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DERIVATIVE UNIFORM SAMPLING VIA WEIERSTRASS $\sigma(z)$. TRUNCATION ERROR ANALYSIS IN $\left[2, \frac{\pi q}{2s^2}\right)$

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Abstract. In the entire functions space $\left[2, \frac{\pi q}{2s^2}\right)$ consisting of at most second order functions such that their type is less than $\pi q/(2s^2)$ it is valid the q-order derivative sampling series reconstruction procedure, reading at the von Neumann lattice $\{s(m+ni)|\ (m,n) \in \mathbb{Z}^2\}$ via the Weierstrass $\sigma(\cdot)$ as the sampling function, $s > 0$. The uniform convergence of the sampling sums to the initial function is proved by the *circular truncation error* upper bound, especially derived for this reconstruction procedure. Finally, the explicit second and third order sampling formulæ are given.

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1. Introduction and Preparation

The well-known sampling representation of some function $f(z)$, $z \in \mathbb{C}$, means that f is completely reconstructed from the values $f(sn)$ obtained by "sampling" f at the points of the one-dimensional lattice $s\mathbb{Z}$, $s > 0$. The reconstruction formula (often called by the names of Whittaker, Kotel'nikov and Shannon) is

$$f(z) = \frac{\sin \pi z/s}{\pi/s} \sum_{n \in \mathbb{Z}} f(sn) \frac{(-1)^n}{z - sn}. \tag{1}$$

One of the most interesting extensions of (1) was done by J. M. Whittaker. Namely, for $f \in [2, \pi/2)$[1] for the two-dimensional so-called *von Neumann lattice* $\{s(m+ni)|\ (m,n) \in \mathbb{Z}^2\}$ one has

[1]The functions space $[\rho, \sigma)$ is introduced as the space of all entire functions of the order less than ρ; when it is equal to ρ, it possesses type less than σ, i.e.,

$$\limsup_{r \to \infty} \frac{\ln M_f(r)}{r^\rho} < \sigma$$

(cf. [1, p. 18]). Here $M_f(r) := \sup_{|z|=r} |f(z)|$ denotes the maximum modulus. Similarly, $[\rho, \sigma]$ denotes the functions spaces of order at most ρ, and the type less than or equal to σ.

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$$f(z) = \sigma(z) \sum_{(m,n)\in\mathbb{Z}^2} (-1)^{m+n+mn} \frac{f(m+ni)}{z-m-ni} e^{-\frac{\pi}{2}(m^2+n^2)}, \tag{2}$$

uniformly in all $z \in \mathbb{C}$, and

$$\sigma(z) = z \prod_{m,n\in\mathbb{Z}}{}' \left(1 - \frac{z}{\zeta_{m,n}}\right) \exp\left(\frac{z}{\zeta_{mn}} + \frac{z^2}{2\zeta_{mn}^2}\right),$$

where the primed product means that the term $\zeta_{m,n} = 0$ is omitted, see [2, pp. 72–73]. (Throughout we use the notation $\zeta_{mn} \equiv m + ni,\ (m,n) \in \mathbb{Z}^2$.)

J. R. Higgins extended (2) in a derivative sampling manner, i.e., when the reconstruction procedure involves not only the sampled values of the initial function f, but also the sampled values of first $q-1$ derivatives $f^{(j)}(z),\ j = \overline{1, q-1}$ of f. When f belongs to the Paley–Wiener type functions space (we deal with the exponential type functions), then the role of the sine function is crucial. But in that case the Weierstrass $\sigma(z)$ replaces the sine in (1). We give this result now. If $f \in \left[2, \frac{\pi q}{2s^2}\right)$ for some positive real s we have

$$\begin{aligned} f(z) = \sigma^q(z/s) &\sum_{(m,n)\in\mathbb{Z}^2} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{f^{(q-1-j-k)}(s(m+ni))}{j!(q-1-j-k)!} \\ &\times \frac{R^q_{mnj}}{(z-s(m+ni))^{k+1}}, \end{aligned} \tag{3}$$

with

$$R^q_{mnj} = s^{q-j} \lim_{w\to\zeta_{mn}} \frac{d^j}{dw^j} \left(\frac{w-m-ni}{\sigma(w)}\right)^q,$$

uniformly on all compact z-sets from $\mathbb{C}$ [3, Ch. 5], [4]; here we follow the usual convention $f^{(j)} = \frac{d^j f}{dz^j} := f$ for $j = 0$.

Our main goal in this paper is the following. Let us introduce

$$\mathbf{N}_\delta(r) := \{(m,n)|\ s|m+ni| < r;\ s(N+\delta)\sqrt{2} < r < s(N+\sqrt{\delta})\sqrt{2};\ \delta \in (0,1)\}.$$

The truncated to $\mathbf{N}_\delta(r)$ variant of the sampling series (3) is

$$\begin{aligned} &\mathcal{I}_N(z; f; \sigma; q) \\ &= \sigma^q(z/s) \sum_{(m,n)\in\mathbf{N}_\delta(r)} \sum_{j=0}^{q-1} \sum_{k=0}^{q-1-j} \frac{f^{(q-1-j-k)}(s(m+ni)) R^q_{mnj}}{j!(q-1-j-k)!(z-s(m+ni))^{k+1}}, \end{aligned} \tag{4}$$

and the so-called circular truncation error is

$$\varepsilon_N(f; z) := f(z) - \mathcal{I}_N(z; f; \sigma; q). \tag{5}$$

To derive a uniform upper bound for $|\varepsilon_N(f; z)|$, when $f \in \left[2, \frac{\pi q}{2s^2}\right)$ we use the contour integration method following the approach by Whittaker and Higgins.

2. Derivation of an Upper Bound of the Truncation Error

The main tool in deriving upper bounds of the truncation error consists in using the contour integral method in complex integration. Most authors used positively oriented rectangles as an integration path like $\mathcal{C}_{M,N,R}$ which surrounds the set $\{z|\ -M-\frac{\delta}{2}<\Re\{z\}<N+\frac{\delta}{2};\ |\Im\{z\}|<R\}$ for some positive integers M,N,R (cf. [3, Ch. 5], [4, p. 215]). In the case of circular truncation there is no need to take $\mathcal{C}_{M,N,R}$ and therefore we choose a more suitable circular path Γ_r, the circle of radius r with center at the origin. Another mathematical tool is the use of the previous results in evaluating the growth of the Weierstrass σ-function. Namely, we have

$$|\sigma(z)| \asymp d e^{\frac{\pi}{2}|z|^2}, \quad z\in\mathbb{C}, \tag{6}$$

where $d=\mathrm{dist}(z,\mathbb{Z})$ is the distance from z to the nearest point in the von Neumann lattice $\mathbb{Z}^2$ (see [5, pp. 346–347]).

Theorem 1. *Assume $f\in[2,\frac{\pi q}{2s^2}\theta_f]$, where the parameter $\theta_f\in[0,1)$ depends of the function f. Then for all $N\ge(s\sqrt{2})^{-1}|\max(\Re\{z\},\Im\{z\})|$ and $\delta\in(0,1)$ we have*

$$|\varepsilon_N(f;z)| \le \frac{|\sigma(z/s)|^q(N+\sqrt{\delta})}{(C_1 d)^q\left(N+\delta-\frac{|z|}{s\sqrt{2}}\right)} e^{-\pi q(1-\theta_f)N^2}, \tag{7}$$

where $d=\mathrm{dist}(z,s\mathbb{Z}^2)$ and the absolute constant C_1 is defined by (6).

Proof. Let us estimate the truncation error of the nonharmonic Fourier series expansion of the function $f(z)$ with respect the complex variable z. By the calculus of residues we find that

$$\varepsilon_N(f;z)=f(z)-\mathcal{I}_N(z;f;\sigma;q)=\frac{\sigma^q(z/s)}{2\pi i}\oint_{\Gamma_r}\frac{f(\zeta)d\zeta}{\sigma^q(\zeta/s)(\zeta-z)}.$$

The positively oriented circle Γ_r forms the integration contour containing and bounding the square

$$-N-\delta<\Re\left\{\frac{z}{s}\right\},\Im\left\{\frac{z}{s}\right\}<N+\delta,\ \delta\in(0,1)$$

and the set $N_\delta(r)$ so, that the radius r satisfies $s(N+\delta)\sqrt{2}<r<s(N+\sqrt{\delta})\sqrt{2}$. Now it is clear that

$$\begin{aligned}|\varepsilon_N(f;z)| &\le \frac{|\sigma(z/s)|^q}{2\pi}\oint_{\Gamma_r}\frac{|f(\zeta)||d\zeta|}{|\sigma(\zeta/s)|^q|\zeta-z|}\\ &\le \frac{r|\sigma(z/s)|^q}{(C_1 d)^q}e^{-[\pi q/(2s^2)]r^2}\sup_{\zeta\in\Gamma_r}|f(\zeta)|\sup_{\zeta\in\Gamma_r}|\zeta-z|^{-1}\\ &\le \frac{r|\sigma(z/s)|^q}{(C_1 d)^q|r-|z||}\exp\left\{-\frac{\pi q(1-\theta_f)}{2s^2}r^2\right\}\end{aligned}$$

$$\leq \frac{|\sigma(z/s)|^q (N+\sqrt{\delta})}{(C_1 d)^q \left(N+\delta-\frac{|z|}{s\sqrt{2}}\right)} e^{-\pi q(1-\theta_f)(N+\delta)^2}, \tag{8}$$

where in (8) it is enough to use tha maximum modulus principle to take the suprema on the countour Γ_r, bearing in mind the Hayman estimate (7). After that straightforward transformations lead to the estimate (7). □

Corrolary 1. *Let the situation be the same as in the previous theorem. Then the qth order derivative plane sampling reconstruction formula* (3) *holds true for all* $f \in \left[2, \frac{\pi q}{2s^2}\right)$, *where, in* (3) *the convergence is uniform on compact subsets of* $\mathbb{C}$. *The uniform convergence rate of* $\varepsilon_N(f;z)$ *defined by* (4) *and* (5) *is* $\mathcal{O}\left(e^{-\pi q(1-\theta_f)N^2}\right)$.

Proof. Taking z fixed in the truncation error upper bound (7), we revisit the Higgins extension (3) of the Whittaker result (2) in the uniform manner as $N \to \infty$. On the other hand, according to the assumptions of Theorem 1, estimate (7) becomes

$$\sup_{z\in\mathbb{C}} |\varepsilon_N(f;z)| \leq C_2 \frac{|\sigma(z/s)|^q (N+\sqrt{\delta})}{N+\delta-\frac{|z|}{s\sqrt{2}}} e^{-\pi q(1-\theta_f)N^2} = \mathcal{O}\left(e^{-C_3 N^2}\right),$$

for N large enough and $C_2, C_3 \in \mathbb{R}_+$. This means an exponential convergence rate of the symmetric partial sums sequence $\mathcal{I}_N(f;z;\sigma;q)$ of the non-harmonic Fourier series type derivative plane sampling reconstruction formula (3) as $N \to \infty$. □

3. Second and Third Order Derivative Sampling

In this section we give the exact derivative plane sampling reconstruction formulae for $q = 3, 4$ (the case $q = 2$ was considered already by Higgins under $s = \sqrt{2}$, see [3, Problem 9.4, p. 101]. Another reason, which restricts ourselves to considereing these formulæ, is that the first four derivatives of the Weierstrass $\sigma(z)$ do not contain the invariants g_2, g_3 of the Weierstrass elliptic function $\wp(\cdot)$. Indeed, since $\sigma(z)$ is entire, we have the following straigthforward extension:

$$\sigma(z) = z - \frac{g_2}{240} z^5 - \frac{g_3}{840} z^7 + \mathcal{O}(z^9), \tag{9}$$

where

$$g_2 = 60 \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m+ni)^4}; \quad g_3 = 140 \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{1}{(m+ni)^6}.$$

Consequently, having in mind the quasi-periodicity property

$$\sigma(z+m+ni) = (-1)^{m+n+mn} \sigma(z) e^{\pi(m-ni)z + \pi(m^2+n^2)/2},$$

we obtain by extension (9) the following results.

Lemma 1.

$$\sigma^{(p)}(m+ni) = (-1)^{m+n+mn} p[\pi(m-ni)]^{p-1} e^{-(\pi/2)(m^2+n^2)}, \quad p = \overline{1,4}, \quad (10)$$

$$R^q_{mnj} = s^{q-j}[-q\pi(m-ni)]^j (-1)^{q(m+n+mn)} e^{-(\pi q/2)(m^2+n^2)}, \; j = \overline{0,3}. \quad (11)$$

For a detailed derivation of these relations consult [6, §4].

Now we are ready to formulate our results on the derivative plane sampling reading on the von Neumann lattice $s(\mathbb{Z})^2$ for $q = 3, 4$.

Corrolary 2. *Let* $f \in \left[2, \frac{3\pi}{2s^2}\right)$. *Then*

$$\begin{aligned} f(z) = s\sigma^3\left(\frac{z}{s}\right) &\sum_{(m,n)\in\mathbb{Z}^2} (-1)^{m+n+mn} e^{-3\pi|\zeta_{mn}|^2/2} \\ &\times \left\{ \left[\frac{s^2}{(z - s\zeta_{mn})^2} - \frac{3\pi s\zeta^*_{mn}}{z - s\zeta_{mn}} + \frac{(3\pi\zeta^*_{mn})^2}{2} \right] \frac{f(s\zeta_{mn})}{z - s\zeta_{mn}} \right. \\ &\left. + \left[\frac{s}{z - s\zeta_{mn}} - 3\pi\zeta^*_{mn} \right] \frac{sf'(s\zeta_{mn})}{z - s\zeta_{mn}} + \frac{s^2 f''(s\zeta_{mn})}{2(z - s\zeta_{mn})} \right\}, \end{aligned} \quad (12)$$

where the convergence is uniform in z, *on all compact subsets of* $\mathbb{C}$; z^* *denotes the complex conjugate of* z.

Proof. Using formulæ (10), (11), the rearrangement of (3) for $q = 3$ gives us the asserted display (12). □

Corrolary 3. *Let* $f \in \left[2, \frac{2\pi}{s^2}\right)$. *Then*

$$\begin{aligned} f(z) = \sum_{(m,n)\in\mathbb{Z}^2} \frac{s\sigma^4(z/s)}{6(z - s\zeta_{mn})} &\left\{ \left[\frac{6s^3}{(z - s\zeta_{mn})^3} - \frac{24\pi s^2\zeta^*_{mn}}{(z - s\zeta_{mn})^2} + \frac{3s(4\pi\zeta^*_{mn})^2}{z - s\zeta_{mn}} \right. \right. \\ &- \left. (4\pi\zeta^*_{mn})^3 \right] f(s\zeta_{mn}) + \left[\frac{6s^2}{(z - s\zeta_{mn})^2} - \frac{24\pi s\zeta^*_{mn}}{z - s\zeta_{mn}} + 48(\pi\zeta^*_{mn})^2 \right] sf'(s\zeta_{mn}) \\ &+ \left. \left[\frac{3s}{z - s\zeta_{mn}} - 12\pi\zeta^*_{mn} \right] s^2 f''(s\zeta_{mn}) + s^3 f'''(s\zeta_{mn}) \right\} e^{-2\pi|\zeta_{mn}|^2}, \end{aligned}$$

where the convergence is uniform in z, *on all compact subsets of* $\mathbb{C}$.

4. Final Remarks

The presented method and the truncation error upper bound results have their counterparts in the derivative plane sampling reconstruction of the so-called *Piranashvili* α-processes defined on the probability space $(\Omega, \mathcal{F}, P)$, and constituting the space

$$L^\alpha(\Omega) := \{\xi(t) | \; \|\xi\|_\alpha < \infty\},$$

such that is with the quasi norm $\|\cdot\|_\alpha := (E|\xi(t)|^\alpha)^{1/\alpha}$ endoved, where $\alpha \in [0,2]$ and E is the expectation operator. The spectral representation of such a stochastic process ξ is given by

$$\xi(t) = \int_\Lambda f(t,\lambda) dZ_\xi(\lambda),$$

where Λ is a linear Borel set, Z_ξ is the spectral measure of the process ξ and $f(z,\lambda)$ is in the entire functions space $[1,\infty)$, i.e., the kernel function $f(z,\lambda)$ is an exponentially bounded finite-type function, see [5] for appropriate definitions and stochastic sampling results. Nevertheless, our reconstruction procedure is applicable to such f, so we get a very powerful tool for the derivative plane sampling reconstruction of the Piranashvili α-processes reading at the von Neumann lattice $s\mathbb{Z}^2$. However it has to be pointed out that the stochastic process results of [6] are not scaled ($s \equiv 1$), since the Weierstrass $\sigma \in [2,\infty)$. So there is no need to introduce some scaling parameter $s > 0$ because the convergence is controlled by the behaviour of $\sigma(\cdot)$ in sampling formulæ similar to (3).

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