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# ON THE EXISTENCE OF SINGULAR SOLUTIONS

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**Abstract**. Sufficient conditions are given, under which the equation $y^{(n)} = f(t, y, y', \dots, y^{(l)})g(y^{(n-1)})$ has a singular solution $y\ [T, \tau) \to \mathbf{R}$, $\tau < \infty$ satisfying $\lim\limits_{t\to\tau_-} y^{(i)}(t) = c_i \in \mathbf{R}$, $i = 0, 1, \dots, l$ and $\lim\limits_{t\to\tau_-} |y^{(j)}(t)| = \infty$ for $j = l+1, \dots, n-1$ where $l \in \{0, 1, \dots, n-2\}$.

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## 1. Introduction

Consider the $n$-th order differential equation

$$y^{(n)} = f(t, y, y', \dots, y^{(l)})g(y^{(n-1)}), \tag{1}$$

where $n \geq 2$, $l \in \{0, 1, \dots, n-2\}$, $f \in C^0(\mathbf{R}_+ \times \mathbf{R}^{l+1})$, $g \in C^0(\mathbf{R})$, $\mathbf{R}_+ = [0, \infty)$, $\mathbf{R} = (-\infty, \infty)$ and there exists $\alpha \in \{-1, 1\}$ such that

$$\alpha f(t, x_1, \dots, x_{l+1})x_1 > 0 \quad \text{for} \quad x_1 \neq 0. \tag{2}$$

A solution $y$ defined on the interval $[T, \tau) \subset \mathbf{R}_+$ is called singular if $\tau < \infty$ and $y$ cannot be defined for $t = \tau$.

The problem of the existence of singular solutions satisfying the Cauchy initial-value problem and their asymptotic behaviour is thoroughly studied in [4] for the second order Emden–Fowler equation

$$y'' = r(t)|y|^\lambda \operatorname{sgn} y, \quad r(t) \geq 0. \tag{3}$$

In the common case for (1), the profound investigations are carried out in [5]. All these results concern the case $\alpha = 1$. For $\alpha = -1$, sufficient conditions are given in [2], under which singular solutions of

$$y^{(n)} = r(t)|y|^\lambda \operatorname{sgn} y, \quad n \geq 2, \quad r \leq 0,$$

exist.

Another problem concerning singular solutions is solved in [3] ($n = 2$, $f(t, x_1, \dots, x_l) \equiv r(t)|x_1|^\sigma \operatorname{sgn} x_1$, $g(x) = |x|^\lambda$) and in [1] in the case $l = n - 2$.

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Let $\tau \in (0,\infty)$. Sufficient and/or necessary conditions are given there, under which a singular solution $y$ exists with given asymptotic behaviour at the left-hand side point $\tau$ of the definition interval $c_i \in \mathbf{R}$,

$$\lim_{t\to\tau_-} y^{(i)}(t) = c_i \quad \text{for} \quad i=0,1,\dots,n-2, \quad \lim_{t\to\tau_-} |y^{(n-1)}(t)| = \infty. \tag{4}$$

In the present paper this result is generalized to the case in which we seek a singular solution $y$ satisfying the condition

$$\begin{gathered} \tau \in (0,\infty), \quad c_i \in \mathbf{R}; \quad \lim_{t\to\tau_-} y^{(i)}(t) = c_i, \quad i=0,1,\dots,l, \\ \lim_{t\to\tau_-} |y^{(j)}(t)| = \infty \quad \text{for} \quad j = l+1,\dots,n-1. \end{gathered} \tag{5}$$

Note that in [3] such solutions are called blackhole solutions (for $n=2$ and $l=0$).

Denote by $[[a]]$ the entire part of the number $a$.

## 2. Main Results

Let $y$ be a solution of (1) satisfying (5). Since, according to (2) $f(t,c_0,c_1,\dots,c_l) \neq 0$ if and only if $c_0 \neq 0$, we ought to divide our investigation into two cases $c_0 \neq 0$ and $c_0 = 0$ for which the results are different.

Let $c_0 \neq 0$. The following theorem gives a necessary condition for the existence of a solution of (1), (5).

**Theorem 1.** *Let $c_0 \neq 0$, $M \in (0,\infty)$, $K \in (0,1]$, $\lambda \le 2$ for $l = n-2$,*

$$\begin{gathered} \lambda \bar{\in} \left(1 + \frac{1}{n-l-1}, 1 + \frac{1}{n-l-2}\right] \quad \textit{for} \quad < n-2, \\ K|x|^\lambda \le g(x) \le |x|^\lambda \quad \textit{for} \quad |x| \ge M. \end{gathered} \tag{6}$$

*Then equation* (1) *has no singular solution $y$ satisfying* (5).

The next theorem shows that in the opposite case in (6) problem (1), (5) is solvable.

**Theorem 2.** *Let $\tau \in (0,\infty)$, $c_0 \neq 0$, $M \in (0,\infty)$, $\beta = \alpha \operatorname{sgn} c_0$, $\lambda > 2$ for $l = n-2$,*

$$1 + \frac{1}{n-l-1} < \lambda \le 1 + \frac{1}{n-l-2} \quad \textit{for} \quad l < n-2 \tag{7}$$

*and*

$$g(x) \ge |x|^\lambda \quad \textit{for} \quad \beta x \ge M. \tag{8}$$

*Then there exists a singular solution $y$ of* (1) *satisfying* (5) *which is defined in a left neighborhood of $\tau$.*

*If, moreover, $\varepsilon > 0$, $g(x) > 0$ for $\beta x \in (0,\varepsilon]$,*

$$l + \frac{1-\alpha}{2} \quad \textit{is odd}, \quad (-1)^i c_i c_0 \ge 0 \quad \textit{for} \quad i = 1,2,\dots,l, \tag{9}$$

*and*

$$\left|\int_0^{\beta\varepsilon} \frac{ds}{g(s)}\right| = \infty, \tag{10}$$

*then $y$ is defined on the interval $[0,\tau)$.*

**Corollary 1.** *Let $c_0 \neq 0$, $M \in (0,\infty)$ and*

$$g(x) = |x|^\lambda \quad \textit{for} \quad |x| \geq M.$$

*Then (1) has a singular solution $y$ satisfying (5) if and only if (7) is valid.*

**Corollary 2.** *Let $\lambda > 1 + \frac{1}{n-1}$ and $M \in \mathbf{R}_+$ be such that*

$$g(x) \geq x^\lambda \quad \textit{for} \quad x \geq M.$$

*Then (1) has a singular solution.*

*Remark.* For $\alpha = 1$ the conclusion of Corollary 2 is known, see, e.g., [6, Theorem 11.3]. For $\alpha = -1$ it generalizes Corollary 1 in [1].

The following two theorems solve the same problem in the case

$$\beta \in \{-1,1\}, \quad c_0 = 0, \quad (-1)^i \beta c_i \geq 0 \quad \text{for} \quad i = 1,2,\dots,l. \tag{11}$$

**Theorem 3.** *Let $\tau \in (0,\infty)$, $\sigma > 0$, $\varepsilon > 0$,$M \in (0,\infty)$, $\bar{M} \in (0,\infty)$,*

$$l - \frac{1-\alpha}{2} \quad \textit{be odd}, \tag{12}$$

$$2 + (n-2)\sigma < \lambda \quad \textit{for} \quad l = n-2,$$

$$1 + \frac{l\sigma + 1}{n-l-1} < \lambda \leq 1 + \frac{(l+1)\sigma + 1}{n-l-2} \quad \textit{for } l < n-2, \tag{13}$$

*(8) and (11) hold. Further, let*

$$|f(t,x_1,\dots,x_{l+1})| \geq \bar{M}|x_1|^\sigma \tag{14}$$

*for $t \in [0,\tau]$, $\beta x_1 \in [0,\varepsilon]$, $(-1)^j \beta x_{j+1} \in [(-1)^j \beta c_j, (-1)^j \beta c_j + \varepsilon]$, $j = 1,\dots,l$.*

*Then there exists a singular solution $y$ of (1) satisfying (5), which is defined in a left neighborhood of $\tau$.*

*If, moreover, $g(x) > 0$ for $\beta x \in (0,\varepsilon]$ and (10) holds, then $y$ is defined on the interval $[0,\tau)$.*

**Theorem 4.** *Let $\sigma > 0$, $c_i = 0$ for $i = 0,1,\dots,l$, $M \in (0,\infty)$, $\varepsilon > 0$, $\alpha \in \{-1,1\}$, $r \in C^0(\mathbf{R}_+)$, $\alpha r(t) > 0$ on $\mathbf{R}_+$, (12) hold and*

$$g(x) = |x|^\lambda \quad \textit{for} \quad |x| \geq M.$$

*Then the equation*

$$y^{(n)} = r(t)|y|^\sigma g\left(y^{(n-1)}\right) \operatorname{sgn} y \tag{15}$$

*has a solution $y$ satisfying (1), (5) if and only if (13) is valid.*

The following proposition shows that assumption (12) in Theorems 3 and 4 is important.

**Proposition.** *Let $c_i = 0$, $i = 0, 1, \dots, l$ and $l - \frac{\alpha-1}{2}$ be even. Let $g(x) \geq 0$ on $\mathbf{R}$. Then equation* (1) *has no solution satisfying* (1), (5).

In this paper the main assumptions are imposed on the function $g$ depending on $y^{(n-1)}$. But solutions of (1), (5) may exist for the equation

$$y^{(n)} = f(t, y, \dots, y^{(j)}), \quad j \in \{l+1, \dots, n-1\}, \tag{16}$$

too. From this we formulate an open problem.

**Open problem.** To study the existence of a solution satisfying (1), (5) of equation (16).

## 3. Lemmas and Proofs

We need the next two lemmas.

**Lemma 1.** *Let $[a, b] \subset \mathbf{R}_+$, $\phi \in C^0[a, b]$ and $\tilde{f} \in C^0([a, b] \times \mathbf{R}^n)$ be such that*

$$\tilde{f}(t, x_1, \dots, x_n)| \leq \phi(t), \quad t \in [a, b], \ \ x_i \in \mathbf{R}, \ \ i = 1, \dots, n.$$

*Then for arbitrary $\gamma_i \in \mathbf{R}$, $i = 0, 1, \dots, n-1$, the equation*

$$u^{(n)} = \tilde{f}(t, u, u', \dots, u^{(n-1)})$$

*has at least one solution satisfying the boundary value conditions*

$$\begin{aligned} u^{(i)}(b) &= \gamma_i \quad \text{for} \quad i = 0, 1, \dots, l+1; \\ u^{(j)}(a) &= \gamma_{j+1} \quad \text{for} \quad j = l+1, \dots, n-2. \end{aligned}$$

*Proof.* It follows, e.g., from [6, Lemma 10.1] since the homogeneous problem

$$u^{(n)} = 0, \quad u^{(i)}(b) = u^{(j)}(a) = 0 \quad \text{for} \quad i = 0, 1, \dots, l+1; \quad j = l+1, \dots, n-2,$$

has a trivial solution only. □

The following Kolmogorov–Horny type inequality is a very useful tool (see, e.g., the proof of Lemma 5.2 in [6]).

**Lemma 2.** *Let $[a, b] \subset \mathbf{R}_+$, $a < b$, $m \geq 2$ be an integer, $u \in C^m[a, b]$, and let $u^{(j)}$ have zero in the interval $[a, b]$ for $j = 1, \dots, m-1$. Then*

$$\rho_i \leq 2^{i(m-i)} \rho_0^{\frac{m-i}{m}} \rho_m^{\frac{i}{m}}, \quad i = 1, 2, \dots, m-1,$$

*where*

$$\rho_i = \max\{|u^{(i)}(t)| \ a \leq t \leq b\}, \quad i = 0, 1, \dots, m.$$

*Proof of Theorem* 1. Let for simplicity $c_0 > 0$ and $\alpha = 1$. Put $\lambda_1 = \frac{1}{\lambda - 1}$ and let $y\ [\tau_1, \tau) \to \mathbf{R}$ be a solution of (1), (5). Then, according to (1) and (2) $\lim_{t\to\tau_-} y^{(j)}(t) = \infty$ for $j = l+1, \dots, n$. Let $T \in [\tau_1, \tau)$ be such that

$$y(t) \ge \frac{c_0}{2} \quad \text{on} \quad [T, \tau), \quad y^{(j)}(T) \ge 0, \quad j = l+1, l+2, \dots, n-2, \\ y^{(n-1)}(T) \ge M. \tag{17}$$

By this and the boundedness of $y^{(i)}(t)$, $i = 0, 1, \dots, l$, we obtain from (1)

$$y^{(n)}(t) \le M_1 \left[y^{(n-1)}(t)\right]^{\lambda}, \quad t \in [T, \tau),$$

where $M_1$ is a suitable constant. Let $\lambda \le 1 + \frac{1}{n-l-1}$. Hence the integration on the interval $[t, \tau)$ yields

$$y^{(n-1)}(t) \ge [(\lambda - 1) M_1 (\tau - t)]^{-\lambda_1}, \quad t \in [T, \tau). \tag{18}$$

Hence $n - l - 1 \le \lambda_1$, and the Taylor Series Theorem, (17) and (18) yield

$$\begin{aligned} c_l = y^{(l)}(\tau) &= \sum_{i=0}^{n-l-2} \frac{y^{(l+i)}(T)}{i!}(\tau - T)^i + \int_T^\tau \frac{(\tau - s)^{n-l-2}}{(n-l-2)!} y^{(n-1)}(s)\, ds \\ &\ge y^{(l)}(T) + M_2 \int_T^\tau (\tau - s)^{n-l-2-\lambda_1}\, ds = \infty, \end{aligned}$$

where

$$M_2 = \frac{[(\lambda - 1) M_1]^{-\lambda_1}}{(n-l-2)!}.$$

Hence a solution $y$ satisfying (1), (5) does not exist in this case.

Let $l < n-2$ and $\lambda > 1 + \frac{1}{n-l-2}$. Hence $n - l - 2 - \lambda_1 > 0$. Then, similarly to (18), we can prove that

$$y^{(n-1)}(t) \le [(\lambda - 1) M_3 (\tau - t)]^{-\lambda_1}, \quad t \in [T, \tau), \tag{19}$$

where $M_3 = \min\left\{K f\left(t, y(t), \dots, y^{(l)}(t)\right)\ T \le t \le \tau\right\} > 0$.

From this the Taylor Series Theorem yields

$$\begin{aligned} \infty = y^{(l+1)}(\tau) &= \sum_{i=0}^{n-l-3} \frac{y^{(l+i+1)}}{i!}(\tau - T)^i + \int_T^\tau \frac{(\tau - s)^{n-l-3}}{(n-l-3)!} y^{(n-1)}(s)\, ds \\ &\le M_4 + M_5 \int_T^\tau (\tau - s)^{n-l-3-\lambda_1}\, ds < \infty \end{aligned}$$

as $n - l - 3 - \lambda_1 > -1$; $M_4$ and $M_5$ are positive constants. The contradiction obtained proves that a singular solution does not exist. $\square$

*Proof of Theorem* 2. For $l = n-2$ we proved the statement in [1]. Thus let $l < n-2$ and, first, we prove the result for (7) with $\lambda \ne 1 + \frac{1}{n-l-2}$.

We prove the statement for $\alpha = 1$ and $c_0 > 0$; thus $\beta = 1$. For the other cases the proof is similar.

Let

$$N > 2\max\left(c_0, |c_1|, \dots, |c_l|\right), \quad k_0 > [[2M]]\,, \tag{20}$$

$D = \{[x_1, \dots, x_{l+1}]\ \frac{c_0}{2} \le x_1 \le c_0,\ |x_j| \le N \quad \text{for} \quad j = 2, \dots, l+1\}$,
$M_1 = \min\{f(t, x_1, \dots, x_{l+1})\ t \in [0, \tau],\ [x_1, \dots, x_{l+1}] \in D\} > 0$,
$M_2 = \max\{f(t, x_1, \dots, x_{l+1})\ t \in [0, \tau],\ [x_1, \dots, x_{l+1}] \in D\}$,
$M_3 = 2[(\lambda - 1)M_1]^{-\lambda_1}$, $\lambda_1 = \frac{1}{\lambda - 1}$, $\bar\lambda = n - l - 1 - \lambda_1 > 0$,
$N_1 = 2^{n-l-2} M_3^{\frac{1}{n-l-1}} \left[1 - \frac{\lambda_1}{n-l-1}\right]^{-1}$.

Further, let $T \in [0, \tau)$ be such that

$$\tau - T < \left(\frac{M_3}{M}\right)^{\lambda - 1}, \quad (\tau - T)^{n-l-2} < \frac{k_0}{M}, \tag{21}$$

$$\tau - T < \frac{1}{M_2} \int\limits_M^{2M} \frac{ds}{g(s)}, \tag{22}$$

$$(\tau - T)^{\bar\lambda} \le (2N_1)^{-n+l+1} N, \tag{23}$$

$$\sum_{r=i+1}^{l-1} |c_r| \frac{(\tau - T)^{r-i}}{(r-i)!} + N \frac{(\tau - T)^{l-i}}{(l-i)!} \le \frac{N}{2}, \quad i = 0, 1, \dots, l-1, \tag{24}$$

$$\sum_{r=1}^{l-1} |c_r| \frac{(\tau - T)^r}{r!} + N \frac{(\tau - T)^l}{l!} \le \frac{c_0}{2}. \tag{25}$$

Denote $J = [T, \tau)$ and note that due to $\bar\lambda > 0$, $T$ exists.

Consider the auxilliary two-point boundary-value problem $k \in \{k_0, k_0 + 1, \dots\}$,

$$\begin{gathered} y^{(n)} = f\left(t, \Phi_0(y), \Phi_1(y'), \dots, \Phi_1(y^{(l)})\right) g\left(\Phi_2(t, y^{(n-1)})\right), \\ y^{(i)}(\tau) = c_i, \quad i = 0, 1, \dots, l; \quad y^{(l+1)}(\tau) = k; \\ y^{(j)}(T) = 0, \quad j = l+1, \dots, n-2, \quad t \in J, \end{gathered} \tag{26}$$

where

$$\Phi_0(s) = \begin{cases} s & \text{for } \frac{c_0}{2} \le s \le N, \\ N & \text{for } s > N, \\ \frac{c_0}{2} & \text{for } s < \frac{c_0}{2}, \end{cases} \tag{27}$$

$$\Phi_1(s) = \begin{cases} s & \text{for } |s| \le N, \\ N \operatorname{sgn} s & \text{for } |s| > N \end{cases} \tag{28}$$

and

$$\Phi_2(t,s) = \begin{cases} s & \text{for } M \le s \le M_3(\tau-t)^{-\lambda_1}, \\ M_3(\tau-t)^{-\frac{1}{\lambda-1}} & \text{for } s > M_3(\tau-t)^{-\lambda_1}, \\ M & \text{for } s < M. \end{cases} \tag{29}$$

Note that due to (21) $\Phi_2$ is well defined.

To prove the existence of a solution of (26), let us consider the sequence of boundary value problems

$$\begin{gathered} \bar{m}_0 > \frac{1}{\tau - t}, \quad m \in \{\bar{m}_0, \bar{m}_0+1, \dots\}, \quad \tau_m = \tau - \frac{1}{m}, \\ z^{(n)} = F(t, z, z', \dots, z^{(l)}, z^{(n-1)}), \\ z^{(i)}(\tau_m) = c_i, \quad i = 0, 1, \dots, l, \quad z^{(l+1)}(\tau_m) = k, \\ z^{(j)}(T) = 0, \quad j = l+1, \dots, n-2, \end{gathered} \tag{30}$$

where

$$F(t, x_1, \dots, x_{l+2}) = f\left(t, \Phi_0(x_1), \Phi_1(x_2), \dots, \Phi_1(x_{l+1})\right) g\left(\Phi_2(t, x_{l+2})\right).$$

Since

$$|F(t, x_1, \dots, x_{l+2})| \le M_2 \max_{T \le \bar{t} \le t} \max_{M \le s \le M_3(\tau - \bar{t})^{-\lambda_1}} g(\bar{t}, s), \quad t \in [T, \tau_m],$$

(30) has a solution $z_m$ according to Lemma 1.

Further, we estimate $z_m^{(n-1)}$. Let $J_m = [T, \tau_m]$. First we prove that

$$z_m^{(n-1)}(t) < M_3(\tau - t)^{-\lambda_1}, \quad t \in [T, \tau_m), \tag{31}$$

for large $m$, say $m \ge \bar{m}_0$. If (31) is not valid, then either

(i) there exists $t_1 \in [T, \tau_m)$ such that

$$z_m^{(n-1)}(t_1) = M_3(\tau - t_1)^{-\lambda_1} \quad \text{and} \quad z_m^{(n-1)}(\tau_m) \le M_3(\tau - \tau_m)^{-\lambda_1} \tag{32}$$

or

(ii)

$$z_m^{(n-1)}(t) > M_3(\tau - t)^{-\lambda_1} \tag{33}$$

in a left neighborhood of $t = \tau_m$.

Let (i) be valid. As (26)–(30) yield $z_m^{(n)}(t) > 0$ and $z_m^{(n-1)}$ is increasing on $J_m$, it follows from (32) and (21) that

$$M \le z_m^{(n-1)}(t), \quad t \in [t_1, \tau_m]. \tag{34}$$

Hence

$$z_m^{(n)}(t) \ge M_1 \left(z_m^{(n-1)}(t)\right)^\lambda, \quad t \in J_m,$$

and the integration and (32) yield

$$\frac{\tau_m - t_1}{M_3^{\lambda-1}} > \frac{1}{[z_m^{(n-1)}(t_1)]^{\lambda-1}} - \frac{1}{[z_m^{(n-1)}(\tau_m)]^{\lambda-1}} \ge M_1(\lambda - 1)(\tau_m - t_1),$$

which contradicts the definition of $M_3$.

Let (33) be valid and let $t_1$, $T \le t_1 < \tau_m$ be such that $z_m^{(n-1)}(t) > M_3(\tau - t)^{-\lambda_1}$ on the interval $[t_1, \tau_m)$. Then the Taylor Series Theorem yields

$$\begin{aligned} k = z_m^{(l+1)}(\tau_m) &\ge \int_{t_1}^{\tau_m} \frac{(\tau_m - s)^{n-l-3}}{(n-l-3)!} z_m^{(n-1)}(s)\, ds \\ &\ge \frac{M_3}{(n-l-3)!} \int_{t_1}^{\tau_m} (\tau_m - s)^{n-l-3} (\tau - s)^{-\lambda_1}\, ds \\ &\ge \frac{-M_3}{(n-l-2)!} \int_{t_1}^{\tau_m} (\tau - s)^{n-l-1-\lambda_1} \frac{d}{ds}\left( (1 - \frac{1}{m(\tau - s)})^{n-l-2} \right) ds \\ &\ge \frac{M_3}{(n-l-2)!} \left(\frac{1}{m}\right)^{n-l-2-\lambda_1} \left(1 - \frac{1}{m(\tau - t_1)}\right)^{n-l-2} \to \infty \quad \text{for } m \to \infty. \end{aligned}$$

Hence (31) holds.

Further, we prove indirectly the following estimation from bellow

$$M < z_m^{(n-1)}(t), \quad t \in J_m. \tag{35}$$

Note that $z_m^{(n-1)}$ is increasing, and first we prove that (35) is valid for $t = \tau_m$. Let, conversely, $z_m^{(n-1)}(\tau_m) \le M$. Then

$$k_0 \le k = z_m^{(l+1)}(\tau_m) = \int_T^{\tau_m} \frac{(\tau_m - s)^{n-l-3}}{(n-l-3)!} z_m^{(n-1)}(s)\, ds \le \frac{M}{(n-l-2)!}(\tau_m - T)^{n-l-2},$$

which contradicts (21). Thus (35) holds. Let $T_1 \in [T, \tau_m)$ exist such that $z_m^{(n-1)}(T_1) = M$. Then $M < z_m^{(n-1)}(t)$ on $J_m$ and

$$z_m^{(n)}(t) \le M_2 g\left(z_m^{(n-1)}(t)\right), \quad t \in J_m.$$

From this, by the integration, we have

$$\int_M^{2M} \frac{ds}{g(s)} \le \int_M^{k} \frac{ds}{g(s)} \le M_2(\tau_m - T) < M_2(\tau - T).$$

The contradiction with (22) proves that (35) is valid and according to (30)

$$z_m^{(j)}(t) \ge 0 \quad \text{on} \quad J_m, \quad j = l+1, l+2, \dots, n. \tag{36}$$

Denote $\rho = \max_{t \in J_m} |z_m^{(l)}(t)|$. Then, by virtue of (31), (36) and Lemma 2 with $[a, b] = [\tau, t]$, $u = z_m^{(l)}$ and $m = n - l - 1$, we have

$$0 \le z_m^{(i+1)}(t) \le 2^{n-l-2} \rho^{\frac{n-l-2}{n-l-1}} \left(z_m^{n-1}(t)\right)^{\frac{1}{n-l-1}} \le 2^{n-l-2} M_3^{\frac{1}{n-l-1}} \rho^{\frac{n-l-2}{n-l-1}} (\tau - t)^{-\frac{\lambda_1}{n-l-1}},$$

and hence, as $\frac{\lambda_1}{n-l-1} \in (0, 1)$, the integration on $J_m$ yields

$$0 \le c_l - z_m^{(l)}(T) \le N_1 (\tau - T)^{1 - \frac{\lambda_1}{n-l-1}} \rho^{\frac{n-l-2}{n-l-1}}. \tag{37}$$

Since $z_m^{(l)}$ is increasing on $J_m$, either $z_m^{(l)}(T) \ge -|c_l|$ and $\rho = |c_l|$ or $z_m^{(l)}(T) < -|c_l|$ and (23) and (37) yield

$$c_l + \rho \le N_1(\tau - T)^{1-\frac{\lambda_1}{n-l-1}} \rho^{\frac{n-l-2}{n-l-1}} \le \frac{1}{2} N^{\frac{1}{n-l-1}} \rho^{\frac{n-l-2}{n-l-1}}.$$

Thus $\rho \le 2|c_l|$ or $\frac{\rho}{2} \le c_l + \rho \le \frac{1}{2} N^{\frac{1}{n-l-1}} \rho^{\frac{n-l-2}{n-l-1}}$ and according to (20) in all cases we have

$$|z_m^{(l)}(t)| \le N, \quad t \in J_m. \tag{38}$$

From this, (31), (36) and Lemma 2 with $[a,b] = [T,t]$, $u = z_m^{(l)}$ and $m = n-l-1$ we have

$$\begin{gathered} |z_m^{(j)}(t)| \le 2^{(j-l)(n-j-1)} N^{\frac{n-j-1}{n-l-1}} M_3^{\frac{j-l}{n-l-1}} (\tau - t)^{\frac{j-l}{n-l-1}}, \\ t \in J_m, \quad j = l+1, \dots, n-2. \end{gathered} \tag{39}$$

Further, (20), (24), (25), (38) and the Taylor Series Theorem yield

$$c_i - z_m^{(i)}(t) = \sum_{r=i+1}^{l-1} \frac{c_r (t - \tau_m)^{r-i}}{(r-i)!} + \int_{\tau_m}^{t} \frac{(t-s)^{l-i-1}}{(l-i-1)!} z_m^{(l)}(s)\, ds,$$

$$\begin{gathered} |z_m^{(i)}(t)| \le \sum_{r=i+1}^{l-1} \frac{c_r|}{(r-i)!} (\tau - T)^{r-i} + \frac{N}{(l-i)!} (\tau - T)^{l-i} + |c_i| \le N, \\ i = 0, 1, \dots, l-1, \quad t \in J_m, \end{gathered} \tag{40}$$

$$|z_m(t)| \ge c_0 - \sum_{r=1}^{l-1} |c_r| \frac{(\tau - T)^r}{r!} - \frac{N}{l!} (\tau - T)^l \ge \frac{c_0}{2}, \quad t \in J_m. \tag{41}$$

Estimations (38), (39) and (40) show that $\{z_m^{(j)}\}$, $j = 0, 1, \dots, n-1$, $m = m_0, m_0 + 1, \dots,$ are uniformly bounded with respect to $j$ and $m$ and hence according to the Arzelá–Ascoli Theorem (see [6], Lemma 10.2) there exists a subsequence that converges uniformly to the solution $y_k$ of (26). At the same time, it is clear that (see (41), too)

$$\frac{c_0}{2} \le y_k(t) \le N, \; |y_k^{(i)}(t)| \le N, \quad i = 1, 2, \dots, l, \tag{42}$$

$$\begin{gathered} |y_k^{(j)}(t)| \le 2^{(j-l)(n-j-1)} N^{\frac{n-j-1}{n-l-1}} M_3^{\frac{j-l}{n-l-1}} (\tau - t)^{\frac{j-l}{n-l-1}}, \\ t \in J, \quad j = l+1, \dots, n-1. \end{gathered} \tag{43}$$

Moreover, (31), (35), (42) yield

$$\begin{gathered} \Phi_0\left(y_k(t)\right) = y_k(t), \quad \Phi_1\left(y_k^{(i)}(t)\right) = y_k^{(i)}(t) \quad \text{for} \quad i = 1, 2, \dots, l, \\ \Phi_2\left(t, y_k^{(n-1)}(t)\right) = y_k^{(n-1)}(t), \quad t \in J, \end{gathered}$$

and hence $y_k(t)$ is a solution of (1) satisfying

$$y_k^{(i)}(\tau) = c_i, \quad i = 0, 1, \dots, l; \quad y_k^{(l+1)}(\tau) = k.$$

As estimations (42) a (43) do not depend on $k$, $i$ and $j$, the Arzelá–Ascoli Theorem implies the existence of a subsequence of $\{y_k(\tau)\}_{k_0}^{\infty}$ that converges uniformly to the solution of (1) satisfying

$$y^{(j)}(T) = 0, \quad j = l+1, \dots, n-2, \tag{44}$$

$$\lim_{t\to\tau_-} y^{(i)}(t) = c_i, \quad i = 0, 1, \dots, l, \quad \lim_{t\to\tau_-} y^{(l)}(t) = \infty. \tag{45}$$

Let $\lambda = 1 + \frac{1}{n-l-2}$. Then there exists a sequence of $\{\lambda_s\}_1^{\infty}$ such that $\lambda_s$ satisfies (7) and $\lim_{s\to\infty} \lambda_s = 1 + \frac{1}{n-l-2}$. Denote by $y_s$ a solution of (1), (5) with $\lambda = \lambda_s$. It follows from (21)–(25) that there exists $T \in [0, \tau)$ such that $y_s$, $s \in \{1, 2, \dots\}$ is defined on the interval $[T, \tau)$. At the same time, since (38)–(41) do not depend on $\lambda$, there exists $\Phi$ such that

$$|y_s^{(i)}(t) \le \Phi(t), \quad t \in [T, \tau), \quad i = 0, 1, \dots, n-1, \quad s = 1, 2, \dots .$$

Hence, according to the Arzelá–Ascoli Theorem, there exists a subsequence of $\{y_s\}_1^{\infty}$ that converges uniformly to a solution of (1), satisfying (5).

Let (9) and (10) be valid. Let $y$ be defined on the interval $(\bar\tau, \tau) \subset [0, \tau)$ and not be extendable to $t = \bar\tau$. Then

$$\limsup_{t\to\bar\tau^+} |y^{(n-1)}(t)| = \infty. \tag{46}$$

First we prove that

$$y^{(n-1)}(t) > 0 \quad \text{on} \quad (\bar\tau, \tau). \tag{47}$$

Suppose that there exists $\tau_1 \in (\bar\tau, \tau)$ such that $y^{(n-1)}(\tau_1) = 0$ and $y^{(n-1)}(t) > 0$ on the interval $(\tau_1, \tau)$. As $\tau_1 < T$, it follows from this and (45) that $y^{(j)}$, $j = 0, 1, \dots, l$, are bounded on the interval $(\tau_1, \tau)$. Let $\tau_2 \in (\tau_1, \tau)$ be such that $y^{(n-1)}(\tau_2) = \varepsilon$. Then by the integration of (1) and (10)

$$\infty = \int_0^{\varepsilon} \frac{ds}{g(s)} = \int_{\tau_1}^{\tau_2} f\left(t, y(t), \dots, y^{(l)}(t)\right) dt < \infty.$$

Hence (47) holds. As $\tau_1 < T$, it follows from (9), (44) and (45) that $y(t) > 0$ on the interval $(\bar\tau, \tau)$ ($y^{(i)}$, $i = 0, 1, \dots, l$ change their signs). Thus (1) yields $y^{(n)}(t) > 0$ on the interval $(\bar\tau, \tau)$, which, together with (47), contradicts (46). Hence $y$ is defined at $t = \bar\tau$ and $\bar\tau = 0$. □

*Proof of Theorem* 3. Let $\alpha = 1$ and $\beta = 1$. The proof is similar to the that of Theorem 2. Since (11) and (12) are valid, we can restrict our investigation to the case

$$D = \{[x_1, \dots, x_{l+1}]\, 0 \le x_1 \le \varepsilon,\, (-1)^j x_{j+1} \in [(-1)^j c_j, (-1)^j c_j + \varepsilon]\}.$$

The only problem is that due to $c_0 = 0$, we have $M_1 = 0$ and $M_3 = \infty$, where $M_1$ and $M_3$ are given as in the proof of Theorem 2. Thus (31) gives us no information and it must be proved in a different way. Hence we prove that (31) is valid with the new values of $\lambda_1$ and $M_3$ given by

$$\lambda_1 = \frac{(n-1)\sigma+1}{\lambda+\sigma-1}, \quad M_3 = \left[\frac{2(n-1)\sigma+2}{(\lambda+\sigma-1)M_1}\right]^{\frac{1}{\lambda+\sigma-1}}$$

where $M_1 = \frac{\bar{M}}{[l!(n-l-2)!(n-1)]^\sigma}$. Note that, similarly to the proof of Theorem 2, $z_m^{(n-1)}$ is positively increasing on the interval $J_m = [T, \tau_m]$, $\tau_m = \tau - \frac{1}{m}$. Note that (13) yields

$$n-l-2 \le \lambda_1 < n-l-1.$$

If (31) is not valid, then either (32) or (33) holds.

Let (32) be valid. It follows similarly to (34) that

$$M \le z_m^{(n-1)}(t), \quad t \in [t_1, \tau_m]. \tag{48}$$

Now we will estimate $z_m$. According to (48) and the Taylor Series Theorem we have

$$\begin{aligned} z_m^{(l+1)}(s) &= \sum_{r=0}^{n-l-3} \frac{z_m^{(l+1+r)}(t)}{r!}(s-t)^r + \int_t^s \frac{(s-\sigma)^{n-l-3}}{(n-l-3)!} z_m^{(n-1)}(\sigma)\, d\sigma \\ &\ge z_m^{(n-1)}(t) \frac{(s-t)^{n-l-2}}{(n-l-2)!}, \quad t \le s \le \tau_m. \end{aligned} \tag{49}$$

Similarly, the Taylor Series Theorem, (11), (12) and (49) yield

$$\begin{aligned} z_m(t) &\ge \int_{\tau_m}^t \frac{(t-s)^l}{l!} z_m^{(l+1)}(s)\, ds \ge z_m^{(n-1)}(t) \int_{\tau_m}^t \frac{(s-t)^{n-2}(-1)^l}{l!(n-l-2)!}\, ds \\ &= \frac{(\tau_m-t)^{n-1}}{l!(n-l-2)!(n-1)} z_m^{(n-1)}(t), \quad t \in [t_1, \tau_m]. \end{aligned}$$

From this, (8), (14), (48) and (49)

$$z_m^{(n)}(t) \ge \bar{M} z_m^\sigma(t) \left(z_m^{(n-1)}(t)\right)^\lambda \ge M_1(\tau_m - t)^{(n-1)\sigma} \left(z_m^{(n-1)}(t)\right)^{\lambda+\sigma}.$$

The integration on the interval $[t_1, \tau_m]$ yields

$$\begin{aligned} &\frac{2(\tau_m-t_1)^{(n-1)\sigma+1}}{M_3^{\lambda+\sigma-1}} \ge \frac{(\tau-t_1)^{(n-1)\sigma+1} - m^{-(n-1)\sigma+1}}{M_3^{\lambda+\sigma-1}} \ge \\ &\ge \frac{1}{\left[z_m^{(n-1)}(t_1)\right]^{\lambda+\sigma-1}} - \frac{1}{\left[z_m^{(n-1)}(\tau_m)\right]^{\lambda+\sigma-1}} \ge \frac{M_1(\lambda+\sigma-1)}{(n-1)\sigma+1}(\tau_m-t_1)^{(n-1)\sigma+1} \end{aligned}$$

for large $m$. The contradiction obtained with the definition of $M_3$, shows that (32) does not hold. The fact that (33) is impossible can be proved similarly to the same case in the proof of Theorem 2. □

*Proof of Theorem* 4. (i) Let $y\,[T,\tau) \to \mathbf{R}$ be a solution of (1), (5) with $\alpha = 1$ and, for simplicity, $y^{(n-1)}(t) \geq M$ on the interval $[T,\tau)$. Put $\lambda_1 = \frac{(n-1)\sigma+1}{\lambda+\sigma-1}$ and $M_1$ as in the proof of Theorem 3.

Let $\lambda \geq 1 + \frac{(l+1)\sigma+1}{n-l-2}$ for $l < n-2$; hence $n-l-2-\lambda_1 \geq 0$. We can prove similarly to (44) – (46) that

$$y^{(n)}(t) \geq M_1(\tau-t)^{(n-1)\sigma}[y^{(n-1)}(t)]^{\lambda+\sigma}, \quad t \in [T,\tau).$$

From this and by the integration we obtain an estimation from above of $y^{(n-1)}$ similar to (19) and the proof is similar to the second part of the proof of Theorem 1.

Let $\lambda < 1 + \frac{l\sigma+1}{n-l-1}$; hence $n-l-1-\lambda_1 < 0$. Then

$$y(t) = \int_\tau^t \frac{(t-s)^l}{l!} y^{(l+1)}(s)\,ds \leq \frac{|y^{(l)}(t)|}{l!}(\tau-t)^l, \quad t \in [T,\tau).$$

From this

$$y^{(n)}(t) \leq M_2 y^\sigma(t)\left[y^{(n-1)}(t)\right]^\lambda \leq M_2(\tau-t)^{l\sigma}\left[y^{(n-1)}(t)\right]^\lambda,$$

and the integration on the interval $[t,\tau)$ yields

$$y^{(n-1)}(t) \geq M_3(\tau-t)^{-\frac{l\sigma+1}{\lambda-1}},$$

where

$$M_2 = \max_{t\in[0,\tau]} r(t), \quad M_3 = \left[\frac{M_2(\lambda-1)}{l\sigma+1}\right]^{-\frac{1}{\lambda-1}}.$$

The proof is similar to the first part of the proof of Theorem 1, only in (18) we take $\lambda_1 = \frac{l\sigma+1}{\lambda-1}$.

(ii) The existence problem is solved by Theorem 3. □

*Proof of Proposition.* Let $y\,[T,\tau) \to \mathbf{R}$ be a solution of (1) and (5) with $\alpha = 1$; hence $l$ is even. Let $\lim_{t\to\tau^-} y^{(n-1)}(t) = \infty$. Then $y^{(l)} < 0$ in a left neighborhood $I$ of $\tau$. From this and from $l$ being even we can conclude that $y < 0$ and $y^{(n)} \leq 0$ on $I$. The contradiction to $\lim_{t\to\tau^-} y^{(n-1)} = \infty$ proves the statement. Other possible cases can be proved similarly. □

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## References

1. M. Bartušek, On Existence of singular solutions of $n$-th order differential equations. *Arch. Math. (Brno)* **36**(2000), 395–404.
2. T. A. Chanturia, On existence of singular and unbounded oscillatory solutions of differential equations of Emden–Fowler type. *Diffrentsial'nye Uraveniya* **28**(1992), 1009–1022.
3. J. Jaroš and T. Kusano, On black hole solutions of second order differential equations with a singularity in the differential operator. *Funk. Ekv.* (*to appear*).
4. I. T. Kiguradze, Asymptotic properties of solutions of a nonlinear differential equation of Emden–Fowler type. *Izv. Akad. Nauk SSSR, Ser. Mat.* **29**(1965), 965–986.
5. I. T. Kiguradze, Some singular boundary-value problems for ordinary differential equations. (Ruissian) *Tbilisi University Press, Tbilisi,* 1975.
6. I. T. Kiguradze and T. Chanturia, Asymptotic properties of solution of nonautonomous ordinary differential equations. *Kluwer Academic Publishers, Dordrecht,* 1993.

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