Gen. Math. Notes, Vol. 2, No. 2, February 2011, pp. 1-6
ISSN 2219-7184; Copyright © ICSRS Publication, 2011
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A Three-Parameter Third-Order Family of Methods for Solving Nonlinear Equations

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(Received: 7-1-11 /Accepted: 16-2-11)

Abstract

In this paper a new family of methods free from second derivative is presented. This new family of methods is constructed such that convergence is of order three and requires two require two evaluations of the function and first derivative per iteration. To illustrate the efficiency and performance of the new family of methods, several numerical examples are presented. Further numerical comparisons are made with several other existing third-order methods to show the abilities of the presented family of methods.

Keywords: *Iterative methods, Nonlinear equations, Newton's method.*

1 Introduction

In this paper, we consider iterative methods to find a simple root α, i.e., $f(\alpha)=0$ and $f'(\alpha)\neq 0$, of a nonlinear equation $f(x)=0$ .The design of iterative formulae for solving these equations are very important and interesting tasks in applied mathematics and other disciplines .In recent years, several variants of the methods with free second-derivative have been proposed and analyzed (see [1-8] and the reference therein). These new methods can be

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considered as alternatives for Newton's method which is a well-known iterative method for finding α by using

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

That converges quadratic ally in some neighborhood of α.

2 Derivation of Method and Convergence Analysis

To develop this new family, let us begin with the following multipoint iteration scheme in the form

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)},\ m \neq 1 \tag{2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{Af(x_n) + Bf(y_n)}{Cf(x_n)^2 + Df(y_n)f(x_n) + Ef(y_n)^2}\frac{f(y_n)f(x_n)}{f'(x_n)}. \tag{3}$$

Where A, B, C, D and E are the parameters to be determined such that iterative method defined by (2) and (3) to be of order convergence three. In the following, the sufficient conditions for these proposed, are presented:

Theorem 2.1. *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \to \Re$ for an open interval, which contains x_0 as, initial approximation of α. If A, B, C, D and E, satisfy the conditions:*

$$B = \frac{-C + D(m-1) - E(m-1)^2}{(m^3 - m^2)}, \qquad A = \frac{-C + D(m-1) - E(m-1)^2}{m^2}$$

And $C^2 + D^2 + E^2 \neq 0$,

Then, the family of methods defined by (2) and (3) is of third-order.

Proof. If α is the root and e_n be the error at n th iteration, than $e_n = x_n - \alpha$, using Taylor's expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + O(e_n^4)], \tag{4}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^4 + O(e_n^4)], \tag{5}$$

where

$$c_k = f^{(k)}(\alpha)/k!f'(\alpha), k = 2,3,\ldots, \text{ and } e_n = x_n - \alpha.$$

Using (4), (5) and (2), we have

$$y_n = \alpha + (1\text{-}m)e_n + mc_2e_n^2 + 2(mc_3 - mc_2^2)e_n^3 + O(e_n^4). \tag{6}$$

Now again by Taylor's series, we have

$$f(y_n) = f'(\alpha)[(1-m)e_n + (m^2 - m + 1)c_2e_n^2 + \left((-2m^2)c_2^2 - (m3 - 3m^2 + m + 1)c_3\right)e_n^3] + O(e_n^4) \tag{7}$$

Finally, using (4)-(7) and (3), we get

$$x_{n+1} = x_n + K_1e_n + K_2e_n^2 + O(e_n^3). \tag{8}$$

Where

$$K_1 = -\frac{C - D(m-1) + E(m-1)^2 + B(m-1)^2 - A(m-1)}{C - D(m-1) + E(m-1)^2},$$

An easy manipulation shows that $K_1 = -1$ when $A = (m-1)B$. Then by inserting it in K_2, we have:

$$K_2 = \frac{C - D(m-1) + E(m-1)^2 + B(m^3 - m^2)}{C - D(m-1) + E(m-1)^2} c_2^2,$$

It can be verified that $K_2 = 0$, when $B = \dfrac{-\left(C - D(m-1) + E(m-1)^2\right)}{(m^3 - m^2)}$.

This implies that:

$$e_{n+1} = O(e_n^3),$$

which completes the proof.

By introducing $\alpha = D/C$, $\beta = E/C$, and some manipulations in (8),for those parameters that satisfy conditions of theorem, following three parameters family of fourth order methods is obtained:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_n)M_1 + f(y_n)M_2}{f(x_n)^2 + \alpha f(x_n)(y_n) + \beta f(y_n)^2} \frac{f(x_n)^2}{f'(x_n)}, \tag{9}$$

Where

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)},$$

$$M_1 = \left(\alpha\frac{(m-1)}{m^2} - \frac{1}{m^2} - \beta\frac{(m-1)^3}{m^2}\right),$$

$$M_2 = \left(\alpha\frac{1}{m^2} + \frac{1}{(m^3 - m^2)} - \beta\frac{(m-1)^2}{m^2}\right),$$

$m \neq 1, \alpha$ and $\beta \in \Re$.

Formula (9) includes, as particular cases, the following ones:

For $m = -2/3, \alpha = -1/2$ and $\beta = 0$, we obtain a new fourth-order method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{3}{20}\frac{[5f(x_n) + 33f(y_n)]}{[2f(x_n) - f(y_n)]}\frac{f(y_n)}{f'(x_n)}, \tag{10}$$

For $m = 1/2, \alpha = -1$ and $\beta = 0$, we obtain another new fourth-order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} + \frac{2[f(x_n) + 6f(y_n)]}{f(x_n) - f(y_n)}\frac{f(y_n)}{f'(x_n)}, \tag{11}$$

Per iteration of these methods require two evaluations of the function and one of its derivative.. If we consider the definition of efficiency index in [9] as $p^{1/m}$, where p is the order of the method and m is the number of functional evaluations, the iteration formula defined by (9) has the efficiency index equal to $\sqrt[3]{3} \approx 1.4422$, which is better than that of Newton's method $\sqrt{2} \approx 1.4142$.

3 Numerical Examples

In this section, we present some examples to illustrate the efficiency of one member of the iterative family which has been introduced in the present paper. We present some numerical test results for various cubically convergent iterative schemes in Table 1. A Comparison has been made between the Newton method (NM), the method of Weerakoon and Fernando [9] (WF) defined by

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_n - f(x_n)/f'(x_n))},$$

And the method derived from midpoint rule [10] (MP) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n - f(x_n)/(2f'(x_n)))},$$

And the method of Homeier [11] (HM) defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(x_n - f(x_n)/f'(x_n))}\right),$$

And the methods of Chun (CM1) Introduced in [12],

$$x_{n+1} = x_n - L_f(x_n)\frac{f(x_n)}{f'(x_n)},$$

$$L_f(x_n) = \frac{3}{2} - \frac{1}{2}\frac{[f^2(x_n) + f'^2(x_n)][f'(x_n + f(x_n)) - f'(x_n) - f'^2(x_n)]}{[f^2(x_n) + f'^2(x_n)]^2},$$

And the new methods (10) (BGM1) and (11) (BGM2) introduced in this contribution.
In the present contribution, we used the following test functions as in [12]

$f_1(x) = x^3 + 4x^2 - 10,$

$f_2(x) = \sin^2(x) - x^2 + 1,$

$f_3(x) = x^2 - 3x - e^x + 2,$

$f_4(x) = \cos(x) - x,$

$f_5(x) = (x-1)^3 - 1,$

$f_6(x) = \sin(x) - x/2.$

All computations were done using MAPLE 10 using 64 digit floating point arithmetic (Digits: = 64). In Table 1 results were obtained by using the following stopping criteria:
(i) $|x_{n+1} - x_n| < 10^{-15}$, (ii) $|f(x_n)| < 10^{-15}$.
Therefore, as soon as the stopping criteria are satisfied, $x_{n+1} = x_{IT}$ is taken as the approximation of solution of $f(x) = 0$. In table 1:
(IT) stands for the number of iterations, (NFE) stands for the number of evaluations of the function and derivative, (δ) stands for distance of two consecutive approximations for finding zero.

4 Conclusion

In this work, we have constructed a new iterative family of methods of order three, for solving nonlinear equations. It has been shown that, the proposed iterative family of order three and can be effectively used for solving nonlinear equations. It can be shown that the proposed methods of this family, has less (IT) in comparison with other methods and also need less computations per iteration, which are valuable advantages of this presented family.

Table 1

Comparison of various third-order convergent iterative methods

	IT	*NFE*	x_{IT}	$f(x_{IT})$	δ
$f_1, x_0 = 1.27$					
NM	5	10	1.365230013414096	2.70-41	1.8e-21
WF	4	12	1.365230013414096	0.0e-1.0	3.0e-35
MP	4	12	1.365230013414096	4.5e-48	2.0e-16
HM	3	9	1.365230013414096	1.0e-63	1.7e-33
CM	4	12	1.365230013414096	0.0e-01	2.4e-26
BGM 1	2	6	1.365230013414096	2.7e-16	1.6e-17
BGM 2	3	9	1.365230013414096	1.2e-35	7.5e-37
$f_2, x_0 = 2.0$					
NM	6	12	1.404491648215341	2.2e-32	1.0e-16
WF	5	15	1.404491648215341	2.0e-63	6.0e-42
MP	5	15	1.404491648215341	2.0e-63	7.1e-41
HM	4	12	1.404491648215341	2.0e-63	1.0e-24
CM	4	12	1.404491648215341	1.3e-63	3.4e-26
BGM 1	3	9	1.404491648215341	4.1e-16	1.6e-16
BGM 2	4	14	1.404491648215341	1.6e-39	6.5e-40
$f_3, x_0 = 2.0$					
NM	6	12	0.257530285439860	2.9e-55	9.1e-28
WF	5	15	0.257530285439860	1.0e-63	1.6e-34
MP	4	12	0.257530285439860	1.0e-63	3.9e-24
HM	5	15	0.257530285439860	0.0e-01	9.3ev43
CM	5	15	0.257530285439860	1.0e-63	3.3e-39
BGM 1	6	18	0.257530285439860	4.4e-33	1.1e-33
BGM 2	4	12	0.257530285439860	2.5e-58	1.3e-58
$f_4, x_0 = 1.4$					
NM	5	10	0.739085133215160	1.2ev32	1.8e-16
WF	4	12	0.739085133215160	0.0e-01	5.3e-28
MP	4	12	0.739085133215160	0.0e-01	6.7e-24
HM	4	12	0.739085133215160	1.0e-64	1.1e-24
CM	5	15	0.739085133215160	3.3e-50	2.3e-17
BGM 1	3	9	0.739085133215160	8.9e-17	5.3e-17
BGM 2	3	9	0.739085133215160	1.7e-21	1.0e-21
$f_5, x_0 = 2.6$					
NM	7	14	2	5.5e-49	4.2e-25
WF	5	15	2	0.0e-01	1.0e-29
MP	5	15	2	0.0e-01	1.4e-32
HM	4	12	2	2.8e-48	1.7e-16
CM	4	12	2	0.0e-01	3.5e-22
BGM 1	3	9	2	3.2e-19	1.0e-19
BGM 2	4	12	2	2.2e-31	7.9e-33
$f_6, x_0 = 2.3$					
NM	6	12	1.895494267033980	2.4e-48	2.2e-24
WF	4	12	1.895494267033980	3.0e-64	1.1e-21
MP	4	12	1.895494267033980	1.3e-59	3.6e-20
HM	4	12	1.895494267033980	3.0e-64	2.2e-38
CM	5	15	1.895494267033980	3.0e-64	1.0e-44
BGM 1	4	12	1.895494267033980	1.4e-19	1.7e-19
BGM 2	4	12	1.895494267033980	2.2e-31	7.9e-33

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