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Convolution Integral Equation of Fredholm Type with Certain Binomial and Transcendental Functions

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Abstract

We derive a solution of a certain class of convolution integral equation of Fredholm type whose kernel involve certain product of binomial and special functions by using Riemann-Liouville and Weyl fractional integral operators. Certain interesting special cases have also been discussed.

Keywords: *Riemann Liouville fractional integral, Weyl fractional integral, Mellin transform technique,* $\overline{H}$ *-function.*

1 Introduction

The following is the special case of Raizada's [13] generalized polynomial set and defined as:

$$S_n^{\alpha,\beta,0}[x;r,q,A,B,k,\ell]=(Ax+B)^{-\alpha}\exp(\beta x^r)T_{k,\ell}^{n}[(ax+b)^{\alpha+qn}\exp(-\beta x^r)]$$

$$=\sum_{e,p,u,v}\phi_1(e,p,u,v)\,x^L, \qquad \text{… (1.1)}$$

Where

$$\phi_1(e,p,u,v)=(B^{qn-p})\ell^n\frac{(-1)^p(-v)_u(-p)_e(\alpha)_p}{u!\;v!\;e!\;p!}\frac{(-\alpha-qn)_e}{(1-\alpha-p)_e}\left(\frac{\alpha+k+rv}{\ell}\right)_n A^p\beta^v, \qquad \text{… (1.2)}$$

$$L=\ell n+p+rv,\ (p,v=0,1,...,n) \qquad \text{… (1.3)}$$

And

$$\sum_{e,p,u,v}=\sum_{v=0}^{n}\sum_{u=0}^{v}\sum_{p=0}^{n}\sum_{e=0}^{p}. \qquad \text{… (1.4)}$$

The polynomial set defined by (1.1) is a very general in nature and it unifies and extends a number of classical polynomials introduced and studied by various research workers such as Chatterjea [1, 2], Gould and Hopper [7], Krall and Frink [11], Srivastav and Singhal [15] etc. We have made an effort, in the present paper, to obtain an exact solution of the following convolution integral equation of Fredholm type

$$\int_0^\infty y^{-\mu}\,u\left(\frac{x}{y}\right)f(y)\,dy=g(x),\ (x>0) \qquad \text{…(1.5)}$$

where g is a recommended function, f is a unknown function to be determined and the kernel u(x) is given by

$$u(x)=(cx^t+d)^\theta S_n^{\alpha,\beta,0}[z\,x^\rho;r,q,A,B,k,\ell]H_{P_1,Q_1}^{M_1,N_1}\left[t(x^\sigma)\left|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right.\right]$$

$$\times\overline{H}_{P,Q}^{M,N}\left[\omega x^\lambda\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j;\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right], \qquad \text{…(1.6)}$$

in (1.6) the $H_{P_1,Q_1}^{M_1,N_1}[z]$ is the well known Fox's H-function [6] and its series representation is given by

$$H^{M_1,N_1}_{P_1,Q_1}\left[z\left|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right.\right]=\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\frac{(-1)^G}{G!\,F_g}\phi_2(\eta_G)\,z^{\eta_G}, \qquad \ldots(1.7)$$

where

$$\phi_2(\eta_G)=\frac{\prod\limits_{\substack{j=1\\ j\neq g}}^{M_1}\Gamma(f_j-F_j\eta_G)\prod\limits_{j=1}^{N_1}\Gamma(1-e_j+E_j\eta_G)}{\prod\limits_{j=M_1+1}^{Q}\Gamma(1-f_j+F_j\eta_G)\prod\limits_{j=N_1+1}^{P_1}\Gamma(e_j-E_j\eta_G)}$$

and

$$\eta_G=\frac{(f_g+G)}{F_g}$$

The $\overline{H}$-function defined by Inayat-Hussain [9, 10] in (1.6) as:

$$\overline{H}^{M,N}_{P,Q}[z]=\overline{H}^{M,N}_{P,Q}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\phi_3(s)\,z^s\,ds, \qquad \ldots(1.8)$$

where

$$\phi_3(s)=\frac{\prod\limits_{j=1}^{M}\Gamma(b_j-\beta_j s)\prod\limits_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j s)\}^{A_j}}{\prod\limits_{j=M+1}^{Q}\{\Gamma(1-b_j+\beta_j s)\}^{B_j}\prod\limits_{j=N+1}^{P}\Gamma(a_j\alpha_j s)} \qquad \ldots(1.9)$$

which contains fractional powers of some of the Γ-functions. Here z may be real or complex but not equal to zero and an empty product is interpreted as unity. P,Q,M and N are integers such that $1 \le M \le Q$, $0 \le N \le P$, α_j (j = 1,…,P), β_j (j = 1,…,Q) are complex numbers. The exponents A_j (j = 1,…, N) and B_j (j = M+1,…,Q) can take non integer values, when these exponents take integer values, the $\overline{H}$-function reduces to the familiar H-function due to Fox.

Srivastav [16] introduced the general class of polynomial as follows

$$S^{m_1}_{n_1}[I]=\sum_{k_1=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k_1}}{k_1!}I^{k_1}A_{n_1,k_1},:[n_1=0,1,2,...] \qquad \ldots(1.10)$$

where m_1 is an arbitrary positive integer and the coefficients $A_{n_1,k_1}(n_1,k_1 \geq 0)$ are arbitrary constants, real or complex.

Let J denote the space of all functions f which are defined on $R^+ = [0,\infty]$ and satisfy

(i) $f \in C^\infty(R^+)$

(ii) $\lim_{x \to \infty} [x^k f^r(x)] = 0,$ for all non negative integer k and r.

(iii) f(x) = 0(1) as $x \to 0$

J corresponds to the space of good functions defined on the whole real line (Miller [12]).

For the present study, we shall also require the Riemann-Liouville fractional integral (of order μ) defined by

$$D^{-\mu}\{f(x)\} = {}_0D_x^{-\mu}\{f(x)\}$$

$$= \frac{1}{\Gamma(\mu)}\int_0^x (x-w)^{\mu-1} f(w)\,dw, (\mathrm{Re}(\mu)>0; f \in J), \quad \ldots(1.11)$$

and the Weyl fractional integral (of order h) defined by

$$W^{-h}\{f(x)\} = {}_xD_\infty^{-h}\{f(x)\}$$

$$= \frac{1}{\Gamma(h)}\int_x^\infty (\xi - x)^{h-1} f(\xi)\,d\xi, (\mathrm{Re}(h) > 0; f \in J). \quad \ldots(1.12)$$

2 Preliminary Results

Lemma 1: *Assuming the following that*

(i) P,Q,M,N are integers such that $1 \leq M \leq Q$, $0 \leq N \leq P$, α_j (j = 1,…,P), β_j (j = 1,…,Q) are complex numbers.

(ii) $\mathrm{Re}(\mu) > \mathrm{Re}(h); \mathrm{Re}\left(h + \rho L + tR + \sigma\eta_G + \lambda\frac{b_j}{\beta_j}\right) > 0,$

where j = 1,…,M, M is a positive integer,

$\lambda \geq 0, (L = \ell n + p + rv), p, v = 0,1,...,n; w \geq 0, \sigma \geq 0;$

(iii) $|\arg w| < \frac{1}{2}\pi\Omega_2$, where

$$\Omega_2 = \sum_{j=1}^{M} |\beta_j| + \sum_{j=1}^{N} |A_j \alpha_j| - \sum_{j=M+1}^{Q} |B_j \beta_j| - \sum_{j=N+1}^{P} |\alpha_j| > 0;$$

then

$$W^{h-\mu}\left\{ y^{-\mu}\left[c\left(\frac{x}{y}\right)+d\right]^{\theta} S_n^{\alpha,\beta,0}\left[z\left(\frac{x}{y}\right)^{\rho}; r,q,A,B,k,\ell\right] H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{y}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]\right.$$

$$\left.\times \overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right\}$$

$$= y^{-h}\sum_{e,p,u,v}\phi_1(e,p,u,v) z^L \sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!\,F_g}\phi_2(\eta_G)T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^R\binom{\theta}{R}\left(\frac{x}{y}\right)^{\rho L+\sigma\eta_G+tR}$$

$$\times \overline{H}_{P+1,Q+1}^{M,N+1}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(1-h-\rho L-\sigma\eta_G-tR,\lambda;1),(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\rho L-\sigma\eta_G-tR,\lambda;1)\end{matrix}\right]. \quad \ldots(2.1)$$

Proof: Making use of definition (1.12) and the series representation (1.1) and for the generalized polynomial set and the Fox's (See skibinski [14]) respectively in the left hand side of (2.1), then expressing $\overline{H}$-function in Mellin-Barnes type contour integral and changing the order of summations and integration (which is justified under the condition stated), we find that left hand side of (2.1).

$$= \frac{1}{\Gamma(\mu-h)2\pi i}\sum_{e,p,u,v}\phi_1(e,p,u,v)\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!F_g}\phi_2(\eta_G)T^{\eta_G}z^L d^{\theta}\left(\frac{c}{d}\right)^R\binom{\theta}{R}$$

$$\times\int_{-i\infty}^{i\infty}\phi_3(s)\,w^s\left(\frac{x}{y}\right)^{\rho L+\sigma\eta_G+tR+\lambda s}\left\{\int_y^{\infty}(\delta-y)^{\mu-h-1}\delta^{-\mu-\rho L-\sigma\eta_G-\lambda s-tR}d\delta\right\}ds. \quad \ldots(2.2)$$

Now evaluating the inner δ integral in (2.2) with the help of known result Erdélyi et al. [5] and the reinterpreting the resulting Mellin-Barnes contour integral in terms of $\overline{H}$-function, we arrive at the desired result (2.1).

Lemma 2: *Under the conditions stated with Lemma 1, we have*

$$\int_0^\infty y^{-h}\sum_{e,p,u,v}\phi_1(e,p,u,v)z^L\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!\,F_g}\phi_2(\eta_G)T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^R\binom{\theta}{R}\left(\frac{x}{y}\right)^{\rho L+\sigma\eta_G+tR}$$

$$\times\overline{H}_{P+1,Q+1}^{M,N+1}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(1-h-\rho L-\sigma\eta_G-tR,\lambda;1),(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\rho L-\sigma\eta_G-tR,\lambda;1)\end{matrix}\right]f(y)\,dy$$

$$=\int_0^\infty \xi^{-\mu}\left[c\left(\frac{x}{\xi}\right)^t+d\right]^{\theta}S_n^{\alpha,\beta,0}\left[z\left(\frac{x}{\xi}\right)^{\rho};r,q,A,B,k,\ell\right]H_{P_1,Q_1}^{M_1,N_1}\left[t\left(\frac{x}{\xi}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]$$

$$\times\overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{\xi}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]D^{(h-\mu)}f(\xi)\,d\xi,\text{provided } f\in J \text{ and } x>0$$

…(2.3)

Proof: Let ∇ denote the left hand side of (2.3). Then using Lemma 1 and applying (1.12), we get

$$\nabla=\int_0^\infty\left\{\frac{1}{\Gamma(\mu-h)}\int_y^\infty(\xi-y)^{\mu-h-1}\xi^{-\mu}\left[c\left(\frac{x}{\xi}\right)^t+d\right]^{\theta}\right.$$

$$\times S_u^{\alpha,\beta,0}\left[z\left(\frac{x}{\xi}\right)^{\rho};r,q,A,B,k,\ell\right]H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{\xi}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]$$

$$\left.\times\overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{\xi}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]d\xi\right\}f(y)\,dy \qquad \text{…(2.4)}$$

$$=\int_0^\infty \xi^{-\mu}\left[c\left(\frac{x}{\xi}\right)^t+d\right]^{\theta}S_n^{\alpha,\beta,0}\left[z\left(\frac{x}{\xi}\right)^{\rho};r,q,A,B,k,\ell\right]H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{\xi}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]$$

$$\times\overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{\xi}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\left\{\int_y^\infty\frac{(\xi-y)^{\mu-h-1}}{\Gamma(\mu-h)}f(y)\,dy\right\}d\xi, \qquad \text{…(2.5)}$$

where it has been assumed that the change of order of integration is permissible as in the proof of lemma 1. Now by using (1.11), we get easily the right hand side of (2.3).

Main Theorem

If $f \in J, D^{\mu-h}\{f\}$ exists, $\lambda > 0, x > 0, |\arg w| < \frac{\pi}{2}\Omega_2, \Omega_2 > 0, \mathrm{Re}(\mu) > \mathrm{Re}(h) > 0,$ then the solution of the integral equation (1.5) is given by

$$f(x) = \frac{\lambda}{2\pi i} x^{\mu-1} \lim_{\gamma\to\infty} \int_{C-i\gamma}^{c+i\gamma} \frac{x^{-s}\phi_4(s)}{\psi(s,z,T,w)} ds, \qquad \ldots(2.6)$$

where

$$\phi_4(s) = \int_0^\infty x^{s-1} g(x)\, dx, \qquad \ldots(2.7)$$

and

$$\psi(s,z,T,w) = \sum_{e,p,u,v} \phi_1(e,p,u,v) z^L \sum_{G=0}^{\infty} \sum_{g=1}^{M_1} \sum_{R=0}^{\infty} \frac{(-1)^G}{G!\, F_g} \phi_2(\eta_G) T^{\eta_G} d^{\theta} \left(\frac{c}{d}\right)^R \binom{\theta}{R}$$

$$\phi_5\left(\frac{-\rho L - s - \sigma\eta_g - tR}{\lambda}\right) - w\left(\frac{s + \rho L + \sigma\eta_G + tR}{\lambda}\right), \qquad \ldots(2.8)$$

provided that

$$-\min_{1\le j\le M} \mathrm{Re}\left(\frac{b_j}{\beta_j}\right) < \mathrm{Re}\left(\frac{s + \rho L + \sigma\eta_G + tR}{\lambda}\right) < \min_{1\le j\le N} \mathrm{Re}\left\{\frac{(1-a_j)}{A_j}\right\}$$

where $L = \ell n + p + rv : p, v = 0,1,\ldots,n.$

Replace f by $D^{\mu-h}\{f\}$ in left hand side of (2.3), we have

$$g(x) = \int_0^\infty y^{-h} \sum_{e,p,u,v} \phi_1(e,p,u,v) z^L \sum_{G=0}^{\infty} \sum_{g=1}^{M_1} \sum_{R=0}^{\infty} \frac{(-1)^G}{G!\, F_g} \phi_2(\eta_G) T^{\eta_G} d^{\theta} \left(\frac{c}{d}\right)^R \binom{\theta}{R}$$

$$\times \left(\frac{x}{y}\right)^{\rho L + \sigma\eta_G + tR} \overline{H}^{M,N+1}_{P+1,Q+1}\left[w\left(\frac{x}{y}\right)^{\lambda} \middle| \begin{matrix} (1-h-\rho L-\sigma\eta_G - tR, \lambda; 1), (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q}, (1-\mu-\sigma\eta_G - \rho L - tR, \lambda; 1) \end{matrix} \right]$$

$$\times D^{\mu-h}\{f(y)\}\, dy. \qquad \ldots(2.9)$$

Now taking Mellin transform of both sides, we get

$$\phi_4(s) = \int_0^\infty \sum_{e,p,u,v} \phi_1(e,p,u,v)\, z^L \sum_{G=0}^\infty \sum_{g=1}^{M_1} \sum_{R=0}^\infty \frac{(-1)^G}{G!\, F_g} \phi_2(\eta_G)\, T^{\eta_G} d^\theta \left(\frac{c}{d}\right)^R \binom{\theta}{R}$$

$$\times y^{-h-\rho L-\sigma\eta_G - tR} \left\{ \int_0^\infty x^{(s+\rho L+\sigma\eta_G+tR)-1}\, \overline{H}^{M,N+1}_{P+1,Q+1}\left[w\left(\frac{x}{y}\right)^\lambda \middle| \begin{matrix} (1-h-\rho L-\sigma\eta_G-tR,\lambda;1), \\ (b_j,\beta_j)_{1,M}, (b_j,\beta_j;B_j)_{M+1,Q}, \end{matrix} \right.\right.$$

$$\left.\left. \begin{matrix} (a_j,\alpha_j;A_j)_{1,N}, (a_j,\alpha_j)_{N+1,P} \\ (1-\mu-\sigma\eta_G-\rho L-tR,\lambda;1 \end{matrix} \right] dx \right\} D^{(\mu-h)}\{f(y)\}\, dy\,, \qquad \ldots(2.10)$$

Now by using elementary property of $\overline{H}$-function, assuming the change of order of integration is permissible under the conditions stated with the theorem and on evaluating the inner integral, (2.10) reduces to

$$\frac{\lambda \phi_4(s)\, \Gamma(\mu - s)}{\Gamma(h-s)\, \psi(s,z,T,w)} = M\left[y^{1-h} \;\; D^{\mu-h}\{f(y)\} : s \right] \qquad \ldots(2.11)$$

which on applying Mellin inversion theorem gives

$$D^{\mu-h}\{f(y)\} = \frac{\lambda}{2\pi i} \lim_{\gamma\to\infty} \int_{c-i\gamma}^{c+i\gamma} y^{h-s-1} \frac{\Gamma(\mu-s)\, \phi_4(s)\, ds}{\Gamma(h-s)\, \psi(s,z,T,w)} \qquad \ldots(2.12)$$

Now operating upon both the sides by $D^{h-\mu}$ defined by (1.11) and then on changing the order of integration which is permissible under the conditions stated with the theorem, we obtain

$$f(y) = \frac{\lambda}{2\pi i\, \Gamma(\mu-h)} \lim_{\gamma\to\infty} \int_{c-i\gamma}^{c+i\gamma} \frac{\Gamma(\mu-s)}{\Gamma(h-s)\, \Psi(s,z,T,w)}$$

$$\times \left\{ \int_0^y \xi^{h-s-1} (y-\xi)^{\mu-h-1}\, d\xi \right\} \phi_4(s)\, ds \qquad \ldots(2.13)$$

which on evaluating the inner integral by appealing to the well known definition of Beta function, finally yield the required result (2.6).

Lemma 3: *If*

(i) P, Q, M, N are integers such that $1\le M\le Q, 0\le N\le P, \alpha_j\,(j=1,...,P)$, $\beta_j\,(j=1,...,Q)$ are complex numbers;

(ii) $\mathrm{Re}(\mu)>\mathrm{Re}(h)\,;\,\mathrm{Re}(h+\rho L+\sigma\eta_G+tR+\delta k_1+\lambda\frac{b_j}{B_j})>0,$

where j = 1,…,M, M is a positive integer,
$\sigma\le 0, \lambda\ge 0, L\,(\ell n+p+rv), (p,v=0,1,...,n);$

(iii) m_1 be an arbitrary positive integer and the coefficients $A_{n_1,k_1}\,(n_1,k_1\ge 0)$ be arbitrary constants, real or complex;

(iv) $|\arg w|<\frac{1}{2}\Omega_2$ where Ω_2 is defined by

$$\Omega_2=\sum_{j=1}^{M}|\beta_j|+\sum_{j=1}^{N}|A_j\alpha_j|-\sum_{j=M+1}^{Q}|B_j\beta_j|-\sum_{j=N+1}^{P}|\alpha_j|>0$$

Then

$$W^{h-\mu}\left\{y^{-\mu}S_n^{\alpha,\beta,0}\left[z\left(\frac{x}{y}\right)^{\rho};r,q,A,B,k,\ell\right]\left[c\left(\frac{x}{y}\right)^{t}+d\right]S_{n_1}^{m_1}\left[A\left(\frac{x}{y}\right)^{\delta}\right]\right.$$

$$\left.H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{y}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]\overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right\}$$

$$=y^{-h}\sum_{e,p,u,v}\phi_1(e,p,u,v)\,z^{L}\sum_{k_1=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k_1}}{k_1!}I^{k_1}A_{n_1,k_1}\beta^{v}z^{rv-n}\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}$$

$$\times\frac{(-1)^G}{G!\,F_g}\phi_2(\eta_G)\,T^{\eta_G}\,d^{\theta}\left(\frac{c}{d}\right)^{R}\binom{\theta}{R}\left(\frac{x}{y}\right)^{\rho L+\sigma\eta_G+tR+\delta k_1}$$

$$\overline{H}_{P+1,Q+1}^{M,\,N+1}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(1-h-\rho L-\sigma\eta_G-tR-\delta k_1,\lambda;1),(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\rho L-\sigma\eta_G-tR-\delta k_1,\lambda;1)\end{matrix}\right]\quad\ldots(2.14)$$

Proof: To prove Lemma 3, firstly we use the definition of Weyl fractional integral given in (1.12), expres the generalized polynomial set, H-function in series representation, a general class of polynomials and $\overline{H}$-function, then we change the order of summations and integration (which is justified under the stated conditions), evaluate the integral and reinterpreting the resulting Mellin-Barnes contour integral in terms of $\overline{H}$-function, we get the required result.

3 Special Cases

(1) If we set $n=q=k=B=0, \ell=r=-1$ and $A=1$ in (2.1), (2.3), (2.6) and (2.14), we get the following results

(1.a)

$$W^{h-\mu}\left\{y^{-\mu}\left[c\left(\frac{x}{y}\right)^{t}+d\right]^{\theta} H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{y}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]\right.$$
$$\left.\times\overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right\}$$

$$=y^{-h}\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!F_g}\phi_2(\eta_G)\,T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^{R}\binom{\theta}{R}\left(\frac{x}{y}\right)^{\sigma\eta_G+tR}$$
$$\times\overline{H}_{P+1,Q+1}^{M,\,N+1}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(1-h-\sigma\eta_G-tR,\lambda;1),(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\sigma\eta_G-tR,\lambda;1)\end{matrix}\right] \quad \ldots(3.1)$$

(1.b)

$$\int_0^{\infty} y^{-h}\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!F_g}\phi_2(\eta_G)\,T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^{R}\binom{\theta}{R}\left(\frac{x}{y}\right)^{\sigma\eta_G+tR}$$
$$\cdot\overline{H}_{P+1,Q+1}^{M,\,N+1}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(1-h-\sigma\eta_G-tR,\lambda;1),(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\sigma\eta_G-tR,\lambda;1)\end{matrix}\right]f(y)\,dy$$

$$=\int_0^{\infty}\xi^{-\mu}\left[c\left(\frac{x}{\xi}\right)^{t}+d\right]^{\theta} H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{\xi}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]$$

$$\times \overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{\xi}\right)^{\lambda} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right] D^{h-\mu} f(\xi)\, d\xi. \qquad \ldots(3.2)$$

(1.c)

$$\int_0^{\infty} y^{-\mu}\left[c\left(\frac{x}{y}\right)^{t} + d \right]^{\theta} H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{y}\right)^{\sigma} \middle| \begin{matrix} (e_{P_1},E_{P_1}) \\ (f_{Q_1},F_{Q_1}) \end{matrix} \right]$$

$$\times \overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{y}\right)^{\lambda} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right] f(y)\, dy = g(x)$$

has its solution given by

$$f(x) = \frac{\lambda x^{\mu-1}}{2\pi i} \underset{\gamma\to\infty}{\mathrm{Lt}} \int_{c-i\gamma}^{c+i\gamma} x^{-s}\left[\sum_{G=0}^{\infty} \sum_{g=1}^{M_1} \sum_{R=0}^{\infty} \frac{(-1)^G}{G!\,F_g} \phi_2(\eta_G) T^{\eta_G} d^{\theta}\left(\frac{c}{d}\right)^{R} \binom{\theta}{R} \right.$$

$$\left. \times \phi_5\left(\frac{-s-\sigma\eta_G - tR}{\lambda}\right) w^{\left(\frac{-s-\sigma\eta_G - tR}{\lambda}\right)} \right]^{-1} \phi_4(s)\, ds. \qquad \ldots(3.3)$$

(1.d)

$$W^{(h-\mu)}\left\{ y^{-\mu}\left[c\left(\frac{x}{y}\right)^{t} + d \right]^{\theta} S_{n_1}^{m_1}\left[A\left(\frac{x}{y}\right)^{\delta} \right] H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{y}\right)^{\sigma} \middle| \begin{matrix} (e_{P_1},E_{P_1}) \\ (f_{Q_1},F_{Q_1}) \end{matrix} \right] \right.$$

$$\left. \times \overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{y}\right)^{\lambda} \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q} \end{matrix} \right] \right\}$$

$$= y^{-h} \sum_{k_1=0}^{[n_1/m_1]} \frac{(-n_1)_{m_1 k_1}}{k_1!} I^{k_1} A_{n_1,k_1} \sum_{G=0}^{\infty} \sum_{g=1}^{M_1} \sum_{R=0}^{\infty} \frac{(-1)^G}{G!\,F_g}$$

$$\phi_2(\eta_G)\, T^{\eta_G} d^{\theta}\left(\frac{c}{d}\right)^{R} \binom{\theta}{R} \left(\frac{x}{y}\right)^{\sigma\eta_G + tR + \delta k_1}$$

$$\times \overline{H}_{P+1,Q+1}^{M,N+1}\left[w\left(\frac{x}{y}\right)^{\lambda} \middle| \begin{matrix} (1-h-\sigma\eta_G - tR - \delta k_1, \lambda; 1), (a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}, (1-\mu-\sigma\eta_G - tR - \delta k_1, \lambda; 1) \end{matrix} \right] \qquad \ldots(3.4)$$

(2) If we set $A=1, B=q=k=0$ and $\ell=-1$ in (2.1), (2.3), (2.6) and (2.14), we obtain the following

(2.a)

$$W^{h-\mu}\left\{y^{-\mu}\left[c\left(\frac{x}{y}\right)^{t}+d\right]^{\theta} H_{n}^{(r)}\left[z\left(\frac{x}{y}\right)^{\rho};\alpha,\beta\right] H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{y}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]\right.$$

$$\left.\times\overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right\}$$

$$=y^{-h}\sum_{v=0}^{n}\sum_{u=0}^{n}\frac{(-v)_u(-\alpha-ru)_n}{u!v!}\beta^{v}z^{rv-n}\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!F_g}$$

$$\phi_2(\eta_G)\,T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^{R}\binom{\theta}{R}\left(\frac{x}{y}\right)^{\sigma\eta_G+tR+\rho(rv-n)}$$

$$\times\overline{H}_{P+1,Q+1}^{M,N+1}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(1-h-\sigma\eta_G-tR-\rho rv+\rho n,\lambda;1),(a_j,\alpha_j)_{1,N};(a_j,\alpha_j;A_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\sigma\eta_G-tR-\rho rv+\rho n,\lambda;1)\end{matrix}\right] \quad \ldots(3.5)$$

(2b)

$$\int_0^{\infty}\xi^{-\mu}\left[c\left(\frac{x}{y}\right)^{t}+d\right]^{\theta} H_{n}^{(r)}\left[z\left(\frac{x}{\xi}\right)^{\rho};\alpha,\beta\right] H_{P_1,Q_1}^{M_1,N_1}\left[T\left(\frac{x}{\xi}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]$$

$$\times\overline{H}_{P,Q}^{M,N}\left[w\left(\frac{x}{\xi}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right] D^{h-\mu}f(\xi)\,d\xi$$

$$=y^{-h}\sum_{v=0}^{n}\sum_{u=0}^{n}\frac{(-v)_u(-\alpha-ru)_n}{u!v!}\beta^{v}z^{rv-n}\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!F_g}$$

$$\phi_2(\eta_G)\,T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^{R}\binom{\theta}{R}\left(\frac{x}{y}\right)^{\sigma\eta_G+tR+\rho(rv-n)}$$

$$\times \overline{H}^{M,N+1}_{P+1,Q+1}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(1-h-\sigma\eta_G-tR-\rho rv+\rho n,\lambda;1),(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\sigma\eta_G-tR-\rho rv+\rho n,\lambda;1)\end{matrix}\right]f(y)\,dy \quad \ldots(3.6)$$

(2.c)

$$\int_0^{\infty} y^{-\mu}\left[c\left(\frac{x}{y}\right)^t+d\right]^{\theta} H_n^{(r)}\left[z\left(\frac{x}{\xi}\right)^{\rho};\alpha,\beta\right] H^{M_1,N_1}_{P_1,Q_1}\left[T\left(\frac{x}{y}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]$$

$$\times \overline{H}^{M,N}_{P,Q}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]f(y)\,dy=g(x)$$

has its solution given by

$$f(x)=\frac{\lambda x^{\mu-1}}{2\pi i}\lim_{\gamma\to\infty}\int_{c-i\gamma}^{c+i\gamma} x^{-s}\left[\sum_{v=0}^{n}\sum_{u=0}^{v}\frac{(-v)_u(-\alpha-ru)_n}{u!\,v!}\beta^v z^{rv-n}\right.$$

$$\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}\frac{(-1)^G}{G!\,F_g}\phi_2(\eta_G)T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^R\binom{\theta}{R}$$

$$\left.\times\phi_5\left(\frac{-s-\sigma\eta_G-tR-\rho rv+\rho n}{\lambda}\right)w^{\left(\frac{-s-\sigma\eta_G-tR-\rho rv+\rho n}{\lambda}\right)}\right]^{-1}\phi_4(s)\,ds. \quad \ldots(3.7)$$

(2.d)

$$W^{h-\mu}\left\{y^{-\mu}\left[c\left(\frac{x}{y}\right)^t+d\right]^{\theta} H_n^{(r)}\left[z\left(\frac{x}{y}\right)^{\rho};\alpha,\beta\right] S^{m_1}_{n_1}\left[A\left(\frac{x}{y}\right)^{\delta}\right] H^{M_1,N_1}_{P_1,Q_1}\left[T\left(\frac{x}{y}\right)^{\sigma}\middle|\begin{matrix}(e_{P_1},E_{P_1})\\(f_{Q_1},F_{Q_1})\end{matrix}\right]\right.$$

$$\left.\times \overline{H}^{M,N}_{P,Q}\left[w\left(\frac{x}{y}\right)^{\lambda}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right]\right\}$$

$$=y^{-h}\sum_{v=0}^{n}\sum_{u=0}^{n}\sum_{k_1=0}^{[n_1/m_1]}\frac{(-v)_u(-\alpha-ru)_n(-n_1)_{m_1k_1}}{u!\,v!\quad k_1!}I^{k_1}A_{n_1,k_1}\beta^v z^{rv-n}\sum_{G=0}^{\infty}\sum_{g=1}^{M_1}\sum_{R=0}^{\infty}$$

$$\frac{(-1)^G}{G!F_g}\phi_2(\eta_G)T^{\eta_G}d^{\theta}\left(\frac{c}{d}\right)^R\binom{\theta}{R}\left(\frac{x}{y}\right)^{\sigma\eta_G+tR+\delta k_1+\rho(rv-n)}$$

$$\times \overline{H}^{M, N+1}_{P+1,Q+1}\left[w\left(\frac{x}{y}\right)^{\lambda} \middle| \begin{matrix} (1-h-\sigma\eta_G-tR-\delta k_1-\rho rv+\rho n,\lambda;1),(a_j,\alpha_j;A_j)_{1,N};(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M};(b_j,\beta_j;B_j)_{M+1,Q},(1-\mu-\sigma\eta_G-tR-\delta k_1-\rho rv+\rho n,\lambda;1) \end{matrix} \right]. \quad \ldots(3.8)$$

(3) The result obtained by Srivastava, H.M. and Raina, R.K. [17] follows as special cases of our result on assigning certain values to parameters in the function involved and by setting $\theta = 0$.

(4) On taking $A_j (j=1,...,N) = B_j (j = M+1,...,Q) = 1, \sigma \to 0$ and $\theta = 0$ in (2.1) and (2.3) the result reduce to known result derived by Goyal, S.P. and Mukherjee, Rohit [8] with t = 0.

(5) Letting $A_j (j=1,...,N) = B_j (j = M+1,...,Q) = 1, \sigma \to 0, n = q = k = B = 0,$ $\ell = r = -1, A = 1$ and $\theta = 0$, the result reduce to a known result derived by Chaurasia, V.B.L. and Patni, Rinku [3]]with n = 0.

(6) Letting $A_j (j=1,...,N) = B_j (j = M+1,...,Q) = 1, \sigma \to 0, n = q = k = B = 0,$ $\ell = r = -1, A = 1$ and $\theta = 0$, the result reduces to a known result derived by Chaurasia, V.B.L. and Shekhawat, Ashok Singh [4] with n = 0.

(7) Taking $A_j (j=1,...,N) = B_j (j = M+1,...,Q) = 1, \sigma \to 0$ and $\theta = 0$ in (2.6), we find a known result of Goyal, S.P. and Mukherjee, Rohit [8] with t = 0.

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