*Gen. Math. Notes, Vol. 17, No. 2, August, 2013, pp.91-102*
*ISSN 2219-7184; Copyright ©ICSRS Publication, 2013*
*www.i-csrs.org*
*Available free online at http://www.geman.in*

# PBIB Designs and Association Scheme Arising from Minimum Total Dominating Sets of Non Square Lattice Graph

**H.G. Shekharappa[1], Shailaja S. Shirkol[2] and Manjula C. Gudgeri[3]**

[1] Visveswaraya Technological University, Belgaum, Karnataka State, India
E-mail: shekharhg@rediffmail.com
[2] Department of Mathematics, SDM College of Engineering and Technology
Dharwad, Karnataka State, India
E-mail: shaila_shirkol@rediffmail.com
[3] Department of Mathematics, KLE Dr. M. S. Sheshagiri College of
Engineering and Technology, Belgaum, Karnataka State, India
E-mail: gudgeri_manjula@rediffmail.com

(Received: 21-4-13 / Accepted: 6-6-13)

**Abstract**

*Square lattice graphs $L_2(n)$ with the parameters $(n^2, 2(n-1), n-2, 2)$ are strongly regular and are unique for all n except n=4. however for n=4, we have two non-isomorphic strongly regular graphs. The non - lattice graph with parameters $(16, 6, 2, 2)$ is known as Shrikhande graph. In this paper we show that every minimum total dominating set in Shrikhande graph induces two $K_2$s. Further we establish that these classes of minimum total dominating sets of Shrikhade graph form Partially Balanced Incomplete Block Designs with the parameters (16, 44, 11, 4, $\lambda_i\, i = 1 or 2$, $\lambda_j\, j = 3 or 4$).*

**Keywords:** *Partially Balanced Incomplete Block Designs, Minimum total dominating set, Total domination number, Strongly Regular Graph.*

## 1 Introduction

In combinatorial mathematics, a block design is a particular kind of set system which has application to finite geometry, cryptography and algebric geometry.

A balanced incomplete block design is one among many variations that have been studied in the block designs and it is a set of v elements arranged in b blocks of k elements each in such a way that each element occurs in exactly r blocks. The combinatorial representation so obtained is called $(v, b, r, k, \lambda)$ design. The relation between graph and designs where first observed by Berge [?]. Motivated by the works of Berge, J. W. D. Paola [12]as given a link between graphs and balanced incomplete block designs (BIBD) whose blocks are maximum independent sets. As the class of BIBD's do not fit many practical situations as these designs require large number of applications, to over come this Bose and Nair [5] introduced a class of binary equireplicate and proper designs called Partially Balanced Incomplete Block Designs (PBIBD) which is included as a special case of BIBD's. They established the relation between PBIBD's and strongly regular graphs with two association scheme having parameters $(v, b, r, k, \lambda_1, \lambda_2)$ as first kind. More about association schemes can be found in Bannai and Ito [2] Godsil and Royal [8] and Bailey [1]. Harary et. al [9] considered the relation between isomorphic factorization of regular graphs and PBIBD with two association scheme. Ioin and M. S. shrikhande [17] studied certain kind of designs called $(v, k, \lambda, \mu)$designs over strongly regular graph. Walikar et. al[16] introduced design called $(v, \beta_0, \mu)$ - designs, whose blocks are maximum independent sets in regular graph on v vertices. Walikar et. al [15]have also established the relation between dominating sets of a graph with blocks of PBIBD's. It is possible to construct the strongly regular graph G with parameters $(v, n_1, P^1_{11}, P^2_{11})$ from a given PBIBD with two association scheme having parameters $(v, b, r, k, \lambda_1, \lambda_2)$ (see Bose [4] and Rao [14]. In this paper we prove that every Minimum Total Dominating Set (MTDS) in Shrikhande graph induces two$K_2s$. Further we establish that the set of all MTDS form a PBIBD with the parameter $(16, 44, 11, 4, 1or2, 3or4)$.

## 2 Definitions and Preliminary Results

Throughout this paper G = $(V, E)$ where V is the vertex set and E is unordered pair of edges, stands for a finite, connected, undirected graph with neither loops nor multiple edges.

Open neighborhood of a vertex v ∈V is N$(v)$= $\{u \in V/uv \in E\}$ and closed neighborhood is N$[v]$= N$(v)\cup\{v\}$.

**Definition 2.1.** A set D⊆V is called total dominating set if every vertex v ∈V, there exists u ∈D, u ≠v such that u is adjacent to v. The minimum cardinality of a total dominating set of G is a total domination number of G denoted by $\gamma_t\,(G)$. or $\gamma_t$- set.

**Definition 2.2.** A strongly regular graph G with the parameters $(n, k, \lambda, \mu)$is a graph on n vertices which is regular with valency k has the following properties.

i. any two adjacent vertices have exactly $\lambda$ common neighbors ;

ii. any two non-adjacent vertices have exactly $\mu$ common neighbors.

**Lemma 2.3.** If G is strongly regular graph with parameters $(n, k, \lambda, \mu)$then $(n-k-1)\mu=$ k $(k-1-\lambda)$.

**Definition 2.4.** Given a set $\{1, 2, ..., v\}$a relation satisfying the following condition is called m-class association (m $\geq$2)

i. Any two symbols are either first associates or second associates ... $m^{th}$associates the relation of association being symmetric.

ii. Each symbol $\alpha$ has $n_i$, $i^{th}$associates, the number $n_i$ being independent of $\alpha$.

iii. If two symbols $\alpha$ and $\beta$ are the $i^{th}$associates, then the number of symbols which are $j^{th}$ associates of $\alpha$ and $k^{th}$associates of $\beta$ is $P^i_{jk}$ and is independent of the pair of $i^{th}$associates $\alpha$ and $\beta$. Also $P^i_{jk}= P^i_{kj}$.

Thus there are 2m + 4 parameters of first kind and $\frac{m^2(m+1)}{2}$ parameters of second kind. The numbers $(v, b, r, k, \lambda_i, i = 1, 2, ..., m)$are called parameters of first kind, where as numbers $n_i$'s and $P^i_{jk}$'s, $(i, j, k = 1, 2, 3, ..., m)$ are called parameters of second kind. It can be easily seen that vr = bk and $\sum_{i=1}^{m} n_i\lambda_i= r\,(k-1)$

**Definition 2.5.** The PBIB design is an arrangement of v symbols in to b sets (called blocks) of size k, k < v such that

i. Every symbol is contained in exactly r blocks.

ii. Each block contains k distinct symbols.

iii. Any two symbols which are $i^{th}$associates occur together in $\lambda_i$ blocks.

# 3 Main Results

## 3.1 Introduction to Shrikhande Graph

A Cayley graph of a group H with respect to S, where S is a Cayley subset of H, denoted by Cay (H; S), is the graph with S $= \{\pm(0,1), \pm(1,1), \pm(1.0)\}$, Then the graph Cay (H; S) in this case is the Shrikhande graph.

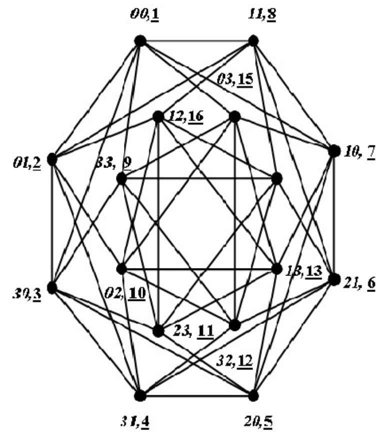



Figure_1. Shrikhande Graph

A simple family of strongly regular graph are called square lattice graph $L_2(n)$. These graphs have parameters $(n^2, 2(n-1), n-2, 2)$. Now strongly regular graph with these parameters are unique for all n except n=4. However n=4 we have two non-isomorphic strongly regular graphs with parameter $(16, 6, 2, 2)$. The non lattice graph with these parameters is known as Shrikhande Graph. Shrikhande graph is $(0, 2)$, locally hexagon graph with girth 3. The independence, chromatic number and total domination number of this graph is 4 (proved in 3.1). The characteristic polynomial of graph is $(x-6)(x-2)^6(x+2)^8$ with -2 as eigen value. There fore it is known as Seidal graph.

## 3.2 Minimum Total Dominating Sets in Shrikhande Graph

**Theorem 3.1.1.** The total domination number of Shrikhande graph is four.

**Proof:** Let G be a Shrikhande graph.

We show that $\gamma_t(G) = 4$. To prove this we show that $\gamma_t(G) \neq 3$.

for if $\gamma_t(G) = 3$, then D = $\{u, v, w\}$ be any minimum total dominating set in G.

Then the possible non-isomorphic subgraphs induced by D are given below.

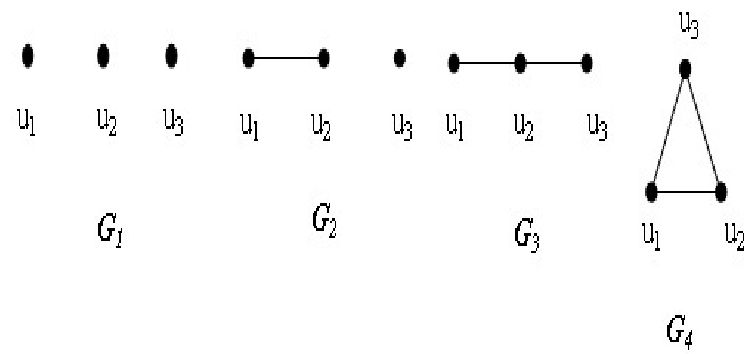



Figure_2

**Property 3.1.a.** Since G is strongly regular graph with the parameters $(16, 6, 2, 2)$ and $K_4$ free graph, any pair of non-adjacent vertices have two common neighbors and any two adjacent vertices have two common neighbors. By this property and regularity of G. We prove the following cases.

**Case 1.** Let $\langle D \rangle = G_1$

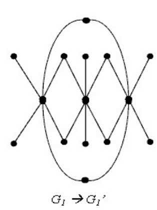


Figure_3

Since G is strongly regular graph with the parameters $(16, 6, 2, 2)$and by the property 3.1.a, we have $\cup_{z \in D} N(z) = 15$

There fore there is at least one vertex which is uncovered by D. Which contradicts the fact that D is a minimum total dominating set. Hence D is not a MTDS.

**Case 2.** Let $\langle D \rangle = G_2$

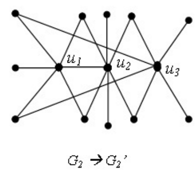
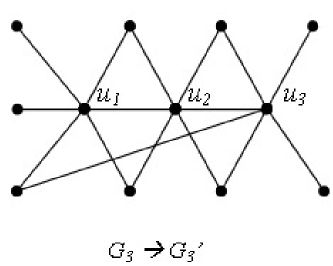



Figure_4

Since G is strongly regular graph with the parameters $(16, 6, 2, 2)$and by the property 3.1.a, we have $\cup_{z \in D} N(z) = 14$

There fore there are at least two vertices which are uncovered by D. Which contradicts the fact that D is a minimum total dominating set. Hence D is not a MTDS.

**Case 3.** $\langle D \rangle = G_3$ or $\langle D \rangle = G_4$



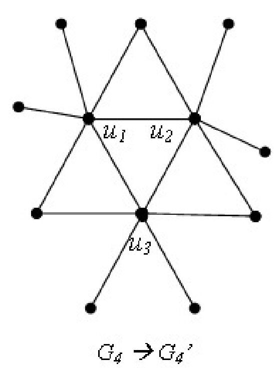




Figure_5

Since G is strongly regular graph with the parameters $(16, 6, 2, 2)$and by the property 3.1.a, we have $\cup_{z \in D} N(z) = 12$

There fore there are at least four vertices which are uncovered by D. Which contradicts the fact that D is a minimum total dominating set. Hence D is not a MTDS.

Thus in all the cases above, we get the contradiction and $\cup_{z\in D}N(z) \neq 16$

Which gives $\gamma_t(G) = 4$

This proves the result.

**Theorem 3.1.2.** If D is a MTDS in Shrikhande graph G, then D induces two$K_2s$.

**Proof:** By the above theorem, we have $\gamma_t(G) = 4$.

Let D = $\{u_1, u_2, u_3, u_4\}$be MTDS in G.

We prove that the set D induces two$K_2s$ in G.

The following are the possible non-isomorphic graphs induced by D as shown in figure - 2.

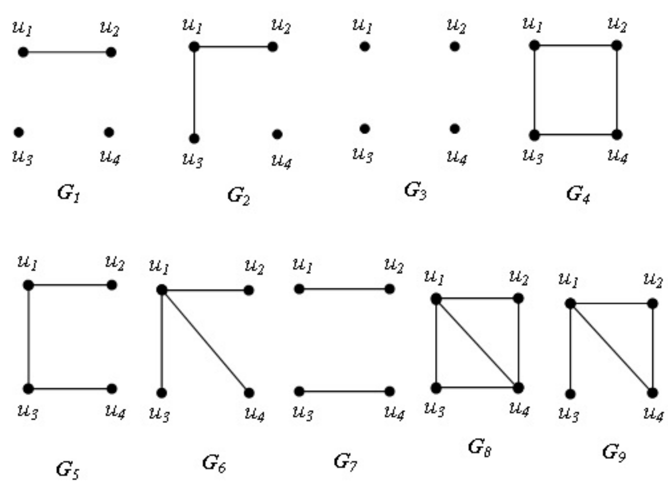



Figure_6

**Case 1.** If $\langle D\rangle = G_1$or $\langle D\rangle = G_2$ or $\langle D\rangle = G_3$

If D induces any of the graphs $G_1$or $G_2$or $G_3$. By the necessary condition of minimum TDS induced subgraph of D should not leave any isolate vertex and also $\cup_{z\in D}N(z) \neq 16 = |V|$ . Hence D is not a MTDS.

**Case 2.** If $\langle D\rangle = G_4$

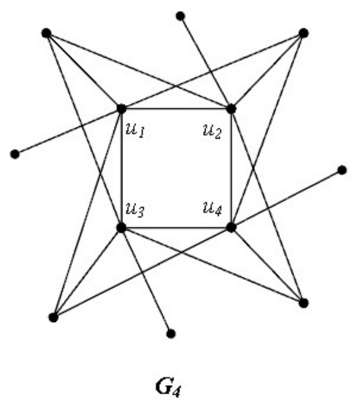



Figure_11

By the property 3.1.a, we have $\cup_{z \in D} N(z) = 12$. There fore there are four vertices, which are still uncovered by D. Which contradicts the fact that D is a minimum total dominating set. Hence D is not a MTDS. Hence D is not a MTDS.

**Case 3.** If $\langle D \rangle = G_5$

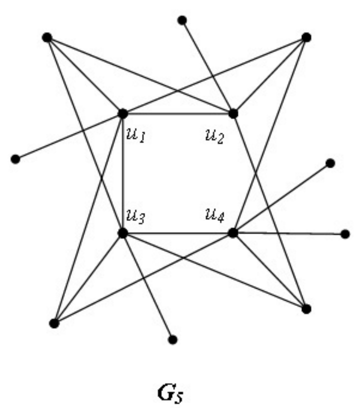



Figure_7

By the property 3.1.a, we have $\cup_{z \in D} N(z) = 13$. There fore there are three vertices, which are still uncovered by D. Which contradicts the fact that D is a minimum total dominating set. Hence D is not a MTDS. Hence D is not a MTDS.

**Case 4.** If $\langle D \rangle = G_6$

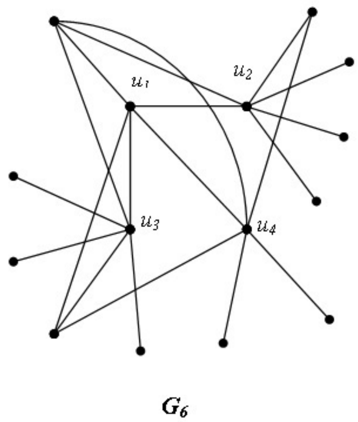



Figure_8

By the property 3.1.a, We have $\cup_{z\in D}N(z) = 15$. There fore there is at least one vertex, which is still uncovered by D. Which contradicts the fact that D is a minimum total dominating set. Hence D is not a MTDS. Hence D is not a MTDS.

**Case 5.** If $\langle D\rangle = G_8$ or $\langle D\rangle = G_9$

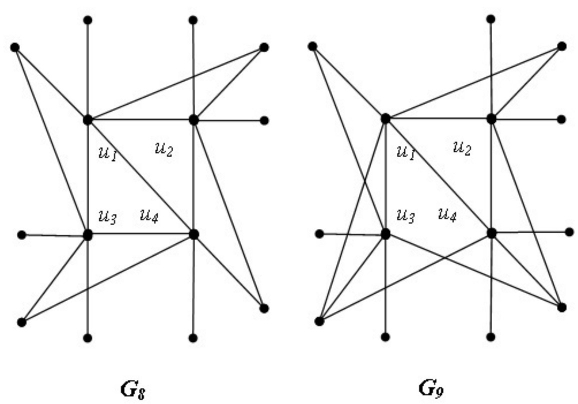



Figure_9

By the property 3.1.a, We have $\cup_{z\in D}N(z) = 14$. There fore there is atleast two vertices, which is still uncovered by D. Which contradicts the fact that D is a minimum total dominating set. Hence D is not a MTDS. Hence D is not a MTDS.

**Case 6.** If $\langle D\rangle = G_7$

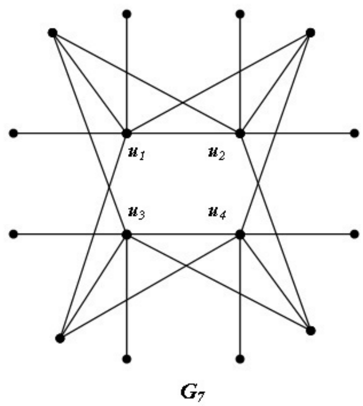



Figure_10

By the property 3.1.a, We have $\cup_{z\in D}N(z) = 16 = |V|$. Thus D covers every vertex of G. Hence $\langle D\rangle = G_7$ is the only MTDS which induces $2 - K_2's$.

This completes the proof.

The following is the list of MTDS which induces two $K_2s$ in Shrikhande graph G.

$\{1,2,5,6\}$, $\{2,3,6,7\}$, $\{3,4,7,8\}$, $\{4,5,8,1\}$, $\{1,5,9,13\}$, $\{2,6,10,14\}$, $\{3,7,11,15\}$, $\{4,8,12,16\}$, $\{9,10,13,14\}$, $\{10,11,14,15\}$, $\{11,12,15,16\}$, $\{12,13,16,9\}$, $\{1,7,12,14\}$, $\{2,8,13,15\}$, $\{3,1,14,16\}$, $\{4,2,15,9\}$, $\{5,3,16,10\}$, $\{6,4,9,11\}$, $\{7,5,10,12\}$, $\{8,6,11,13\}$, $\{1,2,11,14\}$, $\{2,3,12,15\}$, $\{3,4,13,16\}$, $\{4,5,14,9\}$, $\{5,6,15,10\}$, $\{6,7,16,11\}$, $\{7,8,9,12\}$, $\{8,1,10,13\}$, $\{1,3,10,12\}$, $\{2,4,11,13\}$, $\{3,5,12,14\}$, $\{4,6,13,15\}$, $\{5,7,14,16\}$, $\{6,8,15,9\}$, $\{7,1,16,10\}$, $\{8,2,9,11\}$, $\{1,4,9,10\}$, $\{2,5,10,11\}$, $\{3,6,11,12\}$, $\{4,7,12,13\}$, $\{5,8,13,14\}$, $\{6,1,14,15\}$, $\{7,2,15,16\}$, $\{8,3,16,9\}$.

# 4 PBIBDs Associated with MTDS's of Shrikhande Graph

Let us define 2 - class association scheme of Shrikhande graph by using the definition 2.5 as follows,

Let D be the MTDS's of Shrikhande graph which induces two$K_2s$. Then D is the set of blocks of PBIBD with parameters of first kind as $(16, 44, 11, 4, 1or2, 3or4)$ and parameters of second kind as ,

$$P_1 = \begin{pmatrix} P^1_{11} & P^1_{12} \\ P^1_{21} & P^1_{22} \end{pmatrix} = \begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix} n_1 = 12$$

$$P_2 = \begin{pmatrix} P^2_{11} & P^2_{12} \\ P^2_{21} & P^2_{22} \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix} n_2 = 3$$

## References

[1] R.A. Bailey, *Association Schemes Designed Experiments Algebra and Combinatorics*, Cambridge University, Press Cambridge, (2004).

[2] E. Bannai and T. Ito, *Algebraic Combinatorics I.: Association Schemes*, Benjamin, New York, (1984).

[3] C. Berge, *Theory of Graphs and its Applications*, Wiley, New York, (1962).

[4] R.C. Bose, Strongly regular graphs, partial geometries and partially balanced designs, *Pacific J. Math.*, 13(1963), 389-419.

[5] R.C. Bose and K.R. Nair, Partially balanced incomplete block designs, *Sankhya: The Indian Journal of Statistics*, 4(1939),337-372.

[6] R.C. Bose and T. Shimamoto, Classification and analysis of partially balanced incomplete block designs with two associate classes, *J. Amer. Stat. Assn.*, 47(1952), 151-184.

[7] G. Chartr and P. Zhang P. *Introduction to Graph Theory*, Tata Mc graw - Hill Inc., New York, (2006).

[8] C. Godsil and G. Royl, *Algebraic Graph Theory*, Springer-Verlag, New York, (2001).

[9] F. Harary, *Graph Theory*, Addison-Wesley, Reading Mass, (1969).

[10] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, *Fundamentals of Dominations in Graphs*, Marcel Dekker, New York, (1997).

[11] M.A. Henning, A survey of results on total domination in graphs, *Discrete Mathematics*, 309(2009), 32-63.

[12] P.J.W. DI, Block designs in graph theory, *J. of Comb. Theory*, 1(1966), 130-148.

[13] P.N.V. Kumar, D.S. Nandappa and U.S. Mahabaleshwar, Partially balanced incomplete block design associated with minimum perfect dominating sets of Clebesch graph, *Inter Nation Journal Applied Mathematics and Computation*, 4(1) (2012), 39-48.

[14] D. Raghavarao, *Construction and Combinatorial Problems in Design Experiments*, John Wiley, New York, (1971).

[15] H.B. Walikar, H.S. Ramane, B.D. Acharya, H.G. Shekharappa and S. Arumugam, Partially balanced incomplete block design arising from minimum dominating sets of paths and cycles, *AKCE J. Graphs. Combin.*, 4(2) (2007), 223-232.

[16] H.B. Walikar, B.D. Acharya and S. Shirkol, Designs associated with maximum independent sets of graph, *Des. Codes Cryptogr.*, 57(2010), 91-105.

[17] Y.J. Ionin and M.S. Shrikhande, On classification of two class partially balanced designs, *J. Stat. Plan. Inform.*, 95(2001), 209-228.