,3;\ m=1,2,\dots)$$

and for an arbitrary natural m the differential equation

$$\frac{\partial^4 u}{\partial x^2\partial y^2} = p_{0m}(x,y)u + p_{1m}(x,y)\frac{\partial u}{\partial x} + p_{2m}(x,y)\frac{\partial u}{\partial y} + p_{3m}(x,y)\frac{\partial^2 u}{\partial x\partial y}$$

has a solution u_m such that $u_m(x,y)=0$ for $(x,y)\in\Gamma_{ab}$ and

$$\|u_m\|_{C^{1,1}} = 1. \tag{2.55}$$

On the other hand, by Lemma 2.8 the sequence $(u_m)_{m=1}^{\infty}$ contains a subsequence $(u_{m_n})_{n=1}^{\infty}$ converging to the solution of problem $(1.1_0),(1.2_0)$ in the norm of the space $C^{1,1}$. But problem $(1.1_0),(1.2_0)$ has only the trivial solution. Therefore

$$\lim_{n\to\infty}\|u_{m_n}\|_{C^{1,1}} = 0,$$

which contradicts condition (2.55). The obtained contradiction proves the lemma. □

2.6. On an Integral Identity for Functions from the Space $\widetilde{\mathbf{C}}^1(\mathbf{D_{ab}})$ Satisfying the Boundary Condition $(\mathbf{1.2_0})$.

Lemma 2.10. *If the function $u\in\widetilde{C}^1(D_{ab})$ satisfies the boundary condition (1.2_0), then*

$$\int_0^a\int_0^b u(s,t)\frac{\partial^4 u(s,t)}{\partial s^2\partial t^2}\,ds\,dt = \int_0^a\int_0^b \Big[\frac{\partial^2 u(s,t)}{\partial s\partial t}\Big]^2\,ds\,dt. \tag{2.56}$$

Proof. By the formula of integration by parts and condition (1.2_0) we have

$$\int_0^b u(s,t)\frac{\partial^4 u(s,t)}{\partial s^2\partial t^2}\,dt = -\int_0^b \frac{\partial u(s,t)}{\partial t}\,\frac{\partial^3 u(s,t)}{\partial s^2\partial t}\,dt \tag{2.57}$$

and

$$\int_0^a \frac{\partial u(s,t)}{\partial t}\frac{\partial^3 u(s,t)}{\partial s^2 \partial t}\,ds = -\int_0^a \Big[\frac{\partial^2 u(s,t)}{\partial s \partial t}\Big]^2 ds. \tag{2.58}$$

almost everywhere in $[0,a]$ and $[0,b]$.

Now, by integrating (2.57) from 0 to a and taking (2.58) into account we obtain equality (2.56). □

2.7. On an Analogue of Wirtinger's Lemma for Functions of Two Variables.

Lemma 2.11. *If a function $u \in C^{1,1}(D_{ab})$ satisfies the boundary condition* (1.2_0), *then*

$$\int_0^a\int_0^b \Big[\frac{\partial u(s,t)}{\partial s}\Big]^2 ds\,dt \le \frac{b^2}{\pi^2}\int_0^a\int_0^b \Big[\frac{\partial^2 u(s,t)}{\partial s \partial t}\Big]^2 ds\,dt, \tag{2.59}$$

$$\int_0^a\int_0^b \Big[\frac{\partial u(s,t)}{\partial t}\Big]^2 ds\,dt \le \frac{a^2}{\pi^2}\int_0^a\int_0^b \Big[\frac{\partial^2 u(s,t)}{\partial s \partial t}\Big]^2 ds\,dt, \tag{2.60}$$

$$\int_0^a\int_0^b u^2(s,t)\,ds\,dt \le \frac{a^2b^2}{\pi^4}\int_0^a\int_0^b \Big[\frac{\partial^2 u(s,t)}{\partial s \partial t}\Big]^2 ds\,dt. \tag{2.61}$$

Proof. By Wirtinger's lemma (see [25], Lemma 257) and condition (1.2_0) we have the inequalities

$$\int_0^b \Big[\frac{\partial u(s,t)}{\partial s}\Big]^2 dt \le \frac{b^2}{\pi^2}\int_0^b \Big[\frac{\partial^2 u(s,t)}{\partial s \partial t}\Big]^2 dt, \tag{2.62}$$

$$\int_0^a \Big[\frac{\partial u(s,t)}{\partial t}\Big]^2 ds \le \frac{a^2}{\pi^2}\int_0^a \Big[\frac{\partial^2 u(s,t)}{\partial s \partial t}\Big]^2 ds, \tag{2.63}$$

$$\int_0^b u^2(s,t)\,dt \le \frac{b^2}{\pi^2}\int_0^b \Big[\frac{\partial u(s,t)}{\partial t}\Big]^2 dt. \tag{2.64}$$

If we integrate (2.62) and (2.63) from 0 to a and from 0 to b, respectively, then we get inequalities (2.59) and (2.60). If we integrate (2.64) from 0 to a, then with regard to (2.60) we obtain inequality (2.61). □

§ 3. Proofs of the Main Results

Proof of Theorem 1.1′. By virtue of Lemmas 2.3 and 2.9 problem (1.1), (1.2) is uniquely solvable, and beginning from some sufficiently large m_0 problem $(1.1_m), (1.2_m)$ is also uniquely solvable. Denote the solutions of these problems by u and u_m, respectively. Our goal is to prove that

$$\lim_{m\to\infty} \|u_m - u\|_{C^{1,1}} = 0. \tag{3.1}$$

Assume the opposite, i.e., that condition (3.1) does not hold. Then without loss of generality it can be assumed that

$$\|u_m - u\|_{C^{1,1}} \geq \varepsilon \ (m = m_0, m_0 + 1, \dots), \tag{3.2}$$

where ε is a positive constant independent of m. There exist two possibilities: either $(u_m)_{m=1}^{\infty}$ satisfies condition (2.36), or

$$\limsup_{m\to\infty} \|u_m\|_{C^{1,1}} = +\infty. \tag{3.3}$$

Assume first that (2.36) holds. Then by Lemma 2.8 there exists a subsequence $(u_{m_n})_{n=1}^{\infty}$ of the sequence $(u_m)_{m=1}^{\infty}$ satisfying condition (2.37). But this is impossible on account of condition (3.2).

To complete the proof, we have to show that the assumption that condition (3.3) is fulfilled leads us to a contradiction.

Choose a subsequence $(u_{m_n})_{n=1}^{\infty}$ from $(u_m)_{m=1}^{\infty}$ such that

$$\rho_n = \|u_{m_n}\|_{C^{1,1}} > 0 \ \ (n = 1, 2, \dots), \quad \lim_{n\to\infty} \rho_n = +\infty.$$

Put $v_n(x,y) = \frac{1}{\rho_n} u_{m_n}(x,y)$. Then

$$\|v_n\|_{C^{1,1}} = 1 \quad (n = 1, 2, \dots) \tag{3.4}$$

and for every natural n the function v_n is a solution of the problem

$$\frac{\partial^4 v}{\partial x^2 \partial y^2} = p_{0m_n}(x,y)v + p_{1m_n}(x,y)\frac{\partial v}{\partial x} + p_{2m_n}(x,y)\frac{\partial v}{\partial y} + \\ + p_{3m_n}(x,y)\frac{\partial^2 v}{\partial x \partial y} + \widetilde{q}_n(x,y),$$

$$v(x,y) = \overline{\varphi}_n(x,y) \quad \text{for} \quad (x,y) \in \Gamma_{ab},$$

where $\widetilde{q}_n(x,y) = \frac{1}{\rho_n} q_{m_n}(x,y)$, $\quad \widetilde{\varphi}_n(x,y) = \frac{1}{\rho_n} \varphi_{m_n}(x,y)$. Moreover,

$$\lim_{n\to\infty} \int_0^x \int_0^y \widetilde{q}_n(s,t)\, ds\, dt = 0 \quad \text{uniformly on} \ \ D_{ab} \tag{3.5}$$

and the equalities

$$\lim_{n\to\infty}\frac{\partial^k\widetilde{\varphi}_n(x,0)}{\partial x^k}=0,\quad \lim_{n\to\infty}\frac{\partial^k\widetilde{\varphi}_n(x,b)}{\partial x^k}=0\ \ (k=0,1), \tag{3.6}$$

$$\lim_{n\to\infty}\frac{\partial^k\widetilde{\varphi}_n(0,y)}{\partial y^k}=0,\quad \lim_{n\to\infty}\frac{\partial^k\widetilde{\varphi}_n(a,y)}{\partial y^k}=0\ \ (k=0,1) \tag{3.7}$$

hold uniformly on $[0,a]$ and $[0,b]$, respectively.

By Lemma 2.8 the conditions (i), (ii), (3.5)–(3.7) and the unique solvability of the homogeneous problem $(1.1_0),(1.2_0)$ guarantee that there exists a subsequence $(v_{n_j})_{j=1}^{\infty}$ of the sequence $(v_n)_{n=1}^{\infty}$ such that

$$\lim_{j\to\infty}\|v_{n_j}\|_{C^{1,1}}=0.$$

But this contradicts condition (3.4). The obtained contradiction proves the theorem. □

Proof of Theorem 1.1. Assume that the theorem is not true. Then by virtue of Lemmas 2.3 and 2.9 there exist a positive number ε, a nonnegative function $\gamma\in L(D_{ab})$ and sequences of functions $(p_{km})_{m=1}^{\infty}$ $(k=0,1,2,3)$, $(q_m)_{m=1}^{\infty}$ and $(\varphi_m)_{m=1}^{\infty}$ such that the conditions (i)–(iii) hold, where

$$p(x,y)=\sum_{k=0}^{3}|p_k(x,y)|+\gamma(x,y),$$

and for any natural m problem $(1.1_m),(1.2_m)$ has a unique solution u_m satisfying the inequality

$$\|u_m-u\|_{C^{1,1}}>\varepsilon.$$

But this is impossible, since equality (3.1) is valid by Theorem 1.1′. □

Proof of Theorem 1.2. By Theorem 1.1 we have only to show that in the conditions of Theorem 1.2 the homogeneous problem $(1.1_0),(1.2_0)$ has only the trivial solution. Assume the opposite, i.e., that this problem has a nontrivial solution u. Then according to Lemma 2.10 we have

$$\int_0^a\int_0^b\Big[\frac{\partial^2u(s,t)}{\partial s\partial t}\Big]^2ds\,dt=\int_0^a\int_0^b p_0(s,t)u^2(s,t)\,ds\,dt+$$
$$+\int_0^a\int_0^b p_1(s,t)u(s,t)\frac{\partial u(s,t)}{\partial s}\,ds\,dt+\int_0^a\int_0^b p_2(s,t)u(s,t)\frac{\partial u(s,t)}{\partial t}\,ds\,dt+$$
$$+\int_0^a\int_0^b p_3(s,t)u(s,t)\frac{\partial^2u(s,t)}{\partial s\partial t}\,ds\,dt. \tag{3.8}$$

Applying the Schwartz inequality and condition (1.7), we find

$$\int_0^a\int_0^b p_1(s,t)u(s,t)\frac{\partial u(s,t)}{\partial s}dsdt \le \int_0^a\int_0^b p_{10}(s,t)u(s,t)\frac{\partial u(s,t)}{\partial s}dsdt + l_1\rho_0\rho_1,$$

$$\int_0^a\int_0^b p_2(s,t)u(s,t)\frac{\partial u(s,t)}{\partial t}dsdt \le \int_0^a\int_0^b p_{20}(s,t)u(s,t)\frac{\partial u(s,t)}{\partial t}dsdt + l_2\rho_0\rho_2,$$

$$\int_0^a\int_0^b p_3(s,t)u(s,t)\frac{\partial^2 u(s,t)}{\partial s\partial t}\,ds\,dt \le l_3\rho_0\rho_3,$$

where ρ_0, ρ_1, ρ_2 and ρ_3 are the positive constants given by the equalities

$$\rho_0^2 = \int_0^a\int_0^b u^2(s,t)\,ds\,dt, \quad \rho_1^2 = \int_0^a\int_0^b \Big[\frac{\partial u(s,t)}{\partial s}\Big]^2\,ds\,dt,$$

$$\rho_2^2 = \int_0^a\int_0^b \Big[\frac{\partial u(s,t)}{\partial t}\Big]^2\,ds\,dt, \quad \rho_3^2 = \int_0^a\int_0^b \Big[\frac{\partial^2 u(s,t)}{\partial s\partial t}\Big]^2\,ds\,dt.$$

On the other hand, according to the formula of integration by parts and condition (1.2_0) we have

$$\int_0^a\int_0^b p_{10}(s,t)u(s,t)\frac{\partial u(s,t)}{\partial s}\,ds\,dt = -\frac{1}{2}\int_0^a\int_0^b \frac{\partial p_{10}(s,t)}{\partial s}u^2(s,t)\,ds\,dt,$$

$$\int_0^a\int_0^b p_{20}(s,t)u(s,t)\frac{\partial u(s,t)}{\partial t}\,ds\,dt = -\frac{1}{2}\int_0^a\int_0^b \frac{\partial p_{20}(s,t)}{\partial t}u^2(s,t)\,ds\,dt.$$

Taking this and inequality (1.8) into account, from (3.8) we get

$$\rho_3^2 \le \int_0^a\int_0^b \Big[p_0(s,t) - \frac{1}{2}\frac{\partial p_{10}(s,t)}{\partial s} - \frac{1}{2}\frac{\partial p_{20}(s,t)}{\partial t}\Big]u^2(s,t)\,ds\,dt +$$
$$+l_1\rho_0\rho_1 + l_2\rho_0\rho_2 + l_3\rho_0\rho_3 \le l_0\rho_0^2 + l_1\rho_0\rho_1 + l_2\rho_0\rho_2 + l_3\rho_0\rho_3. \qquad (3.9)$$

By Lemma 2.11 equalities (2.59)–(2.61) are valid, i.e.,

$$\rho_1 \le \frac{b}{\pi}\rho_3, \quad \rho_2 \le \frac{a}{\pi}\rho_3, \quad \rho_0 \le \frac{ab}{\pi^2}\rho_3. \qquad (3.10)$$

Hence for $l_0 \geq 0$ (3.9) implies

$$0 < \rho_3^2 \leq \Big(\frac{a^2b^2}{\pi^4}l_0 + \frac{ab^2}{\pi^3}l_1 + \frac{a^2b}{\pi^3}l_2 + \frac{ab}{\pi^2}l_3\Big)\rho_3^2,$$

which contradicts inequality (1.9).

When $l_0 < 0$, by (1.10) and (3.10), inequality (3.9) again gives the contradiction

$$\rho_3^2 + |l_0|\rho_0^2 \leq \Big(\frac{b}{\pi}l_1 + \frac{a}{\pi}l_2 + l_3\Big)\rho_0\rho_3 < 2\sqrt{|l_0|}\rho_0\rho_3 \leq \rho_3^2 + |l_0|\rho_0^2. \quad \square$$

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