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About Uniform Limitation of Normalized Eigen Functions of T. Regge Problem in the Case of Weight Functions, Satisfying to Lipschitz Condition

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Abstract

In this work, we estimate normalize eigenfunctions to the T.Regge problem whenever the weight functions satisfies Lipschitz condition.

KeyWords: Eigenfunctions, normalize, Lipschitz condition

1. INTRODUCTION

Let's consider spectral problem ($q(x) \in C_{[0;a]}$

, $\rho(x) \in Lip1$ *and* $m \le \rho(x) \le M$)

$$-y''(x)+q(x)y(x)=\lambda^2\rho(x)y(x) \qquad (0<x<a)$$

(1)

$$y(0)=0, \qquad y'(a)-i\lambda y(a)=0$$

(2)

$$\left(\int_0^a \rho(x)|y(x)|^2\,dx\right)^{\frac{1}{2}}=1,$$ where λ - is spectral parameter.

(3)

The problem (1) - (2) arises in different questions of mathematical physics. T.Regge [1], who studied it (in case of $\rho(x)\equiv 1$) in connection with the theory of dispersion has shown, that if $q(x)$ in left semi-neighborhood of point a satisfies to condition $q(x)\sim\ C_\mu(a-x)^\mu\ x\to a-0;\ \mu\geq 0\ C_\mu\neq 0$, the problem has discrete spectrum λ_n and system of eigenfunctions of problem (1) - (2) is full of century $L^2{}_{[0,a]}$. In work [2] is studied asymptotes of proper values and received 2 multiple decomposition in uniform converging numbers on eigenfunctions from which 2 multiple completeness of eigenfunctions of century $L^2{}_{[0,a]}$. In case of equation of $2n$ order and $\rho(x)\equiv 1$ similar problem is considered in works [3] - [4]. And for $\rho(x)\not\equiv 1$ asymptotic of eigenvalues for more general problem is studied in works [5] - [6].

In work [7] is considered the case of weight functions close to Holder class where maximal growth rate of eigenfunctions of problem (1) - (3) is studied.

The purpose of the present work is reception of uniform estimations for normalized eigenfunctions of problem (1) - (3) in case of weight functions, satisfying to Lipschitz condition.

The following is true:

Lemma: For any $\rho(x)\in Lip\,1$ and $\varepsilon>0$ there is function $\rho_\varepsilon(x)\in C^2{}_{[0,a]}$ such, that $\rho_\varepsilon(a)=\rho(a)$, $\rho_\varepsilon(0)=\rho(0)$, $\int_0^a\sqrt{\rho_\varepsilon(x)}dx=\int_0^a\sqrt{\rho(x)}dx$,

$\max\limits_{x\in[0,a]}|\rho(x)-\rho_\varepsilon(x)|\le\varepsilon, \max\limits_{x\in[0,a]}|\rho'_\varepsilon(x)|\le 2N$ and $\max\limits_{x\in[0,a]}|\rho''_\varepsilon(x)|\le\frac{c}{\varepsilon}$, where C is constant independent from $\rho(x)$ and ε.

Proof:

Let's divide interval $[0,a]$ on m equal parts (m-arbitrary) by points $0=x_0<x_1<...<x_{m-1}<x_m=a$, and middle of intervals $[x_{i-1},x_i]$ we shall designate through x'_i (such points m, namely $x'_1,x'_2,...,x'_m$). We shall consider broken line, that connected points $(x_0,\rho(x_0)),(x_1,\rho(x_1)),...,(x_m,\rho(x_m))$.

Obviously, this broken line is function graph $\rho_0(x)$, satisfying to inequalities $\max\limits_{x\in[0,a]}|\rho(x)-\rho_0(x)|\le\frac{N.a}{m}$ and $|\rho'_0(x)|\le N$ there, where $\rho'_0(x)$ exists.

Let's consider now other broken line connecting points $(x_0,\rho_0(x_0)),(x'_1,\rho_0(x'_1)),(x'_2,\rho_0(x'_2))...,(x'_m,\rho_0(x'_m)),(x_m,\rho_0(x_m))$.

Obviously, this broken line is the schedule of function $\rho_1(x)$, satisfying to parities $|\rho'_1(x)|\le N$ there, where $\rho'_1(x)$ exists and $\max\limits_{x\in[0,a]}|\rho(x)-\rho_1(x)|\le\frac{N.a}{m}$.

On sites $[x'_i,x'_{i+1}]$ where $i=1,2,...,m-1$ we shall construct curve $\overline{\rho_\varepsilon}(x)$ as polynomials parities $\overline{\rho_\varepsilon}(x)=\rho_1(x)+\frac{p_i}{8\Delta^3}(x-x_i)^4-\frac{3p_i}{4\Delta}(x-x_i)^2$ where $\Delta=x_i-x'_i=\frac{a}{2m},\quad p_i=\frac{\rho'_0(x'_i)-p'_0(x'_{i+1})}{2}$. On sites $[0,x'_1]$ and $[x'_m,a]$ we shall get $\overline{\rho_\varepsilon}(x)=\rho_1(x)$ (in the same place $\rho_1(x)=\rho_0(x)$).

Let's put $m=2.[\frac{Na}{\varepsilon}]+2$. Direct check shows, that all conditions of lemma except for equality $\int\limits_0^a\sqrt{\overline{\rho_\varepsilon}(x)}dx=\int\limits_0^a\sqrt{\rho(x)}dx$ are executed. In addition, inequality $|\overline{\rho'_\varepsilon}(x)|\le N$ takes place (N and $2N$ in condition of lemma) now let's find number δ from condition $\int_0^a \sqrt{\overline{\rho_\varepsilon}(x)}(1+\delta\sin\frac{\pi}{a}x)dx = \int_0^a \sqrt{\rho(x)}dx$,

hence $\delta = \dfrac{\int_0^a (\sqrt{\rho(x)} - \sqrt{\rho_\varepsilon(x)})dx}{\int_0^a \sin\frac{\pi}{a}x\sqrt{\rho_\varepsilon(x)}dx}$. Obviously at small ε number δ is also not enough, and function $\rho_\varepsilon(x) = \overline{\rho_\varepsilon}(x)(1+\delta\sin\frac{\pi}{a}x)^2$ satisfies to conditions of lemma.

Let's designate through $Q_{[0,a]}$ class of continuous on $[0,a]$ functions $q(x)$, satisfying to inequality $\left|\int_{a_0}^{a_1} q(x)dx\right| < C_Q$, where $C_Q = cont$ and $[a_0, a_1] \subseteq [0,a]$.

Let's consider countable subset $\{q_i(x) |\ i \in N\} \equiv \overline{Q}_{[0,a]}$ of class $Q_{[0,a]}$ satisfying to condition $\underset{i\to\infty}{Lim}\int_0^x\int_0^t q_i(s)dsdt \equiv f_o(x)$, where $f_o(x)$ function satisfying to Lipschitz condition, and convergence is uniform on $[0,a]$.

Let $\rho \neq 1, \rho > 0, \lambda \in C -$ is complex, $\mathrm{Im}(\lambda) < const$ (that is ρ - is fixed and λ - is arbitrary of strip $\mathrm{Im}(\lambda) < const$ of complex plane).

Let's designate through $y(x,\lambda,q)$ solution of Cauchy problem

$$-y''(x) + q(x)y(x) = \lambda^2\rho\, y(x)\ ,\ x \in (0,a)$$
$$y(0) = 0, y'(0) = 1.$$

Then the following is true:

Theorem: There is constant $C_0 \equiv C_0(Q_{[0,a]})$ (uniform for all class $Q_{[0,a]}$) such, that

$$\max_{x\in[0,a]} \frac{|y(x,\lambda,q)|}{(\int_0^a \rho|y(x,\lambda,q)|^2 dx)^{\frac{1}{2}}} < C_0 \text{ for every value large enough by module } \lambda.$$

From this theorem and previous lemma follows important consequence

Consequence: Let $q(x)$ - is continuous function, and $\rho(x) \in Lip\, 1$. Then solution of Cauchy problem

$$-y''(x) + q(x)y(x) = \lambda^2 \rho\, y(x)\ ,\ x \in (0,a), \rho(a) \neq 1$$
$$y(0) = 0, y'(0) = 1.$$

Satisfies to parity $\max\limits_{x \in [0,a]} \dfrac{|y(x)|}{(\int_0^a \rho(x).|y(x)|^2 dx)^{\frac{1}{2}}} < const < \infty$

For every value large enough λ from strip $\operatorname{Im}(\lambda) < const$.

Proof:

As solution of Cauchy problem continuously depends on weight function $\rho(x)$ and functional $(\int_0^a \rho(x)|y(x,\lambda)|^2 dx)^{\frac{1}{2}}$ also continuously depends on $\rho(x)$.

Hence, functional $\dfrac{\max\limits_{x\in[0,a]} |y(x,\lambda)|}{(\int_0^a \rho(x).|y(x,\lambda)|^2 dx)^{\frac{1}{2}}}$ also continuously depends on weight function $\rho(x)$. Hence, there is number $\varepsilon(R)$ such, that

$$\frac{\max\limits_{x\in[0,a]} |y(x,\lambda,\bar{\rho})|}{(\int_0^a \bar{\rho}(x).|y(x,\lambda,\bar{\rho})|^2 dx)^{\frac{1}{2}}} \geq \frac{1}{2} . \frac{\max\limits_{x\in[0,a]} |y(x,\lambda,\rho)|}{(\int_0^a \rho(x).|y(x,\lambda,\rho)|^2 dx)^{\frac{1}{2}}}, \quad \text{if} \quad |\lambda| \leq R \quad \text{and}$$

$|\rho(x) - \bar{\rho}(x)| \leq \varepsilon(R)$

Where $R > 0$ is arbitrary constant . Let's take some R and by $\varepsilon(R)$ and lemma let's plot function $\rho_\varepsilon(x)$ approaching $\rho(x)\,(|\rho(x) - \bar{\rho}(x)| \leq \varepsilon(R))$ $\rho_\varepsilon(a) = \rho(a)$. Now let's consider Cauchy problem with weight function $\rho_\varepsilon(x)$ instead of $\rho(x)$. In this problem $\rho_\varepsilon(x) \in C^2{}_{[0,a]}$ and consequently we can make double replacement $\xi = \int_0^x \frac{dt}{A^2(t)}$, $y(x) = A(x).\eta(\xi(x))$,

where $A(x) = \rho^{-\frac{1}{4}}{}_{\varepsilon}(x).\rho^{\frac{1}{4}}(a)$.

As a result of such replacement we shall obtain problem:

$$-\eta''(\xi) + (q(x)A(x) - A''(x)).A^3(x)\eta(\xi) = \lambda^2\rho(a)\eta(\xi)\ ,\ \xi \in (0, \int_0^a \frac{dt}{A^2(t)}),$$

$\eta(0) = 0,$

$\eta'(0) = A(0)$, As $A(0) = \rho^{-\frac{1}{4}}(0).\rho^{\frac{1}{4}}(a)$ and

$$\int_0^a \frac{dt}{A^2(t)} = \int_0^a \frac{dt}{(\frac{\sqrt{\rho(a)}}{\sqrt{\rho_\varepsilon(t)}})} = \frac{1}{\sqrt{\rho(a)}}\int_0^a \sqrt{\rho_\varepsilon(t)}dt$$

$= \frac{\int_0^a \sqrt{\rho(t)}dt}{\sqrt{\rho(a)}} = \overline{a} = const$ (Independent from ε), designating

$A^3(x)[q(x)A(x) - A''(x)] \equiv \overline{q_\varepsilon}(\xi)$ we shall obtain problem:

$$-\eta''(\xi) + \overline{q_\varepsilon}(\xi)\eta(\xi) = \lambda^2\rho(a)\eta(\xi), \xi \in (0, \overline{a}),$$
$$\eta(0) = 0, \eta'(0) = \sqrt[4]{\frac{\rho(a)}{\rho(0)}}$$

Estimated in theorem functional does not depend on value $y'(0)$ (as all solutions of our equation, satisfying to condition $y(0) = 0$, can be obtained from solution of problem with conditions $y(0) = 0, y'(0) = 1$, by multiplication to constant, which will be reduced in our functional) and consequently if to show, that $\left|\int_0^t q_\varepsilon(\xi)d\xi\right|$ in regular intervals on ε and $t \in [0, \overline{a}]$ is limited for all small $\varepsilon > 0$ under theorem there will be constant $C_0 > 0$ such, that

$$\max \frac{|\eta(\xi)|}{(\int_0^{\bar{a}} \rho(a).|\eta(\xi)|^2 d\xi)^{\frac{1}{2}}} \le C_0,$$

From here and from parities $y(x) = A(x).\eta(\xi(x)),\ \xi(x) = \int_0^x \frac{dt}{A^2(t)}$ obviously follows, that exists $\overline{C_0} > 0$ such, that $\max\limits_{x\in[0,a]} \frac{|y(x,\lambda,\rho)|}{(\int_0^a \rho(x).|y(x,\lambda,\rho)|^2 dx)^{\frac{1}{2}}} < \overline{C_0}$

For every $|\lambda| \le R$, and by arbitrariness R, and for all considered λ (let's remind, that $\operatorname{Im}(\lambda) < const$).

Let's estimate $\left|\int_0^t q_\varepsilon(\xi)d\xi\right|$. Passing to variable x in integral we shall get

$$\left|\int_0^t q_\varepsilon(\xi)d\xi\right| = \left|\int_0^s [q(x)A(x) - A''(x)]A^3(x).\xi'(x)dx\right| = \left|\int_0^s [q(x)A(x) - A''(x)]A^3(x)\frac{dx}{A^2(x)}\right| =$$

$$\left|\int_0^S \frac{q(x)}{A^{-2}(x)}dx - \int_0^s A(x)A''(x)dx\right| \le \left|\int_0^S \frac{q(x)}{A^{-2}(x)}dx\right| + \left|\int_0^s A(x)A''(x)dx\right| =$$

$$\left|\int_0^S \frac{q(x)}{A^{-2}(x)}dx\right| + \left|[A(x)A'(x)]\Big|_0^s - \int_0^s [A'(x)]^2 dx\right| \le \left|\int_0^S \frac{q(x)}{A^{-2}(x)}dx\right| + |A(s)A'(s)| +$$

$$|A(0)A'(0)| + \left|\int_0^s [A'(x)]^2 dx\right|, where\ \ s \in [0,a].$$

From definition $A(x) = \sqrt[4]{\frac{\rho(a)}{\rho_\varepsilon(x)}}$ follows, that

$$A(0) = \sqrt[4]{\frac{\rho(a)}{\rho_\varepsilon(0)}} = \sqrt[4]{\frac{\rho(a)}{\rho(0)}},$$

$$A'(x) = -\frac{1}{4}\rho^{\frac{1}{4}}(a).\rho_{\varepsilon}^{-\frac{5}{4}}(x).\rho'_{\varepsilon}(x) = -\frac{\sqrt[4]{\rho(a)}.\rho'_{\varepsilon}(x)}{4\sqrt[4]{\rho^{5}{}_{\varepsilon}(x)}},$$

$$A'(0) = -\frac{\sqrt[4]{\rho(a)}.\rho'_{\varepsilon}(0)}{4\sqrt[4]{\rho^{5}{}_{\varepsilon}(0)}} = \frac{-\rho'_{\varepsilon}(0)}{4\rho(0)}.\sqrt[4]{\frac{\rho(a)}{\rho(0)}}$$ and consequently

$$\left|\int_0^t q_{\varepsilon}(\xi)d\xi\right| \le \int_0^a \frac{|q(x)|}{\sqrt[4]{\rho(a)}}.\sqrt[4]{\rho_{\varepsilon}(x)}dx + \left|\frac{\sqrt{\rho(a)}.\rho'_{\varepsilon}(s)}{4.\sqrt{\rho_{\varepsilon}{}^{3}(s)}}\right| + \left|\frac{\rho'_{\varepsilon}(0)}{4.\rho(0)}.\sqrt{\frac{\rho(a)}{\rho(0)}}\right| + \int_0^a \frac{\sqrt{\rho(a)}.[\rho'_{\varepsilon}(s)]^2}{16.\sqrt{\rho_{\varepsilon}{}^{5}(s)}}$$

On lemma $|\rho'_{\varepsilon}(x)| \le 2.N$ and $\rho(x) - \varepsilon \le \rho_{\varepsilon}(x) \le \rho(x) + \varepsilon$, hence

$$\left|\int_0^t q_{\varepsilon}(\xi)d\xi\right| \le \int_0^a \frac{|q(x)|}{\sqrt[4]{\rho(a)}}.\sqrt[4]{\rho(x)+\varepsilon}dx + \frac{\sqrt{\rho(a)}.N}{2.[\min\limits_{x\in[0,a]}\rho(x)-\varepsilon]^{\frac{3}{2}}} + \frac{N}{2\rho(0)}.\sqrt{\frac{\rho(a)}{\rho(0)}}$$

$$+\int_0^a \frac{\sqrt{\rho(a)}.N^2}{4.\sqrt{[\rho(x)-\varepsilon]^5}}dx.$$

As ε is not enough, consequence is proved.

Let's prove now theorem for what we shall evaluate $\varphi'_0(x,\lambda),\varphi(x,\lambda)$ and $\varphi'(x,\lambda)$ ($\varphi(x,\lambda)$ is solution of equation (1) satisfying to entry conditions $\varphi(0,\lambda)=0, \varphi'(0,\lambda)=1$, and $\varphi_0(x,\lambda)$ is solution of such Cauchy problem with constant coefficient $\rho(x)\equiv\rho$). As it has been established [8, p. 22]

$$\varphi_0(x,\lambda) = \frac{-\sigma_1 - i\delta_1}{\sigma_1^2+\delta_1^2}(-\sinh\sigma_1 x.\cos\delta_1 x + i\cosh\sigma_1 x.\sin\delta_1 x)$$ and

consequently $\varphi'_0(x,\lambda) = (\cosh\sigma_1 x.\cos\delta_1 x - i\sinh\sigma_1 x.\sin\delta_1 x)$ (simple transformations are lowered). As at greater $|\lambda|$ parities $\sigma_1 \approx \sqrt{\rho}.\sigma$ and $\delta_1 \approx \sqrt{\rho}.\delta$ take place, then obviously $|\varphi'_0(x,\lambda)| < const$ is regular on $x\in[0,a]$ and λ from considered strip. Let's consider number

$$\varphi(x,\lambda)=\varphi_0(x,\lambda)+\int_o^x[q(\tau_1)-q].\varphi_0(x-\tau_1,\lambda)\,\varphi_0(\tau_1,\lambda)d\tau_1+\sum_{i=2}^{\infty}\int_0^x[q(\tau_1)-q].\varphi_0(x-\tau_1,\lambda)\int_0^{\tau_1}\dots$$
$$\int_0^{\tau_{i-1}}[q(\tau_i)-q]\varphi_0(\tau_{i-1}-\tau_i,\lambda)\,\varphi_0(\tau_i,\lambda)\,d\tau_i\dots.d\tau_1$$

Let's enter designations $f_i(x)$ for $i-$ member of series

$(\,f_1(x)=\int_0^x[q(\tau_1)-q].\varphi_0(x-\tau_1,\lambda)d\tau_1,\dots)$, As result we shall get:

$$\varphi(x,\lambda)-\varphi_0(x,\lambda)=\sum_{i=1}^{\infty}f_i(x)$$

(4)

If $i\geq 1$, then obviously

$f_{i+1}(x)=\int_0^x[q(\tau_1)-q]\varphi_0(x-\tau_1,\lambda).f_i(\tau_1)d\tau_1$ and consequently, integrating in parts, we shall obtain

$$f_{i+1}(x)=\left\{\varphi_0(x-\tau_1,\lambda).f_i(\tau_1).\int_0^{\tau_1}[q(s)-q]ds\right\}\Bigg|_0^x-\int_0^x\{[\varphi_0(x-\tau_1,\lambda).f_i'(\tau_1)-\varphi_0'(x-\tau_1,\lambda).f_i(\tau_1)].$$
$$\left.\int_0^x[q(s)-q]ds\right\}d\tau_1$$

Or

$$f_{i+1}(x)=\int_0^x[\varphi_0'(x-\tau,\lambda)\,f_i(\tau)-\varphi_0'(x-\tau,\lambda)\,f_i'(\tau)].\int_0^{\tau}[q(s)-q]dsd\tau\,,$$

(5)

Absolutely similarly for $f_1(x)$ it is possible to get parity

$$f_1(x)=\int_0^x[\varphi_0'(x-\tau,\lambda)\varphi_0(\tau)-\varphi_0'(x-\tau,\lambda)\varphi_0'(\tau)].\int_0^{\tau}[q(s)-q]dsd\tau \qquad (6)$$

Differentiating parities (5) and (6) on x we shall get parities for derivatives

$$f'_{i+1}(x) = f_i(x).\int_0^x [q(\tau)-q]d\tau + [\int_0^x [q-\lambda^2\rho]\varphi_0(x-\tau,\lambda) f_i(\tau) -$$
$$\varphi'_0(x-\tau,\lambda) f'_i(\tau)].\int_0^\tau [q(\tau)-q]dsd\tau,$$
(7)

$$f'_1(x) = \varphi_0(x,\lambda).\int_0^x [q(\tau)-q]d\tau + [\int_0^x [q-\lambda^2\rho]\varphi_0(x-\tau,\lambda)\varphi_0(\tau,\lambda) -$$
$$\varphi'_0(x-\tau,\lambda)\varphi'_0(\tau,\lambda)].\int_0^\tau [q(\tau)-q]dsd\tau,$$

(8)

(Here, it is considered, that $\varphi''_0(x,\lambda) \equiv (q-\lambda^2\rho)\varphi_0(x,\lambda).$)

From choice of class $Q_{[0,a]}$ follows, that $\left|\int_0^\tau [q(s)-q]ds\right| < Q = const < \infty$

and consequently from (5) and (8) follows, that

$|f_1(x)| \le \frac{2Q.C_0.C'_0}{|\lambda|}x$, $|f'_1(x)| \le \frac{Q.C_0}{|\lambda|} + Q[(\rho + \frac{q}{|\lambda|^2})C_0^2 + C_0'^2]$, where C_0 and C'_0

Such constants for which inequalities $|\varphi_0(x,\lambda)| \le \frac{C_0}{|\lambda|}$ and $|\varphi'_0(x,\lambda)| \le C'_0$ are executed (as $|\varphi_0(x,\lambda)| \le \sqrt{\frac{\sinh^2\sigma_1 x + \sin^2\delta_1 x}{\delta_1^2+\sigma_1^2}}$ then, obviously C_0 and C'_0 exist). Using recurrent parities (5) and (7) we shall obtain, that number (4) converges and moreover $|\varphi(x,\lambda)-\varphi_0(x,\lambda)| \le \frac{\overline{C_0}}{|\lambda|}$ (evaluations $|f_2(x)|, |f_3(x)|,...,$ and $|f'_2(x)|, |f'_3(x)|,...$ are made consistently for $i=2,3,$ …).

Then $\varphi'(x,\lambda)=\varphi_0'(x,\lambda)+\int_0^x[q(\tau)-q]\varphi_0'(x-\tau,\lambda)d\tau$, or, integrating in parts, we shall get

$$\varphi'(x,\lambda)-\varphi_0'(x,\lambda)=\left\{\varphi_0'(x-\tau,\lambda)\varphi(\tau,\lambda).\int_0^\tau[q(s)-q]ds\right\}\Bigg|_0^x-$$
$$-\int_0^x\left\{[\varphi_0'(x-\tau,\lambda)\varphi'(\tau,\lambda)-\varphi_0''(x-\tau,\lambda).\varphi(\tau,\lambda)]\int_0^\tau[q(s)-q]ds\right\}d\tau,$$

As $\varphi_0'(0,\lambda)\equiv 1$ and $\varphi_0''(x-\tau,\lambda)\equiv(q-\lambda^2\rho)\varphi_0(x-\tau,\lambda)$,

$$\varphi'(x,\lambda)-\varphi_0'(x,\lambda)=\varphi(x,\lambda).\int_0^x[q(s)-q]ds-\int_0^x\{[\varphi_0'(x-\tau,\lambda)\varphi'(\tau,\lambda)-$$
$$(q-\lambda^2\rho)\varphi_0(x-\tau,\lambda)\varphi(\tau,\lambda)].\int_0^a[q(s)-q]ds\}d\tau=$$
$$\varphi(x,\lambda).\int_0^x[q(s)-q]ds+\int_0^x\left\{(\lambda^2\rho-q)\varphi_0(x-\tau,\lambda)\varphi(\tau,\lambda).\int_0^a[q(s)-q]ds\right\}d\tau$$
$$-\int_0^x\left\{\varphi_0'(x-\tau,\lambda)\varphi(\tau,\lambda)\int_0^\tau[q(s)-q]ds\right\}d\tau,$$

Subtracting and adding in last integral $\varphi_0'(\tau,\lambda)$ to $\varphi'(\tau,\lambda)$ and representing integral in the form of sum of two integrals we shall obtain

$$\varphi'(x,\lambda)-\varphi_0'(x,\lambda)=\varphi(x,\lambda).\int_0^x[q(s)-q]ds+(\lambda^2\rho-q)\int_0^x\left\{\varphi_0(x-\tau,\lambda)\varphi(\tau,\lambda).-\int_0^\tau[q(s)-q]ds\right\}d\tau$$

$$-\int_0^x\left\{\varphi_0'(x-\tau,\lambda)[\varphi_0'(\tau,\lambda).\int_0^\tau[q(s)-q]ds\right\}d\tau-\int_0^x\left\{\varphi_0'(x-\tau,\lambda)[\varphi'(\tau,\lambda)-\varphi_0'(\tau,\lambda)].\int_0^\tau[q(s)-q]ds\right\}d\tau$$

From this equality follows, that

$$\left|\varphi'(x,\lambda)-\varphi_0'(x,\lambda)\right| \le \left|\varphi(x,\lambda)\right| . \left|\int_0^x [q(s)-q]ds\right| + \left|\lambda^2\rho - q\right| . \int_0^x \left|\varphi_0(x-\tau,\lambda)\right| . \left|\varphi(\tau,\lambda)\right| .$$

$$\left|\int_0^\tau [q(s)-q]ds\right| d\tau + \int_0^x \left|\varphi_0'(x-\tau,\lambda)\right| . \left|\varphi_0'(\tau,\lambda)\right| . \left|\int_0^\tau [q(s)-q]ds\right| d\tau +$$

$$\int_0^x \left|\varphi_0'(x-\tau,\lambda)\right| . \left|\int_0^\tau [q(s)-q]ds\right| . \left|\varphi'(\tau,\lambda)-\varphi_0'(\tau,\lambda)\right| d\tau$$

And consequently, using estimations obtained before for $\left|\varphi_0(x,\lambda)\right|$, $\left|\varphi(x,\lambda)\right|$, $\left|\varphi_0'(x,\lambda)\right|$ and considering, that $\left|\int_0^\tau [q(s)-q]ds\right| \le \left|\int_0^\tau q(s)ds\right| + \left|\int_0^\tau q ds\right| = q\tau + \left|\int_0^\tau q(s)ds\right| < const < \infty,$ we shall obtain

$$\left|\varphi'(x,\lambda)-\varphi_0'(x,\lambda)\right| < R + \int_0^x B(\tau) . \left|\varphi'(\tau,\lambda)-\varphi_0'(\tau,\lambda)\right| d\tau ,$$

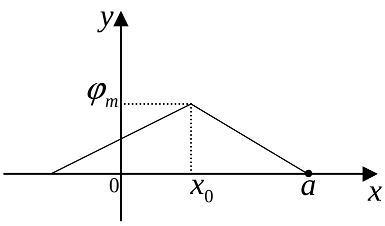
where $R > 0$ and $B(\tau) > 0$ are limited.

Hence, by Gronwall's lemma $\left|\varphi'(x,\lambda)-\varphi_0'(x,\lambda)\right| \le R.e^{\int_0^x B(\tau)d\tau} \le const < \infty$.

As $\left|\varphi_0'(x,\lambda)\right| \le const < \infty$, from last inequality follows, that $\left|\varphi'(x,\lambda)\right| \le const < \infty$ is regular on $x \in [0,a]$ and λ from considered strip.

Let now $\left|\varphi(x,\lambda)\right|$ reaches maximum φ_m in point $x_0 \in [0,a]$,

Maximum $\left|\varphi'(x,\lambda)\right|$ is equal φ_m'.Then function graph $\left|\varphi(x,\lambda)\right|$ lies above triangle with top in point (x_0,φ_m) and lateral faces with angular coefficients φ_m' ,$-\varphi_m'$ accordingly.



And consequently

$$\int_0^a \rho(x)|\varphi(x,\lambda)|^2 dx \ge \int_0^a m.|\varphi(x,\lambda)|^2 dx = m.\int_0^a |\varphi(x,\lambda)|^2 dx \ge m\int_0^{x_0} |\varphi_m + \varphi'_m(x-x_0)|^2 dx$$

$$+ m.\int_{x_0}^a |\varphi_m - \varphi'_m(x-x_0)|^2 dx = m.\int_0^{x_0} [\varphi^2{}_m + 2\varphi_m.\varphi'_m(x-x_0) + \varphi'_m{}^2(x-x_0)^2]dx +$$

$$m.\int_{x_0}^a [\varphi_m^2 - 2\varphi_m.\varphi'_m(x-x_0) + \varphi'^2_m(x-x_0)^2]dx$$

$$= m.\varphi_m^2 \left\{ a + \varphi'_m{}^2 [\frac{(a-x_0)^3 + x_0^3}{3\varphi_m^2}] - \frac{\varphi'_m}{\varphi_m}[x_0^2 + (a-x_0)^2] \right\}.$$

From inequality we got follows, that

$$\frac{\max\limits_{x\in[0,a]}|\varphi(x,\lambda)|}{(\int_0^a \rho(x)|\varphi(x,\lambda)|^2 dx)^{\frac{1}{2}}} \le \frac{1}{\sqrt{m}.\sqrt{a + \frac{(a-x_0)^3 + x_0^3}{3}.(\frac{\varphi'_m}{\varphi_m})^2 - [x_0^2 + (a-x_0)^2]\frac{\varphi'_m}{\varphi_m}}}$$

.

If to enter designations $x_0 = \varepsilon.a$ and $\frac{\varphi'_m}{\varphi_m} = z$ we shall get

$$a + \frac{(a-x_0)^3 + x_0^3}{3}.(\frac{\varphi'_m}{\varphi_m})^2 - [x_0^2 + (a-x_0)^2]\frac{\varphi'_m}{\varphi_m} =$$

$$= a + \frac{a^3(1-\varepsilon)^3 + \varepsilon^3 a^3}{3} z^3 - [a^2\varepsilon^2 + a^2(1-\varepsilon)^2]z =$$

$$= a[az(az-2)(\varepsilon - \frac{1}{2})^2 + \frac{1}{12}(az-3)^2 + \frac{1}{4}].$$

Where $\varepsilon \in [0,1]$ and $z \in (0,\infty)$. Let's enter now designations

$f(\varepsilon,z) = az(az-2)(\varepsilon - \frac{1}{2})^2 + \frac{1}{12}(az-3)^2 + \frac{1}{4}$ and estimate from below $f(\varepsilon,z)$. It is obvious, that

$$f(0,z) = f(1,z) = \frac{1}{4}(a^2z^2 - 2az) + \frac{1}{12}(a^2z^2 - 6az + 9) + \frac{1}{4}$$

$$= \frac{1}{3}[(az - \frac{3}{2})^2 + \frac{3}{4}] \ge \frac{1}{4}.$$

Inside of interval $0 \le \varepsilon \le 1$ is unique critical point $\varepsilon = \frac{1}{2}$, in this point we have $f(\frac{1}{2}, z) = \frac{1}{12}(az-3)^2 + \frac{1}{4} \ge \frac{1}{4}$.. Hence, for any $\varepsilon \in [0,1]$ and $z \in (0,\infty)$ estimation $f(\varepsilon, z) \ge \frac{1}{4}$ is fair, therefore inequality is

$$\frac{\max\limits_{x\in[0,a]} |\varphi(x,\lambda)|}{(\int\limits_0^a \rho(x)|\varphi(x,\lambda)|^2 dx)^{\frac{1}{2}}} \le \frac{1}{\sqrt{m}.\sqrt{a.\frac{1}{4}}} = \frac{2}{\sqrt{m.a}}$$

So theorem is proved.

Thus we have proved, that normalized eigenfunctions of problem (1) - (3) in case of weight functions satisfying to Lipschitz condition are limited in regular intervals, the obtained result is proved by statement proved in [7] at $\alpha = 1$, as

$$\overline{\lim_{n\to\infty}} \frac{\|y_n(x)\|_{C_{[0,a]}}}{|\lambda_n|^{\frac{1-\alpha}{2}}} = C_0 > 0.$$

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