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# CENTRAL SERIES FOR GROUPS WITH ACTION AND LEIBNIZ ALGEBRAS

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**Abstract.** The notion of central series for groups with action on itself is introduced. An analogue of Witt's construction is given for such groups. A certain condition is found for the action and the corresponding category is defined. It is proved that the above construction defines a functor from this category to the category of Lie–Leibniz algebras and in particular to Leibniz algebras; also the restriction of this functor on the category of groups leads us to Lie algebras and gives the result of Witt.

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## Introduction

The well-known construction of Witt defines a functor from the category of groups to the category of Lie algebras [6], [5]. The aim of this paper is to define a category and to give an analogue of Witt's construction for its objects, which will lead us to the category of Leibniz algebras. This problem was stated by J.-L. Loday; later an analogous question for the possibly defined partial Leibniz algebras was proposed, which was inspired by the work of Baues and Conduché [1]. Since the main interest lies in the absolute case, the author decided to begin with this one.

In Section 1 we define the category of groups with action on itself $\mathbb{Gr}^\bullet$, the category of abelian groups with action on itself $\mathbb{Ab}^\bullet$ and the category of groups with bracket operation $\mathbb{Gr}^{[\,]}$. This kind of groups are $\Omega$-groups in the sense of [2]. We construct adjoint pairs of functors relating categories $\mathbb{Gr}^\bullet$, $\mathbb{Ab}^\bullet$, $\mathbb{Gr}^{[\,]}$, $\mathbb{Gr}$. In Section 2 we define ideals and commutators for the objects of $\mathbb{Gr}^\bullet$ (similarly for $\mathbb{Gr}^{[\,]}$) and show that these notions are equivalent to the special case of the known definitions for $\Omega$-groups [2]. In Section 3 we define central series of groups with action on itself and a category of Lie–Leibniz algebras $\mathbb{LL}$. We consider the category of groups with action on itself satisfying a certain condition $\mathbb{Gr}^c$. We give an analogue of Witt's construction [6] and prove that it defines a functor $LL : \mathbb{Gr}^c \longrightarrow \mathbb{LL}$, in particular this gives a functor $\mathbb{Gr}^c \longrightarrow \mathbb{Leibniz}$. In a similar way one can construct a functor $\mathbb{Ab}^c \longrightarrow \mathbb{Leibniz}$, which is actually the restriction of $LL$ on $\mathbb{Ab}^c$. The functorial relations with the classical situation ($\mathbb{Gr} \longrightarrow \mathbb{Lie}$) is considered, namely by the restriction of $LL$ on $\mathbb{Gr}$ we obtain the result of Witt [6], [5].

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## 1. Groups with Action on Itself

Let $G$ be a group which acts on itself from the right side; i.e. we have a map $\varepsilon : G \times G \longrightarrow G$ with

$$\begin{aligned} &\varepsilon(g, g' + g'') = \varepsilon(\varepsilon(g, g'), g''), \\ &\varepsilon(g, 0) = g, \\ &\varepsilon(g' + g'', g) = \varepsilon(g', g) + \varepsilon(g'', g), \\ &\varepsilon(0, g) = 0, \end{aligned} \tag{1.1}$$

for $g, g', g'' \in G$. Denote $\varepsilon(g, h) = g^h$, for $g, h \in G$. We denote the group operation additively, nevertheless the group is not commutative in general. If $(G', \varepsilon')$ is another group with action, then a homomorphism $(G, \varepsilon) \longrightarrow (G', \varepsilon')$ is a group homomorphism $\varphi : G \longrightarrow G'$ for which the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \varepsilon\ } & G \\ {\scriptstyle (\varphi,\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ G' \times G' & \xrightarrow{\ \varepsilon'\ } & G' \end{array}$$

commutes. In other words, we have

$$\varphi(g^h) = \varphi(g)^{\varphi(h)}, \quad g, h \in G.$$

If we consider an action as a group homomorphism $G \xrightarrow{\nu} \operatorname{Aut} G$, then a homomorphism between two groups with action means the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\qquad \nu \qquad} & \operatorname{Aut} G \subset \operatorname{Hom}(G, G) \\ \Big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}(G,\varphi)} \\ {\scriptstyle \varphi} & & \operatorname{Hom}(G, G') \\ \Big\downarrow & & \big\uparrow{\scriptstyle \operatorname{Hom}(\varphi,G')} \\ G' & \xrightarrow{\qquad \nu' \qquad} & \operatorname{Aut} G' \subset \operatorname{Hom}(G', G') \end{array}$$

so that $\varphi \cdot (\nu(h)) = \nu'(\varphi(h)) \cdot \varphi$, $h \in G$.

Recall [2] that an $\Omega$-group is a group with a system of $n$-ary algebraic operations $\Omega$ $(n \geq 1)$, which satisfies the condition

$$00 \cdots 0\omega = 0, \tag{1.2}$$

where 0 is the identity element of $G$, and 0 on the left side occurs $n$ times if $\omega$ is an $n$-ary operation. In special cases $\Omega$-groups give groups, rings and groups with action on itself. In the latter case $\Omega$ consists of one binary operation, an action; or $\Omega$ consists of only unary operations, elements of $G$, and this operation is an action again. In both cases the condition (1.2) is satisfied. We shall denote the category of groups with action on itself by $\mathbb{Gr}^\bullet$. Let $\mathbb{Ab}^\bullet$ denote the category of abelian groups with action on itself. We have functors

$$\mathbb{Ab}^{\bullet} \;\; \underset{A}{\overset{E}{\rightleftarrows}} \;\; \mathbb{Gr}^{\bullet} \;\; \begin{array}{c} \xrightarrow{Q_1} \\ \xleftarrow{T} \\ \xrightarrow{Q_2} \\ \xleftarrow{C} \end{array} \;\; \mathbb{Gr},$$

where $Q_1(G)$, for $G \in \mathbb{Gr}^{\bullet}$, is the greatest quotient group of $G$ which makes the action trivial; $Q_2(G)$ is a quotient of $G$ by the equivalence relation generated by the relation $g^h \sim -h+g+h$, $g, h \in G$; $A$ is the abelianization functor, thus $A(G) = G/(G,G)$, where $(G,G)$ is the ideal of $G$ generated by the commutator normal subgroup of $G$ (for the definition of an ideal see Section 2). $A(G)$ has the induced operation of action on itself. Each group can be considered as a group with the trivial action or with the action by conjugation, these give functors $T$ and $C$, respectively. Every object of $\mathbb{Ab}^{\bullet}$ can be considered as an object of $\mathbb{Gr}^{\bullet}$; this functor is denoted by $E$. It is easy to see that the functors $Q_1, Q_2$ and $A$ are left adjoints to the functors $T, C$ and $E$ respectively. Let $G \in \mathbb{Gr}^{\bullet}$. Define the operation of square brackets $[\,,\,] : G \times G \longrightarrow G$ on $G$ by

$$[g,h] = -g + g^h, \quad g, h \in G.$$

**Proposition 1.1.** *For the operation* $[\,,\,]$ *we have the following identities:*
(i) $[g, h_1 + h_2] = [g, h_1] + [g + [g, h_1], h_2]$;
(ii) $[g + g', h] = -g' + [g, h] + g' + [g', h]$;
(iii) $[g, 0] = [0, g] = 0$.

*Proof.* These identities follow directly from (1.1). □

**Corollary 1.2.** *For* $g, h \in G$

$$[g^h, -h] = -[g,h];$$
$$[-g, h] = g - [g,h] - g.$$

Denote by $\mathbb{Gr}^{[\,]}$ the category of groups with an additional bracket operation $[\,,\,]$ satisfying the conditions (i)–(iii) of Proposition 1.1; morphisms of $\mathbb{Gr}^{[\,]}$ are group homomorphisms preserving the bracket operation. We shall denote the objects of $\mathbb{Gr}^{[\,]}$ by $G^{[\,]}$.

Conversely, if $G^{[\,]} \in \mathbb{Gr}^{[\,]}$, we can define an action of $G^{[\,]}$ on itself due to the bracket operation by

$$g^h = g + [g,h], \quad g, h \in G^{[\,]}.$$

It is easy to prove that these two procedures are converse to each other and actually we have an isomorphism of categories

$$\mathbb{Gr}^{\bullet} \approx \mathbb{Gr}^{[\,]}.$$

## 2. Ideals and Commutators in $\mathbb{Gr}^{\bullet}$

Let $G \in \mathbb{Gr}^{\bullet}$.

**Definition 2.1.** A nonempty subset $A$ of $G$ is called an ideal of $G$ if it satisfies the following conditions:

1. $A$ is a normal subgroup of $G$ as a group;
2. $a^g \in A$, for $a \in A$, $g \in G$;

3. $-g + g^a \in A$, for $a \in A$ and $g \in G$.

**Definition 2.2** (Kurosh [2])**.** A nonempty subset $A$ of an $\Omega$-group $G$ is called an ideal if

(a) $A$ is an additive normal subgroup of $G$;

(b) For any $n$-any operation $\omega$ from $\Omega$, any element $a \in A$ and elements $x_1, x_2, \dots, x_n \in G$

$$-(x_1 \cdots x_n\omega) + x_1 \cdots x_{i-1}(a + x_i)x_{i+1} \cdots x_n\omega \in A,$$

for $i = 1, 2, \dots, n$.

This definition in the case of groups is the definition of a normal subgroup of a group, and in the case of rings is the definition of a two-sided ideal of a ring.

**Proposition 2.3.** *For a group $G \in \mathbb{Gr}^\bullet$ considered as an $\Omega$-group, where $\Omega$ consists of one binary operation of action, Definitions* 2.1 *and* 2.2 *are equivalent.*

*Proof.* The condition (b) of Definition 2.2 has the forms:

$$-x_1^{x_2} + (a + x_1)^{x_2} \in A, \quad \text{for} \quad i = 1; \tag{2.1}$$

$$-x_1^{x_2} + x_1^{a+x_2} \in A, \quad \text{for} \quad i = 2. \tag{2.2}$$

Taking $x_1 = 0$ in (2.1), we obtain $a^{x_2} \in A$, which is condition 2 of Definition 2.1. Taking $x_2 = 0$ in (2.2), we have $-x_1 + x_1^a \in A$, which is condition 3 of Definition 2.1.

Conversely, we shall show that conditions 2 and 3 of Definition 2.1 imply conditions (2.1) and (2.2). From condition 2 we have $a^{x_2} \in A$; also

$$-x_1^{x_2} + (a + x_1)^{x_2} = -x_1^{x_2} + a^{x_2} + a_1^{x_2},$$

and it is an element of $A$ since $A$ is a normal subgroup of $G$. By condition 3 of Definition 2.1, $-x_1 + x_1^a \in A$. We have $-x_1^{x_2} + x_1^{a+x_2} = (-x_1 + x_1^a)^{x_2}$ and this is an element of $A$ due to condition 2, which ends the proof. □

Thus an ideal of $G$ is a subobject of $G$ in $\mathbb{Gr}^\bullet$. It is clear that $G$ itself and the trivial subobject of $G$ are ideals of $G$. An intersection of any system of ideals of $G$ is an ideal, and therefore we conclude that there exists the ideal generated by a system of elements of $G$.

**Proposition 2.4.** *Let $A$ be an ideal of $G$. For $a_1, a_2 \in A$, $g_1, g_2 \in G$ we have*

$$(a_1 + g_1)^{a_2+g_2} \in g_1^{g_2} + A.$$

*Proof.* Since $A$ is an ideal of $G$ there exist $a_1', a_2' \in A$, such that $a_1 + g_1 = g_1 + a_1'$, $a_2 + g_2 = g_2 + a_2'$. Therefore

$$\begin{aligned}(a_1 + g_1)^{a_2+g_2} &= (g_1 + a_1')^{g_2+a_2'} = (g_1^{g_2})^{a_2'} + a_1'^{\,g_2+a_2'} \\ &= g_1^{g_2} - g_1^{g_2} + (g_1^{g_2})^{a_2'} + a_1'^{\,g_2+a_2'} \in g_1^{g_2} + A;\end{aligned}$$

here we apply $-g_1^{g_2} + (g_1^{g_2})^{a_2'} \in A$. □

Let $A$ and $B$ be subobjects of $G$. Denote by $\{A, B\}$ the subobject of $G$ generated by $A$ and $B$, and let $A + B$ denote the subset of $G$

$$A + B = \{a + b \,|\, a \in A,\ b \in B\}.$$

**Proposition 2.5.** *If $A$ is an ideal of $G$ and $B$ is a subobject of $G$, then*

$$\{A, B\} = A + B.$$

*Proof.* It is obvious that $A + B \subset \{A, B\}$. Since $A$ is an ideal, it follows that $A + B$ is a subgroup of $G$. By Proposition 2.4, $(a_1 + b_1)^{a_2+b_2} \in b_1^{b_2} + A$. Since $B$ is a subobject, $b_1^{b_2} \in B$, and since $A$ is an ideal, $b_1^{b_2} + A = A + b_1^{b_2} \in A + B$ which ends the proof. □

For Ω-groups see Propositions 2.4 and 2.5 in [2].

**Proposition 2.6.** *If $A$ and $B$ are ideals of $G$, then $A + B$ is also an ideal.*

*Proof.* For $g \in G$, $a \in A$ and $b \in B$ we have

$$g + (a + b) = (a' + g) + b = a' + b' + g \in A + B + g,$$

for certain $a' \in A$ and $b' \in B$. Thus $g + (A + B) \subset (A + B) + g$. In the same way we show that $(A + B) + g \subset g + (A + B)$ and thus $g + (A + B) = (A + B) + g$. It is obvious that $(a + b)^g \in A + B$. Now we have to show that $-g + g^{a+b} \in A + B$. We have

$$-g + g^{a+b} = -g + g^a - g^a + (g^a)^b \in A + B$$

since $-g + g^a \in A$, $-g^a + (g^a)^b \in B$. □

It is easy to verify that the ideal generated by a system of ideals of $G$ coincides with the additive subgroup of $G$ generated by these ideals. For Ω-groups see [2].

**Definition 2.1′.** Let $G^{[\,]} \in \mathbb{Gr}^{[\,]}$ and $A$ be a nonempty subset of $G^{[\,]}$. $A$ is called an ideal of $G^{[\,]}$ if

1′. $A$ is a normal subgroup of $G^{[\,]}$ as of an additive group;
2′. $[a, g] \in A$, for $a \in A$, $g \in G^{[\,]}$;
3′. $[g, a] \in A$, for $a \in A$, $g \in G^{[\,]}$.

It is easy to see that the isomorphism of categories $\mathbb{Gr}^{\bullet} \approx \mathbb{Gr}^{[\,]}$ carries ideals to ideals.

**Proposition 2.7.** *If $A$ is an ideal of $G$, then the quotient group $G/A$ with the induced action on itself is an object of $\mathbb{Gr}^{\bullet}$.*

*Proof.* Straightforward verification. □

In what follows, for $G \in \mathbb{Gr}^{\bullet}$ and $g, g' \in G$, $[g, g']$ will indicate the element $-g + g^{g'}$ of $G$ and $(g, g')$ the commutator $-g - g' + g + g'$. Let $A$ and $B$ be subobjects of $G$.

**Definition 2.8.** A commutator $[A, B]$ of $G$ generated by $A$ and $B$ is the ideal of $\{A, B\}$ generated by the elements

$$\{[a, b], [b, a], (a, b) \mid a \in A,\ b \in B\}.$$

**Definition 2.9** ([2])**.** Let $G$ be an $\Omega$-group, $A, B$ be $\Omega$-subgroups of $G$ and $\{A, B\}_\Omega$ be the $\Omega$-subgroup of $G$ generated by $A$ and $B$. The commutator $[A, B]_\Omega$ is the ideal of $\{A, B\}_\Omega$ generated by elements of the form

$$(a, b) = -a - b + a + b, \quad a \in A, \quad b \in B,$$

and

$$\begin{aligned}[a_1, \dots, a_n; b_1, \dots, b_n; \omega] = -a_1 a_2 \cdots a_n \omega - b_1 b_2 \cdots b_n \omega \\ + (a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)\omega,\end{aligned} \tag{2.3}$$

where $\omega$ is an $n$-any operation from $\Omega$, $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$.

If $G$ is a group with the trivial action on itself or with the action by conjugation, then $[A, B]$ in Definition 2.8 is the normal subgroup of $G$ generated in $\{A, B\}$ by commutators $(a, b)$, $a \in A$, $b \in B$, i. e. the usual commutator for the case of groups. The same is true for Definition 2.9; if an $\Omega$-group is a group without multioperations, then the commutator $[A, B]_\Omega$ is the usual commutator $(A, B)$ of a group [2].

**Proposition 2.10.** *In the case of groups with action on itself Definitions* 2.8 *and* 2.9 *are equivalent.*

*Proof.* For groups with action (2.3) has the form

$$-a^{a_2} - b_1^{b_2} + (a_1 + b_1)^{a_2 + b_2}. \tag{2.4}$$

Take $a_1 = a$, $a_2 = b_1 = 0$, $b_2 = b$, then $-a + a^b \in [A, B]_\Omega$. Take in (2.4) $a_1 = b_2 = 0$, $a_2 = a$, $b_1 = b$, then we obtain

$$-b + b^a \in [A, B]_\Omega.$$

Thus we have shown that $[A, B] \subset [A, B]_\Omega$. Conversely, for $x = -a_1^{a_2} - b_1^{b_2} + (a_1 + b_1)^{a_2 + b_2} \in [A, B]_\Omega$ we have $x = -a_1^{a_2} - b_1^{b_2} + (a_1^{a_2})^{b_2} + (b_1^{a_2})^{b_2} \in \{A, B\}$. Let $\overline{\{A, B\}} = \{A, B\}/[A, B]$ and let $\overline{g}$ be the class of the element $g \in \{A, B\}$ in $\overline{\{A, B\}}$. We have $\overline{a^b} = \overline{a}$, $\overline{b^a} = \overline{b}$ in $\overline{\{A, B\}}$. Thus

$$\begin{aligned}\overline{x} = \overline{-a_1^{a_2} - b_1^{b_2} + (a_1^{a_2})^{b_2} + (b_1^{a_2})^{b_2}} = \overline{a_1^{a_2}} - \overline{b_1^{b_2}} + \overline{a_1^{a_2}} + \overline{b_1^{a_2}}^{\overline{b_2}} \\ = -\overline{a_1^{a_2}} - \overline{b_1^{b_2}} + \overline{a_1^{a_2}} + \overline{b_1^{b_2}} = \overline{-a_1^{a_2} - b_1^{b_2} + a_1^{a_2} + b_1^{b_2}} = 0,\end{aligned}$$

which means that $x \in [A, B]$. □

Below we formulate without proofs two statements for $\Omega$-groups from [2], which in the case of groups with action give the corresponding results.

**Proposition 2.11.** *For any $\Omega$-subgroups $A$ and $B$ in $G$ we have*

$$[A, B]_\Omega = [B, A]_\Omega.$$

**Proposition 2.12.** *An $\Omega$-subgroup $A$ is an ideal of $G$ if and only if*

$$[A, G]_\Omega \subseteq A.$$

**Corollary 2.13.** *Any $\Omega$-subgroup $A$ of an $\Omega$-group $G$ which contains the commutator $[G, G]_\Omega$ is an ideal of $G$.*

*Proof.* It follows from the inclusions

$$[A, G]_\Omega \subset [G, G]_\Omega \subset A. \qquad \square$$

## 3. Central Series in $\mathbb{Gr}^\bullet$ and the Main Result

Let $G \in \mathbb{Gr}^\bullet$.

**Definition 3.1.** The (lower) central series

$$G = G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots$$

of the object $G$ is defined inductively by

$$G_n = [G_1, G_{n-1}] + [G_2, G_{n-2}] + \cdots + [G_{n-1}, G_1].$$

By definition, we have $[G_n, G_m] \subset G_{n+m}$.

**Proposition 3.2.** *For each $n \geq 1$, $G_{n+1}$ is an ideal of $G_n$.*

*Proof.* We have $G_2 = [G_1, G_1]$, which is an ideal of $G_1$, by definition. $G_3 = [G_1, G_2] + [G_2, G_1]$. By Proposition 2.11, $[G_1, G_2] = [G_2, G_1]$. We have

$$[G_1, G_2] \subset [G_1, G_1] = G_2 \subset \{G_1, G_2\}$$

and $[G_1, G_2]$ is an ideal of $\{G_1, G_2\}$; from this it follows that $[G_1, G_2]$ is an ideal of $G_2$ and therefore, by Proposition 2.6, $G_3$ is an ideal of $G_2$. We have

$$G_{n+1} = [G_1, G_n] + [G_2, G_{n-1}] + \ldots [G_{n-1}, G_2] + [G_n, G_1].$$

For $1 \leq k \leq n$, $[G_k, G_{n-k+1}]$ is an ideal of $\{G_k, G_{n-k+1}\}$; $G_n \subseteq G_k$ from which it follows that $G_n \subseteq \{G_k, G_{n-k+1}\}$. At the same time

$$[G_k, G_{n-k+1}] \subset [G_k, G_{n-k}] \subset G_n.$$

Therefore $[G_k, G_{n-k+1}]$ is an ideal of $G_n$ for each $1 \leq k \leq n$. Thus each summand of $G_{n+1}$ is an ideal of $G_n$. By Propositions 2.6 and 2.11 we conclude that $G_{n+1}$ is an ideal of $G_n$. $\square$

Since $(G_i, G_i) \subset G_{2i} \subset G_{i+1}$, each $G_i/G_{i+1}$ has an abelian group structure. Let

$$LL_G = G_1/G_2 \oplus G_2/G_3 \oplus \cdots \oplus G_n/G_{n+1} \oplus \cdots, \tag{3.1}$$

where $\oplus$ denotes the direct sum of abelian groups.

Let $k$ be a commutative ring with the unit, and $A$ a $k$-module. We recall the definitions of Lie and Leibniz algebras.

**Definition 3.3.** A Lie algebra $(A, (\ ,\ ))$ over $k$ is given by a $k$-module $A$ and a $k$-module homomorphism $(\ ,\ ) : A \otimes_k A \longrightarrow A$ called a round bracket such that the equation

$$(x, x) = 0$$

and the Jacobi identity

$$(x, (y, z)) + (y, (z, x)) + (z, (x, y)) = 0 \tag{3.2}$$

hold for $x, y, z \in A$.

Let $\mathbb{Lie}$ be the category of Lie algebras. Morphisms in $\mathbb{Lie}$ are $k$-module homomorphisms $\varphi$ with

$$\varphi(x,y) = (\varphi(x), \varphi(y)).$$

**Definition 3.4** ([3])**.** A Leibniz algebra $A$ over $k$ is a $k$-module $A$ equipped with a $k$-module homomorphism called a square bracket

$$[\ ,\ ] : A \otimes_k A \longrightarrow A,$$

satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] \tag{3.3}$$

for $x, y, z \in A$.

This is in fact a right Leibniz algebra. The dual notion of a left Leibniz algebra is made out of the dual relation

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]],$$

for $x, y, z \in A$.

A morphism of Leibniz algebras is a $k$-module homomorphism $f : A \longrightarrow A'$ with $\varphi[x,y] = [\varphi(x), \varphi(y)]$.

In this paper we deal with right Leibniz algebras. Denote this category by $\mathbb{Leibniz}$.

**Definition 3.5.** A Lie–Leibniz algebra is a $k$-module $A$ together with two $k$-module homomorphisms

$$(\ ,\ ), [\ ,\ ] : A \otimes_k A \longrightarrow A$$

called round and square brackets, respectively, such that $(x,x) = 0$ for $x \in A$ and both Jacobi and Leibniz identities ((3.2) and (3.3)) hold.

A morphism of Lie–Leibniz algebras is a $k$-module homomorphism $\varphi : A \longrightarrow A'$ with

$$\varphi(x,y) = (\varphi(x), \varphi(y)),$$
$$\varphi[x,y] = [\varphi(x), \varphi(y)].$$

We denote the corresponding category by $\mathbb{LL}$.

**Condition 1.** For each $x, y, z \in G$, $G \in \mathbb{Gr}^{\bullet}$

$$x - x^{(z^x)} + x^{y+z^x} - x + x^z - x^{z+y^z} = 0.$$

It is straightforward to verify that if $G$ satisfies Condition 1, then the group $G^{[\,]}$, which corresponds to $G$ (i.e. $[\ ,\ ]$ is defined by $[g,h] = -g + g^h$, $g, h \in G$) satisfies the following condition.

**Condition 1′.**

$$[x^y, [y,z]] = [[x,y], z^x] + [-[x,z], y^z], \quad x, y, z \in G^{[\,]}.$$

Let $G$ be a group. Consider $G$ as a group with the (right) action by conjugation, i.e. $g^{g'} = -g' + g + g'$. Then $G$ satisfies Condition 1 and in this case Condition 1′ is equivalent to the Witt–Hall identity for groups. Each group with the trivial action on itself (i.e. $g^{g'} = g,\ g, g' \in G$) also satisfies Condition 1. For an arbitrary set $X$ let $\mathscr{F}_X$ be the free group with action on itself generated by $X$. The quotient $\mathscr{F}_X/_\sim$ of $\mathscr{F}_X$ by the equivalence relation generated by the relation corresponding to Condition 1 is obviously a group which satisfies Condition 1. See also an example at the end of the proof of Theorem 3.6.

Denote by $\mathbb{Gr}^{\mathrm{c}}$ a category of groups with action on itself satisfying Condition 1. In an analogous way we define the category $\mathbb{Ab}^{\mathrm{c}}$. It is easy to see that the functors $E, A, T, C, Q_1, Q_2$, defined in Section 1, give the functors between categories $\mathbb{Ab}^{\mathrm{c}}$, $\mathbb{Gr}^{\mathrm{c}}$ and $\mathbb{Gr}$. We shall denote below these functors by the same letters. $\mathscr{F}_X/_\sim$ is a free object in $\mathbb{Gr}^{\mathrm{c}}$ and consequently the action in it is neither the trivial one nor the conjugation.

Let $G \in \mathbb{Gr}^c$. Denote $\overline{G}_m = G_m/G_{m+1}$, then $LL_G = \sum\limits_{m\geq 1} \overline{G}_m$.

Consider maps $(\ ,\ )_{mn}, [\ ,\ ]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ defined by round and square brackets in $G$, respectively:

$$\begin{aligned} x, y &\longmapsto (x, y), \\ x, y &\longmapsto [x, y]. \end{aligned}$$

By the definition of $G_i$, it is clear that if $x \in G_m$, $y \in G_n$, then $(x, y), [x, y] \in G_{m+n}$. For $x \in G_m$, denote by $\overline{x}$ the corresponding class in $\overline{G}_m$.

**Theorem 3.6.** *Let $G$ be a group with action on itself satisfying Condition* 1. *Then we have:*

(a) $\overline{x^y} = \overline{x}$, $\overline{-y + x + y} = \overline{x}$, *for each $x \in G_m$, $y \in G_n$;*

(b) *The maps $(\ ,\ )_{mn}$ and $[\ ,\ ]_{mn} : G_m \times G_n \longrightarrow G_{m+n}$ induce bilinear maps $\alpha_{mn}, \beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$;*

(c) *The maps $\alpha_{mn}, \beta_{mn},\ m, n \geq 1$ define bilinear maps $(\ ,\ ), [\ ,\ ] : LL_G \times LL_G \longrightarrow LL_G$, which give a Lie–Leibniz structure on $LL_G$.*

*Proof.* (a) Let $x \in G_m$, $y \in G_n$, $m, n \geq 1$. Then $[x, y] = -x + x^y \in G_{m+n} \subset G_m$ and since $x \in G_m$ we obtain that $x^y \in G_m$. In $\overline{G}_m$ we have $\overline{[x, y]} = -\overline{x} + \overline{x^y}$, but since $[x, y] \subset G_{m+n} \subset G_{m+1}$ we have $\overline{[x, y]} = 0$ in $\overline{G}_m$ and thus in $\overline{G}_m$ we have $\overline{x} = \overline{x^y}$. In the same way we show for the action with conjugation that $\overline{-y + x + y} = \overline{x}$. (see also [5]).

(b) We shall check this condition for a square bracket; for a round bracket the proof is similar [5]. First we shall show that the map $\beta_{mn} : \overline{G}_m \times \overline{G}_n \longrightarrow \overline{G}_{m+n}$ is defined correctly. Let $\overline{x} \in \overline{G}_m$, $\overline{y} \in \overline{G}_n$, where $x \in G_m$, $y \in G_n$. By definition, $\beta_{mn}(\overline{x}, \overline{y}) = [\overline{x}, \overline{y}] = \overline{[x, y]}$, where $[x, y] \in G_{n+m}$. Let $\overline{x} = \overline{x'}$ for $x' \in G_m$, thus $x - x' \in G_{m+1}$. For simplicity, suppose that $x - x' \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$ (a more general case is treated similarly). Then $x = [a, b] + x'$, where $a \in G_{i+1}$, $b \in G_{m-i}$. From this we have in $\overline{G}_{m+n}$:

$$\begin{aligned} \overline{[x, y]} = \overline{[[a, b] + x', y]} &= \overline{-x' + [[a, b], y] + x' + [x', y]} \\ &= \overline{-x' + [[a, b], y] + x'} + \overline{[x', y]}. \end{aligned} \tag{3.4}$$

$[[a,b],y] \in G_{m+n+1} \subset G_{m+n}$. Applying the condition (a), we obtain

$$\overline{-x' + [[a,b],y] + x'} = \overline{[[a,b],y]} = 0 \quad \text{in} \quad \overline{G}_{m+n}.$$

Thus from (3.4) we have $\overline{[x,y]} = \overline{[x',y]}$. If $x - x' = (a,b) \in [G_{i+1}, G_{m-i}] \subset G_{m+1}$, then by the same argument we have

$$\begin{aligned}\overline{[x,y]} &= \overline{[x' + (a,b), y]} = \overline{-x' + [(a,b),y] + x'} + \overline{[x',y]} \\ &= \overline{[(a,b),y]} + \overline{[x',y]} = \overline{[x',y]}, \ \text{ since } \ \overline{[(a,b),y]} = 0 \ \text{ in } \ \overline{G}_{m+n}.\end{aligned}$$

The correctness of $\beta_{mn}$ for the second argument is proved in an analogous way.

Now we shall show that the maps $\beta_{mn}$ are bilinear. Let $\overline{x}_1, \overline{x}_2 \in \overline{G}_m$ and $\overline{y} \in \overline{G}_n$. We have in $\overline{G}_{m+n}$

$$\begin{aligned}[\overline{x}_1 + \overline{x}_2, \overline{y}] = \overline{[x_1 + x_2, y]} &= \overline{-x_2 + [x_1, y] + x_2} \\ &+ \overline{[x_2, y]} = \overline{[x_1, y]} + \overline{[x_2, y]};\end{aligned}$$

here we again apply the condition (a). Let $\overline{x} \in \overline{G}_m$ and $\overline{y}_1, \overline{y}_2 \in \overline{G}_n$. We have in $\overline{G}_{m+n}$

$$\begin{aligned}[\overline{x}, \overline{y}_1 + \overline{y}_2] &= \overline{[x, y_1 + y_2]} = \overline{[x, y_1] + [x^{y_1}, y_2]} \\ = \overline{[x, y_1]} + \overline{[x^{y_1}, y_2]} &= [\overline{x}, \overline{y}_1] + [\overline{x^{y_1}}, \overline{y}_2] = [\overline{x}, \overline{y}_1] + [\overline{x}, \overline{y}_2],\end{aligned}$$

since, by the condition (a) $\overline{x^{y_1}} = \overline{x}$. This proves that maps $\beta_{mn}$ are bilinear.

(c) The maps $\alpha_{mn}$, $\beta_{mn}$ can be continued linearly in a natural way up to the bilinear maps $(\ ,\ ), [\ ,\ ] : LL_G \times LL_G \longrightarrow LL_G$. The proof of the fact that $(\ ,\ )$ satisfies the condition (3.2) and $(l,l) = 0$ for any $l \in LL_G$ is similar to the proof of the corresponding statement in Witt's theorem (see [5], Proposition 2.3; [6]). It remains to show that the square bracket operation $[\ ,\ ]$ satisfies the Leibniz identity (3.3).

The object $G$ satisfies Condition 1, therefore we have Condition 1′ for the square bracket in $G$. Since the square bracket operation in $LL_G$ is linear for both arguments, we can limit ourself to the case where $\overline{x} \in G_m$, $\overline{y} \in \overline{G}_n$, $\overline{z} \in \overline{G}_t$. Applying the conditions (a) and (b) of the theorem we have

$$\begin{aligned}[\overline{x}, [\overline{y}, \overline{z}]] &= [\overline{x^y}, [\overline{y}, \overline{z}]] = \overline{[x^y, [y,z]]}; \\ [[\overline{x}, \overline{y}], \overline{z}] &= [[\overline{x}, \overline{y}], \overline{z^x}] = \overline{[[x,y], z^x]}; \\ -[[\overline{x}, \overline{z}], \overline{y}] &= [-[\overline{x}, \overline{z}], \overline{y^z}] = \overline{[-[x,z], y^z]}.\end{aligned}$$

By Condition 1′ we obtain

$$[\overline{x}, [\overline{y}, \overline{z}]] = [[\overline{x}, \overline{y}], \overline{z}] - [[\overline{x}, \overline{z}], \overline{y}] \ \text{ in } \ \overline{G_{m+n+t}},$$

which completes the proof of the theorem. □

The following example is due to the referee.

**Example.** Let $G$ be the abelian group of integers $\mathbb{Z}^\bullet$, which acts on itself in the following way: $x^y = (-1)^y x$. We have $[x,y] = 0$ for $y$ even, $[x,y] = -2x$ for $y$ odd and $G_n = 2^{n-1}\mathbb{Z}^\bullet$. It is easy to see that $\mathbb{Z}^\bullet \in \mathbb{G}\mathrm{r}^{\mathrm{c}}$ and $LL_{\mathbb{Z}^\bullet}$ is a free

Leibniz algebra generated by a single element over a two element field (see also [4]).

It is easy to see that by Theorem 3.6 we have actually constructed the functor $LL : \mathbb{Gr}^{c} \longrightarrow \mathbb{LL}$. In an analogous way one can construct the functor $L : \mathbb{Ab}^{c} \longrightarrow \mathbb{Leibniz}$. For $A \in \mathbb{LL}$ let $S_1(A)$ denote the greatest quotient algebra of $A$ which makes square bracket in $A$ trivial. Then $S_1(A) \in \mathbb{Lie}$ and we have a functor $S_1 : \mathbb{LL} \longrightarrow \mathbb{Lie}$. Similarly, we construct the functor $S_2 : \mathbb{LL} \longrightarrow \mathbb{Leibniz}$. $S_1$ and $S_2$ are left adjoints to the embedding functors $E_1$ and $E_2$ respectively. Denote by $W : \mathbb{Gr} \longrightarrow \mathbb{Lie}$ the functor defined by Witt's theorem [6], [5]. Thus we have the following functors between the well defined categories:

$$\begin{array}{ccccc} \mathbb{Ab}^{c} & \underset{A}{\overset{E}{\rightleftarrows}} & \mathbb{Gr}^{c} & \overset{Q_1,\ T,\ Q_2,\ C}{\rightleftarrows\rightleftarrows} & \mathbb{Gr} \\ L\downarrow & & \downarrow LL & & \downarrow W \\ \mathbb{Leibniz} & \underset{S_2}{\overset{E_2}{\rightleftarrows}} & \mathbb{LL} & \underset{S_1}{\overset{E_1}{\leftrightarrows}} & \mathbb{Lie}, \end{array}$$

where $LL \circ C = E_1 \circ W$, $E_2 \circ L = LL \circ E$. A more detailed account of this diagram will be given in the forthcoming paper, where free objects in $\mathbb{Gr}^{\bullet}$ and free Leibniz algebras are studied.


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## References


1. H. J. Baues and D. Conduché, The central series for Peiffer commutators in groups with operators. *J. Algebra* **133**(1990), No. 1, 1–34.
2. A. G. Kurosh, Lectures in general algebra. (Russian) *Nauka, Moscow,* 1973.
3. J.-L. Loday, Cyclic homology. *Grundlehren der mathematischen Wissenschaften,* 301, *Springer,* 1992.
4. J.-L. Loday and T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology. *Math. Ann.* **296**(1993), 139–158.
5. J.-P. Serre, Lie algebras and Lie groups. *W. A. Benjamin, INC. London, Amsterdam, Tokyo*, 1965.
6. E. Witt, Treue Darstellung Liescher Ringe. *J. Reine Angew. Math.* **1**(1937), 152–160.


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