Gen. Math. Notes, Vol. 25, No. 1, November 2014, pp.103-111
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Local Existence of the Solution for Stochastic Functional Differential Equations with Infinite Delay

Le Anh Minh[1], Nguyen Xuan Thuan[2] and Hoang Nam[3]

[1,2,3]Department of Mathematical Analysis
Hong Duc University, Vietnam
[1]E-mail: leanhminh@hdu.edu.vn
[2]E-mail: thuannx7@gmail.com
[3]E-mail: hoangnam@hdu.edu.vn

(Received: 28-6-14 / Accepted: 14-8-14)

Abstract

In this paper we present and prove the existence of solution for stochastic functional differential equations with infinite delay in a separable Hilbert space respects to a local Lipchitz condition.

Keywords: *Local existence, stochastic functional differential equation, local Lipchitz condition, infinite delay.*

1 Introduction

a class of stochastic functional differential equations in a separable Hilbert space H which has the form:

$$\begin{cases} dX(t) = AX(t)dt + f(t, X_t)dt + g(t, X_t)dW(t), & t \geq 0 \\ X(t) = \varphi(t), & t \leq 0 \end{cases} \tag{1}$$

where $A : \mathcal{D}(A) \subset H \to H$ is a linear (possibly unbound) operator, φ is in the phase space $\mathcal{B}$, and X_t is defined as

$$X_t(\theta) = X(t + \theta), \quad -\infty < \theta \leq 0,$$

$f : \mathbb{R}_+ \times \mathcal{B} \to H$, $g : \mathbb{R}_+ \times \mathcal{B} \to L_2^0$ are continuous functions.

In this paper, we present the condition for the local existence of solutions for (1)

2 Preliminaries

2.1 Basic Concepts of Stochastic Analysis

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$ ie. a right continuous, increasing family of sub σ-fields of $\mathcal{F}$ ($\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$, for all $0 \leq t < s < \infty$).

Definition 2.1. *[2] An H - valued random variable is an $\mathcal{F}$ - measurable function $X : \Omega \to H$ and a collection of random variables $X = \{X(t,\omega) : \Omega \to H | 0 \leq t \leq T\}$ is called a stochastic process.*

Note. In this paper, we write $X(t)$ instead of $X(t,\omega)$.

Definition 2.2. *[2] A stochastic process X is said to be adapted if for every t, $X(t)$ is $\mathcal{F}_t$ - measurable.*

Let K be a separable Hilbert space, Q be a nonnegative difinite symmetric trace-class operator on K, and $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in K, and let the corresponding eigenvalues of Q be λ_n i.e $Qe_n = \lambda_n e_n$, for $n = 1, 2, ...$ Let $w_n(t)$ be a sequence of real valued independent Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.3. *[2] The process*

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} w_n(t) e_n \tag{2}$$

is called a Q - Weiner process in K.

Let $K_Q = Q^{1/2}K$ is a Hilbert space equipped with the norm

$$||u||_{K_Q} = ||Q^{1/2}u||_K, \quad u \in K_Q$$

Clearly, K_Q is separable with complete orthonormal basis $\{\sqrt{\lambda_n} e_n\}_{n=1}^{\infty}$.

Now, let $L_2^0 = L_2^0(K_Q, H)$ be the space of all Hilbert - Schimidt operators from K_Q to H. Then L_2^0 is a separable Hilbert space with norm

$$||L||_{L_2^0} = \sqrt{tr\left((LQ^{1/2})(LQ^{1/2})^*\right)}, \quad L \in L_2^0.$$

Remark 2.4. *For $\kappa \in B(K, H)$ this norm reduce to*

$$||\kappa||_{L_2^0} = \sqrt{tr(\kappa Q \kappa^*)}$$

Now, for any $T \geq 0$, if $\Phi = \{\Phi(t), t \in [0,T]\}$ be an $\mathcal{F}_t$ - adapted, L_2^0- valued process such that

$$E\left(\int_0^T tr\left((\Phi Q^{1/2})(\Phi Q^{1/2})^*\right) ds\right) < \infty$$

then the stochastic integral $\int_0^t \Phi(s)dW(s) \in H$ be well defined by

$$\int_0^t \Phi(s)dW(s) = \lim_{n\to\infty} \sum_{i=1}^n \int_0^t \Phi(s)\sqrt{\lambda_i}e_i dw_i(s) \tag{3}$$

2.2 Phase Space

Let $\mathcal{E}$ be a Banach space, we assume that the phase space $(\mathcal{B}, ||.||_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0]$ into $\mathcal{E}$ satisfying the following fundamental axioms

(A_1) For $a > 0$, if X is a function mapping $(-\infty, a]$ into $\mathcal{E}$, such that $X \in \mathcal{B}$ and X is continuous on $[0, a]$, then for every $t \in [0, a]$ the following conditions hold:

(i) X_t is in $\mathcal{B}$;

(ii) $||X(t)|| \leq \mathcal{H}||X_t||_{\mathcal{B}}$;

(iii) $||X_t||_{\mathcal{B}} \leq K(t) \sup_{s\in[0,t]} ||X(s)|| + M(t)||X_0||_{\mathcal{B}}$;

where $\mathcal{H}$ is a possitive constant, $K, M : [0,\infty) \to [0,\infty)$, K is continuous, M is locally bounded, and they are independent of X.

(A_2) For the function X in (A_1), X_t is a $\mathcal{B}$ - valued continuous function for t in $[0, a]$.

(A_3) The space $\mathcal{B}$ is complete.

Example 2.5. *We recall some useful phase space $\mathcal{B}$.*

(i) Let BC be the space of bounded continuous functions from $(-\infty, 0]$ to $\mathcal{E}$, we define

$$C^0 := \{\varphi \in BC : \lim_{\theta\to-\infty} \varphi(\theta) = 0\}$$

and

$$C^\infty := \{\varphi \in BC : \lim_{\theta\to-\infty} \varphi(\theta) \text{ exists in } \mathcal{E}\}$$

endowed with the norm

$$||\varphi||_{\mathcal{B}} = \sup_{\theta \in (-\infty, 0]} ||\varphi(\theta)||$$

then C^0, C^∞ *satisfies* (A_1) - (A_3). *However, BC satisfies* (A_1), (A_3) *but* (A_2) *is not satisfied.*

(ii) *For any real constant* γ, *we define the functional spaces* C_γ *by*

$$C_\gamma = \left\{ \varphi \in C((-\infty, 0], X) : \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } \mathcal{E} \right\}$$

endowed with the norm

$$||\varphi|| = \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta}||\varphi(\theta)||.$$

Then conditions (A_1) - (A_3) *are satisfied in* C_γ.

We prefer the reader to [3] for more comprehensive properties of phase space.

3 Main Results

Definition 3.1. *[1] For* $\tau > 0$, *a stochastic process* X *is said to be a strong solution of* (1) *on* $(-\infty, \tau]$ *if the following conditions holds*

a) $X(t)$ *is* $\mathcal{F}_t$ *- adapted for all* $0 \le t \le \tau$;

b) $X(t)$ *is almost surely continuous in* t;

c) *for all* $0 \le t \le \tau$, $X(t) \in \mathcal{D}(A)$, $\int_0^t ||AX(s)||ds < +\infty$ *almost surely, and*

$$X(t) = X(0) + \int_0^t AX(s)ds + \int_0^t f(s, X_s)ds + \int_0^t g(s, X_s)dW(s) \quad (4)$$

with probability one;

d) $X(t) = \varphi(t)$ *with* $-\infty < t \le 0$ *almost surely.*

Definition 3.2. *[1] For* $\tau > 0$, *a stochastic process* X *is said to be a mild solution of* (1) *on* $(-\infty, \tau]$ *if the following conditions holds*

a) $X(t)$ *is* $\mathcal{F}_t$ *- adapted for all* $0 \le t \le \tau$;

b) $X(t)$ *is almost surely continuous in* t*;*

c) for all $0 \leq t \leq \tau$, $X(t)$ *is measurable* , $\int_0^t ||X(s)||^2 ds < +\infty$ *almost surely, and*

$$X(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, X_s)ds + \int_0^t T(t-s)g(s, X_s)dW(s) \quad (5)$$

with probability one;

d) $X(t) = \varphi(t)$ *with* $-\infty < t \leq 0$ *almost surely.*

Remark 3.3. *In [4], we proved that if* A *generates a strongly semi-group* $(T(t))_{t\geq 0}$ *in* H *and* $\varphi(0) \in \mathcal{D}(A)$ *then* (5) *can be written as follow*

$$X(t) = T(t)\varphi(0) + \int_0^t T(t-s)f(s, X_s)ds + \int_0^t T(t-s)g(s, X_s)dW(s)$$

This means a strong solution to be a mild one.

We asumme that

(M_1) A generates a strongly semigroup $(T(t))_{t\geq 0}$ in H.

(M_2) $f(t,x)$ and $g(t,x)$ satisfy local Lipchitz conditions respects to second argument i.e. for any $\alpha > 0$ be a given real number, there exists $C_1(\alpha), C_2(\alpha) > 0$ such that

$$||f(t,x) - f(t,y)|| \leq C_1(\alpha)||x-y||_{\mathcal{B}},$$

$$||g(t,x) - g(t,y)||_{L_2^0} \leq C_2(\alpha)||x-y||_{\mathcal{B}}$$

for all $t \geq 0$, $x, y \in \mathcal{B}$ which satisfy $||x||_{\mathcal{B}}, ||y||_{\mathcal{B}} \leq \alpha$.

Since Remark 3.3 we have our main result on the local existence of solution for (1).

Theorem 3.4. *If* (M_1) *and* (M_2) *are satisfied then* (1) *has only local mild solution.*

Proof. Let $T > 0$ be a fixed given real number. Since f, g satisfy Local Lipchitz condition then for each $\alpha > 0$ there exists $\varphi \in \mathcal{B}$ ($||\varphi||_{\mathcal{B}} \leq \alpha$), such that

$$||f(t,\varphi)|| \leq C_1(\alpha)||\varphi||_{\mathcal{B}} + ||f(t,0)|| \leq \alpha C_1(\alpha) + \sup_{s\in[0,T]} ||f(s,0)|| \leq C,$$

$$||g(t,\varphi)|| \leq C_2(\alpha)||\varphi||_{\mathcal{B}} + ||g(t,0)|| \leq \alpha C_2(\alpha) + \sup_{s\in[0,T]} ||g(s,0)|| \leq C.$$

where

$$C = \max\left\{\alpha C_1(\alpha) + \sup_{s\in[0,T]} ||f(s,0)||, \alpha C_2(\alpha) + \sup_{s\in[0,T]} ||g(s,0)||\right\}$$

For $\varphi \in \mathcal{B}$, we chose $\alpha = ||\varphi||_{\mathcal{B}}+1$. Let C_{ad} be a spaces of all functions X which adapted with$\{\mathcal{F}_t\}_{t\geq 0}$ such that $X_0 \in \mathcal{B}$ and $X : [0,T] \to H$ is continuous. C_{ad} is a Banach space with norm

$$||X||_{ad} = ||X_0||_{\mathcal{B}} + \max_{0\leq t\leq T} \left(E||X(t)||^2\right)^{1/2}$$

Let Z be a closed subset of C_{ad} which is defined by

$$Z = \{X \in C_{ad} : X(s) = \varphi(s) \text{ for } s \in (-\infty, 0] \text{ and } \sup_{0\leq s\leq T} ||X(s) - \varphi(0)||_H \leq 1\}$$

Let $U : Z \to Z$ be the operator defined by

$$U(X)(t) = \\ = \begin{cases} T(t)\varphi(0) + \int_0^t T(t-s)f(s,X_s)ds + \int_0^t T(t-s)g(s,X_s)dW(s) & \text{for } t \in [0,T] \\ \varphi(t) & \text{for } t \leq 0 \end{cases}$$

then $U(Z) \subseteq Z$. Indeed,

$$\begin{aligned} &\|U(X)(t) - \varphi(0)\|_H^2 = E||U(X)(t) - \varphi(0)||^2 \\ &= E\left(\left\|T(t)\varphi(0) - \varphi(0) + \int_0^t T(t-s)f(s,X_s)ds + \int_0^t T(t-s)g(s,X_s)dW(s)\right\|\right)^2 \\ &\leq 3E||T(t)\varphi(0) - \varphi(0)||^2 + 3E\left\|\int_0^t T(t-s)f(s,X_s)ds\right\|^2 \\ &\qquad + 3E\left\|\int_0^t T(t-s)g(s,X_s)dW(s)\right\|^2 \\ &\leq 3E||T(t)\varphi(0) - \varphi(0)||^2 + 3MT\int_0^t E||f(s,X_s)||^2 ds + 3M\int_0^t E||g(s,X_s)||^2_{L_2^0} ds. \end{aligned}$$

Since $||X(s)-\varphi(0)|| \leq 1$ for $s \in [0,T]$ and $\alpha = ||\varphi||_{\mathcal{B}}+1$ we have $||X(s)|| \leq \alpha$, implies $||X_s||_{\mathcal{B}} \leq \alpha$ for $s \in [0,T]$. Furthermore,

$$||f(s,X_s)|| \leq C \quad \text{and} \quad ||g(t,X_s)|| \leq C.$$

Hence

$$||U(X)(t)-\varphi(0)||_H^2 \le 3E||T(t)\varphi(0)-\varphi(0)||^2+3MC^2(T^2+T)$$

where $M=\sup\limits_{0\le t\le T}||T(t)||^2$. If T is small enough, such that

$$\sup_{0\le s\le T}\left\{3E||T(s)\varphi(0)-\varphi(0)||^2+3MC^2(T^2+T)\right\}\le 1.$$

then for any $t\in[0,T]$ we have $||U(X)(t)-\varphi(0)||\le 1$. In other words, $U(Z)\subseteq Z$.

Now, for any $X,Y\in Z$,

$$\begin{aligned}
&E||U(X)(t)-U(Y)(t)||^2\\
&=E||\int_0^t T(t-s)[f(s,X_s)-f(s,Y_s)]ds+\int_0^t T(t-s)[g(s,X_s)-g(s,Y_s)]dW(s)||^2\\
&\le 2E\left(\int_0^t ||T(t-s)\left[f(s,X_s)-f(s,Y_s)\right]||ds\right)^2\\
&\qquad\qquad +2E\left(\int_0^t ||T(t-s)\left[g(s,X_s)-g(s,Y_s)\right]||dW(s)\right)^2\\
&\le 2ME\left(\int_0^t ||f(s,X_s)-f(s,Y_s)||ds\right)^2+2ME\left(\int_0^t ||g(s,X_s)-g(s,Y_s)||dW(s)\right)^2\\
&\le 2MC^2T\int_0^t E||X(s)-Y(s)||^2ds+2MC^2\int_0^t E||X(s)-Y(s)||^2ds\\
&\le 2MC^2(T+1)\int_0^t E||X(s)-Y(s)||^2ds.
\end{aligned}$$

Now, for any $a>0$, and $t\in[0,T]$ we have

$$\begin{aligned}
&e^{-at}E||U(X)(t)-U(Y)(t)||^2\\
&\le 2MC^2(T+1)\int_0^t e^{-a(t-s)}e^{-as}E||X(s)-Y(s)||^2ds\\
&\le 2MC^2(T+1)\max_{0\le s\le t}e^{-as}E||X(s)-Y(s)||^2\int_0^t e^{-a(t-s)}ds\\
&\le 2a^{-1}MC^2(T+1)\max_{0\le s\le t}e^{-as}E||X(s)-Y(s)||^2.
\end{aligned}$$

Therefore,

$$\max_{0\le t\le T} e^{-at}E||U(X)(t)-U(Y)(t)||^2 \\ \le 2a^{-1}MC^2(T+1)\max_{0\le s\le T} e^{-as}E||X(s)-Y(s)||^2.$$

Finally, if $a > 2MC^2(T+1)$ then U be a contraction mapping on Z respects to the norm

$$|||X||| = ||X_0||_{\mathcal{B}} + \max_{0\le t\le T}\left(e^{-at}E||X(t)||^2\right)^{1/2}, \quad X \in C_{ad}.$$

Since the norm $|||.|||$ is equivalent to the norm $||.||_{ad}$ then by applying fixed point theorem we conclude that (1) has only local mild solution. □

4 Conclusion

Our main results is theTheorem 3.4, in which we present and prove the local existence of solution to a class of stochastic functional differential equations with infinite delay in a separable Hilbert space has the form (1). In this Theorem, we can replace Local Lipchitz condition (M_2) by some other conditions, for example

(M_3) For any $\alpha > 0$ be a given real number, there exists a constant $C(\alpha) > 0$ such that

$$||f(t,x)-f(t,y)|| + ||g(t,x)-g(t,y)||_{L_0^2} \le C(\alpha)||x-y||_{\mathcal{B}}$$

or

(M_3') For any $\alpha > 0$ be a given real number, there exists a constant $C(\alpha) > 0$ such that

$$\max\{||f(t,x)-f(t,y)||, ||g(t,x)-g(t,y)||_{L_0^2}\} \le C(\alpha)||x-y||_{\mathcal{B}}$$

Acknowledgements: The authors thank to all our coworkers for their valued comments.

References

[1] P.H. Bezandry and T. Diagana, *Almost Periodic Stochastic Processes*, Springer, (2011).

[2] L. Gawarecki and V. Mandrekar, *Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations*, Springer, (2011).

[3] J.K. Hale and J. Kato, Phase space for retarded equations with infinite delay, *Fukcialaj Ekvacioj*, 21(1978), 11-41.

[4] L.A. Minh, H. Nam and N.X. Thuan, Local existence of solution to a class of stochastic differential equations with finite delay in Hilbert space, *Applied Mathematics*, 4(2013), 97-101.