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Some New Common Fixed Point Results in a Dislocated Metric Space

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Abstract

The aim of this paper is to establish several new common fixed point results for four self-mappings of a dislocated metric space.

Keywords: *Fixed point, Common fixed point, Dislocated metric space, Weak compatibility.*

1 Introduction

The notion of dislocated metric, introduced in 2000 by P. Hitzler and A.K. Seda, is characterized by the fact that self distance of a point need not be equal to zero and has useful applications in topology, logical programming and in electronics engineering. For further details on dislocated metric spaces, see, for example [2]-[6]. During the recent years, a number of fixed point results have been established by different authors for single and pair of mappings in dislocated metric spaces. In 2012, Jha and Panthi [4] have established the following result

Theorem 1.1 *Let (X, d) be a complete d-metric space. let A, B, T and S be four continuous self-mappings of X such that*

1. *$TX \subset AX$ and $SX \subset BX$*
2. *The pairs (S, A) and (T, B) are weakly compatible and*
3. *$d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx) + \gamma d(Ax, By)$ for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{2}$*

Then A, B, T and S have a unique common fixed point in X.

Our purpose in this paper is to prove that this theorem can be improved without any continuity requirement. Furthermore, we will give some other results when $\alpha + \beta + \gamma \leq \frac{1}{2}$. We begin by recalling some basic concepts of the theory of dislocated metric spaces.

Definition 1.2 *Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions*

1. *$d(x, y) = d(y, x)$*
2. *$d(x, y) = d(y, x) = 0$ implies $x = y$*
3. *$d(x, y) \leq d(x, z) + d(z, y)$ forall $x, y, z \in X$*

Then d is called dislocated metric(or simply d-metric) on X.

Definition 1.3 *A sequence $\{x_n\}$ in a d-metric space(X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in IN$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$*

Definition 1.4 *A sequence in a d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0 as\ n \to \infty$*
In this case, x is called limit of $\{x_n\}$ (in d)and we write $x_n \to x$.

Definition 1.5 *A d-metric space (X, d) is called complete if every Cauchy sequence is convergent.*

Remark 1.6 *It is easy to verify that in a dislocated metric space, we have the following technical properties*

- *A subsequence of a cauchy sequence in d-metric space is a cauchy sequence.*
- *A cauchy sequence in d-metric space which possesses a convergent subsequence, converges.*

- *Limits in a d-metric space are unique.*

Definition 1.7 *Let A and S be two self-mappings of a d-metric space (X,d). A and S are said to be weakly compatible if they commute at their coincident point; that is,* $Ax = Sx$ *for some* $x \in X$ *implies* $ASx = SAx$.

2 Main Result

Theorem 2.1 *Let* (X, d) *be a d-metric space. let* A, B, T *and* S *be four self-mappings of* X *such that*

1. $TX \subset AX$ *and* $SX \subset BX$
2. *The pairs* (S, A) *and* (T, B) *are weakly compatible and*
3. $d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx) + \gamma d(Ax, By)$
   for all $x, y \in X$ *where* $\alpha, \beta, \gamma \geq 0$ *satisfying* $\alpha + \beta + \gamma < \dfrac{1}{2}$
4. *The range of one of the mappings A, B, S or T is a complete subspace of X*

Then A, B, T and S have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X. Choose $x_1 \in X$ such that $Bx_1 = Sx_0$. Choose $x_2 \in X$ such that $Ax_2 = Tx_1$. Continuing in this fashion, choose $x_n \in X$ such that $Sx_{2n} = Bx_{2n+1}$ and $Tx_{2n+1} = Ax_{2n+2}$ for $n = 0, 1, 2, ...$ To simplify, we consider the sequence (y_n) defined by $y_{2n} = Sx_{2n}$ and $y_{2n+1} = Tx_{2n+1}$ for $n = 0, 1, 2, ...$
We claim that (y_n) is a Cauchy sequence. Indeed, for $n \geq 1$, we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha d(Ax_{2n}, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Sx_{2n}) + \gamma d(Ax_{2n}, Bx_{2n+1}) \\ &\leq \alpha d(y_{2n-1}, y_{2n+1}) + \beta d(y_{2n}, y_{2n}) + \gamma d(y_{2n-1}, y_{2n}) \\ &\leq \alpha \left(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\right] + \beta \left[d(y_{2n}, y_{2n-1}) + d(y_{2n-1}, y_{2n})\right] + \gamma d(y_{2n-1}, y_{2n}) \\ &\leq (\alpha + 2\beta + \gamma) d(y_{2n-1}, y_{2n}) + \alpha d(y_{2n}, y_{2n+1}) \end{aligned}$$

Therefore

$$d(y_{2n}, y_{2n+1}) \leq h\, d(y_{2n-1}, y_{2n})$$

where $h = \dfrac{\alpha + 2\beta + \gamma}{1 - \alpha} \in [0, 1[$. Hence (y_n) is a Cauchy sequence in X and therefore, according to Remarks 1.1, (Sx_{2n}), (Bx_{2n+1}), (Tx_{2n+1}) and (Ax_{2n+2}) are also Cauchy sequence. Suppose that SX is a complete subspace of X, then the sequence (Sx_{2n}) converges to some Sa such that $a \in X$. According to Remark 1.1, (y_n), (Bx_{2n+1}), (Tx_{2n+1}) and (Ax_{2n+2}) converge to Sa. Since

$SX \subset BX$, there exists $u \in X$ such that $Sa = Bu$. We show that $Bu = Tu$. Indeed, we have

$$d(Sx_{2n}, Tu) \leq \alpha d(Ax_{2n}, Tu) + \beta d(Bu, Sx_{2n}) + \gamma d(Ax_{2n}, Bu)$$

and therefore, on letting n to infty, we get

$$\begin{array}{rcl} d(Bu, Tu) & \leq & \alpha d(Bu, Tu) + \beta d(Bu, Bu) + \gamma d(Bu, Bu) \\ & \leq & \alpha d(Bu, Tu) + 2\beta d(Bu, Tu) + 2\gamma d(Bu, Tu) \\ & \leq & (\alpha + 2\beta + 2\gamma)\ d(Bu, Tu) \end{array}$$

which implies that

$$(1 - \alpha - 2\beta - 2\gamma)\ d(Bu, Tu) \leq 0$$

and therefore $d(Bu, Tu) = 0$, since $(1 - \alpha - 2\beta - 2\gamma) < 0$, which implies that $Tu = Bu$. Since $TX \subset AX$, there exists $v \in X$ such that $Tu = Av$. We show that $Sv = Av$. Indeed, we have

$$\begin{array}{rcl} d(Sv, Av) & = & d(Sv, Tu) \\ & \leq & \alpha d(Av, Tu) + \beta d(Bu, Sv) + \gamma d(Av, Bu) \\ & \leq & \alpha d(Av, Av) + \beta d(Av, Sv) + \gamma d(Av, Av) \\ & \leq & 2\alpha d(Av, Sv) + \beta d(Av, Sv) + 2\gamma d(Av, Sv) \\ & \leq & (2\alpha + \beta + 2\gamma)\ d(Av, Sv) \end{array}$$

which implies that

$$(1 - 2\alpha - \beta - 2\gamma)\ d(Av, Sv) \leq 0$$

and therefore $d(Av, Sv) = 0$, since $1 - 2\alpha - \beta - 2\gamma < 0$, which implies that $Av = Sv$. Hence $Bu = Tu = Av = Sv$.
Using the fact that (S, A) is weakly compatible, we deduce that $ASv = SAv$, from which it follows that $AAv = ASv = SAv = SSv$.
The weak compatibility of B and T implies that $BTu = TBu$, from which it follows that $BBu = BTu = TBu = TTu$.
Let us show that Bu is a fixed point of T. Indeed, we have

$$\begin{array}{rcl} d(Bu, TBu) & = & d(Sv, TBu) \\ & \leq & \alpha d(Av, TBu) + \beta d(BBu, Sv) + \gamma d(Av, BBu) \\ & \leq & \alpha d(Bu, TBu) + \beta d(TBu, Bu) + \gamma d(Bu, TBu) \\ & \leq & (\alpha + \beta + \gamma)\ d(Bu, TBu) \end{array}$$

and therefore $d(Bu, TBu) = 0$, since $1 - \alpha - \beta - \gamma < 0$, which implies that $TBu = Bu$. Hence Bu is a fixed point of T. It follows that $BBu = TBu = Bu$,

which implies that Bu is a fixed point of B.
On the other hand, we have

$$\begin{aligned} d(SBu, Bu) &= d(SBu, TBu) \\ &\leq \alpha d(ABu, TBu) + \beta d(BBu, SBu) + \gamma d(ABu, BBu) \\ &\leq \alpha d(SBu, Bu) + \beta d(Bu, SBu) + \gamma d(SBu, Bu) \\ &\leq (\alpha + \beta + \gamma)\ d(Bu, SBu) \end{aligned}$$

which implies $d(Bu, SBu) = 0$ and therefore $SBu = Bu$. Hence Bu is a fixed point of S. It follows that $ABu = SBu = Bu$, which implies that Bu is also a fixed point of S. Thus Bu is a common fixed point of S, T, A and B.
Finally to prove uniqueness, suppose that there exists $u, v \in X$ such that $Su = Tu = Au = Bu$ and $Su = Tu = Au = Bv$. If $d(u, v) \neq 0$, then

$$\begin{aligned} d(u, v) &= d(Su, Tv) \\ &\leq \alpha d(Au, Tv) + \beta d(Bv, Su) + \gamma d(Au, Bv) \\ &\leq \alpha d(u, v) + \beta d(v, u) + \gamma d(u, v) \\ &\leq (\alpha + \beta + \gamma)\, d(u, v) \end{aligned}$$

which is a contradiction. Hence $d(u, v) = 0$ and therefore $u = v$.
The proof is similar when TX or AX or BX is a complete subspace of X. This completes the proof of the Theorem.

For $A = B$ and $S = T$, we have the following result

Corollary 2.2 *Let (X, d) be a d-metric space. let A and S be two self-mappings of X such that*

1. *$SX \subset AX$*
2. *The pair (S, A) is weakly compatible and*
3. *$d(Sx, Sy) \leq \alpha d(Ax, Sy) + \beta d(Ay, Sx) + \gamma d(Ax, Ay)$*
   for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{2}$
4. *The range of A or S is a complete subspace of X*

Then A and S have a unique common fixed point in X.

For $A = B = Id_X$, we get the following corollary

Corollary 2.3 *Let (X, d) be a d-metric space. let T and S be two self-mappings of X such that*

1. *$d(Sx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y)$*
   for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{2}$

2. The range of S or T is a complete subspace of X

Then T and S have a unique common fixed point in X.

For $S = T = Id_X$, we have the following result

Corollary 2.4 *Let (X, d) be a complete d-metric space. let A and B be two surjective self-mappings of X such that*

$$d(x, y) \leq \alpha d(Ax, y) + \beta d(By, x) + \gamma d(Ax, By)$$

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0$ satisfying $\alpha + \beta + \gamma < \frac{1}{2}$.
Then A and B have a unique common fixed point in X.

Remark 2.5 *Following the procedure used in the proof of Theorem 2.1, we have the next new result in which we remplace the condition $\alpha + \beta + \gamma < \frac{1}{2}$ by $\alpha + \beta + \gamma \leq \frac{1}{2}$ for $\alpha, \beta, \gamma > 0$*

Theorem 2.6 *Let (X, d) be a d-metric space. let A, B, T and S be four self-mappings of X such that*

1. *$TX \subset AX$ and $SX \subset BX$*
2. *The pairs (S, A) and (T, B) are weakly compatible and*
3. *$d(Sx, Ty) \leq \alpha d(Ax, Ty) + \beta d(By, Sx) + \gamma d(Ax, By)$ for all $x, y \in X$ where $\alpha, \beta, \gamma > 0$ satisfying $\alpha + \beta + \gamma \leq \frac{1}{2}$*
4. *The range of one of the mappings A, B, S or T is a complete subspace of X*

Then A, B, T and S have a unique common fixed point in X.

Example 2.7 *Let $X = [0, 1]$ and $d(x, y) = |x| + |y|$. We consider A, B, S and T defined by:*

$$\text{For all } x \in X,\ Sx = 0,\ Tx = \frac{x}{5},\ \text{and } Ax = Bx = x$$

Then, for $\alpha = \beta = \gamma = \frac{1}{6}$, it is easy to see that all assumptions of Theorem 2.2 are verified, $\alpha + \beta + \gamma = \frac{1}{2}$ and 0 is the unique common fixed point of A, B, S and T.

As consequences of the Theorem 2.2, we have the following new results

Corollary 2.8 *Let (X, d) be a d-metric space. let A and S be two self-mappings of X such that*

1. $SX \subset AX$
2. *The pair (S, A) is weakly compatible and*
3. $d(Sx, Sy) \leq \alpha d(Ax, Sy) + \beta d(Ay, Sx) + \gamma d(Ax, Ay)$ *for all $x, y \in X$ where $\alpha, \beta, \gamma > 0$ satisfying $\alpha + \beta + \gamma \leq \frac{1}{2}$*
4. *The range of A or S is a complete subspace of X*

Then A and S have a unique common fixed point in X.

Corollary 2.9 *Let (X, d) be a d-metric space. let T and S be two self-mappings of X such that*

1. $d(Sx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Sx) + \gamma d(x, y)$ *for all $x, y \in X$ where $\alpha, \beta, \gamma > 0$ satisfying $\alpha + \beta + \gamma \leq \frac{1}{2}$*
2. *The range of S or T is a complete subspace of X*

Then T and S have a unique common fixed point in X.

Corollary 2.10 *Let (X, d) be a complete d-metric space. let A and B be two surjective self-mappings of X such that*

$$d(x, y) \leq \alpha d(Ax, y) + \beta d(By, x) + \gamma d(Ax, By)$$

for all $x, y \in X$ where $\alpha, \beta, \gamma > 0$ satisfying $\alpha + \beta + \gamma \leq \frac{1}{2}$.
Then A and B have a unique common fixed point in X.

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