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Linear Maps Preserving Quasi-Unitary Operators

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Abstract

Let $\mathcal{H}$ be a infinite separable complex Hilbert space and $\mathscr{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. We give the concrete forms of surjective linear maps $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ preserving quasi-unitary operators and using this result for giving a form of ϕ when it preserves operator pairs whose products or triple Jordan products are nonzero quasi-unitary operators in both directions.

Keywords: *Linear preserver, Jordan homomorphisms, quasi-unitary operators.*

1 Introduction and Preliminaries

Linear preserver problems is an active research area in Matrix, operator theory and Banach algebras, it has attracted the attention of many mathematicians in the last few decades [2, 3, 4, 5, 7, 10, 12, 13, 14, 15, 19]. A linear preserver is a linear map of an algebra $\mathscr{A}$ into itself which, roughly speaking, preserves certain properties on some elements in $\mathscr{A}$. Linear preserver problems concern the characterization of such maps. Automorphisms and anti-automorphisms certainly preserve various properties of the elements. Therefore, it is not surprising that these two types of maps often appear in the conclusions of the results. In this paper, we shall concentrate on the case when $\mathscr{A} = \mathscr{B}(\mathcal{H})$, the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We should point out that a great deal of work has been devoted to the case when $\mathcal{H}$ is finite dimensional, it is the case when $\mathscr{A}$ is a matrix algebra (see survey articles [10, 16, 22]).The first papers concerning this case date back to the previous century [6].

In 1971 Palme has considered the concept of U^* algebra in terms of quasi-unitary element, according to him follows this definition

Definition 1.1. *Let $\mathscr{A}$ be an $*-$algebra, an element $x \in \mathscr{A}$ is quasi-unitary if*

$$x^*x = xx^* = x + x^*.$$

Due to this definition, in 1977 Phadke et al. in [17] have introduced the notion of a quasi-unitary operator on a Hilbert space as follows.

Definition 1.2. *An operator T on a Hilbert space $\mathcal{H}$ is called quasi-unitary if*

$$TT^* = T^*T = T + T^*.$$

Theorem 1.3. *An operator T is a quasi-unitary operator on a Hilbert space if and only if $I - T$ is a unitary operator.*

We say that a linear maps on $\mathscr{B}(\mathcal{H})$ into it self, preserves quasi-unitary operators in both directions, If for any $A \in \mathscr{B}(\mathcal{H})$, $\phi(A)$ is a quasi-unitary operator if and only if A is, preserves pairs whose products are nonzero quasi-unitary operator in both direction, If for any $A, B \in \mathscr{B}(\mathcal{H})$, $\phi(A)\phi(B)$ is a quasi-unitary operator if and only if AB is and preserves pairs whose triple Jordan products are nonzero quasi-unitary operator in both direction, If for any $A, B \in \mathscr{B}(\mathcal{H})$, $\phi(A)\phi(B)\phi(A)$ is a quasi-unitary operator if and only if ABA is. The aim of this paper is to characterize surjective linear maps $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ that preserves the quasi-unitary operators in both directions. At the end, we use this result to characterize the form of surjective linear maps $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ that preserves operator pairs whose products or triple Jordan products are nonzero quasi-unitary operators in both directions.

2 Main Result

First we prove some elementary lemmas which are useful in the proofs of main theorems.

Lemma 2.1. *Let $\mathcal{H}$ be a complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a linear surjective map. Assume that ϕ preserves quasi-unitary operators in both directions, then ϕ is injective.*

Proof. Suppose there exists $A \in \mathscr{B}(\mathcal{H})$ such that $\phi(A) = 0$, then

$$\phi(\lambda A) = 0,$$

since 0 is quasi-unitary operator, then λA is a quasi-unitary operator for all $\lambda \in \mathbb{C}$. This implies that

$$|\lambda|^2 A^* A = |\lambda|^2 A^* A = \overline{\lambda} A^* + \lambda A.$$

Taking successively $\lambda = 1$ and $\lambda = 2$, we get

$$A^* A = 0.$$

So $A = 0$, hence the proof is complete. □

Lemma 2.2. *Let $\mathcal{H}$ be a complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a linear bijective map. If ϕ preserves quasi-unitary operators in both directions, then$\phi(I) = I$.*

Proof. Suppose there exists $A \in \mathscr{B}(\mathcal{H})$ such that $\phi(A) = I$, since $2I$ is a quasi-unitary operator, then $2A$ is also a quasi-unitary operator. Assume now that $A \neq I$ and consider the quasi-unitary operator $S = 2I$, we know that $2A - S$ is not quasi-unitary. Then by the properties of ϕ and the fact that S is a quasi-unitary operator we get that $2I - \phi(S)$ is a quasi-unitary operator. Thus, $\phi(2A - S)$ is a quasi-unitary operator, this is impossible consequently $A = I$. □

Lemma 2.3. *Let $\mathcal{H}$ be a complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a linear bijective map. If ϕ preserves quasi-unitary operators in both directions, then ϕ preserve orthogonal projections in both directions.*

Proof. Let p is an orthogonal projection in $\mathscr{B}(\mathcal{H})$, we consider a scalar $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = \overline{\lambda} + \lambda$. Thus, λp is a quasi-unitary operator, $\lambda\phi(p)$ is also a quasi-unitary operator and we get

$$|\lambda|^2 \phi(p)\phi(p)^* = |\lambda|^2 \phi(p)^* \phi(p) = \overline{\lambda}\phi(p)^* + \lambda\phi(p). \tag{1}$$

If we replace λ by 2 in (1), we obtain

$$2\phi(p)\phi(p)^* = 2\phi(p)^*\phi(p) = \phi(p)^* + \phi(p). \tag{2}$$

In view of (1) and (2), we get

$$\frac{\lambda + \overline{\lambda}}{2} \left(\phi(p) + \phi(p)^*\right) = \overline{\lambda}\phi(p)^* + \lambda\phi(p) \tag{3}$$

which gives afterward by a simple computation that

$$\frac{\lambda-\overline{\lambda}}{2}\phi(p)=\frac{\lambda-\overline{\lambda}}{2}\phi(p)^*.$$

So, if we take $\lambda\in\mathbb{C}-\mathbb{R}$, we have $\phi(p)=\phi(p)^*$. Replace the result obtained in (2), we get $\phi(p)^2=\phi(p)$ and consequently ϕ preserves orthogonal projections in first direction. Now, for ϕ preserves orthogonal projections in second direction. Repeating the same with ϕ^{-1}, completes the proof. $\square$

Lemma 2.4. *ϕ preserves the orthogonality of projections in both directions.*

Proof. Let p and q tow mutually orthogonal projections, so $p+q$ is a orthogonal projection and so $\phi(p)+\phi(q)$ is a orthogonal projection, hence

$$(\phi(p)+\phi(q))^2=\phi(p)+\phi(q). \tag{4}$$

Consequently,

$$\phi(p)\phi(q)+\phi(q)\phi(p)=0. \tag{5}$$

Now, left multiplication by $\phi(p)$ gives

$$\phi(p)\phi(q)+\phi(p)\phi(q)\phi(p)=0, \tag{6}$$

and right multiplication by $\phi(p)$ gives

$$\phi(q)\phi(p)+\phi(p)\phi(q)\phi(p)=0, \tag{7}$$

therefore, formula (6) and (7) yields

$$\phi(p)\phi(q)=\phi(q)\phi(p), \tag{8}$$

formula (5) and (8) yields

$$\phi(p)\phi(q)=\phi(q)\phi(p)=0. \tag{9}$$

Now, repeating the same with ϕ^{-1}, which completes the proof. $\square$

Theorem 2.5. *Let $\mathcal{H}$ be a separable infinite complex Hilbert space and let $\phi:\mathscr{B}(\mathcal{H})\to\mathscr{B}(\mathcal{H})$ be a linear surjective map. Suppose that ϕ preserves quasi-unitary operators in both directions. Then there exists a unitary operator $U\in\mathscr{B}(\mathcal{H})$ such that*

$$\phi(A)=UAU^*$$

or

$$\phi(A)=UA^tU^*$$

for all $A\in\mathscr{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of $\mathcal{H}$.

Proof. By using Lemmas 2.1, 2.2, 2.3 and 2.4, we get that ϕ is a bijection on the set of all projections of $\mathscr{B}(\mathcal{H})$ preserving orthogonality in both directions. It follows from the Uhlhorn's Theorem in [21] that there is a unitary operator U on $\mathcal{H}$ such that $\phi(E) = UEU^*$ for all E projection in $\mathscr{B}(\mathcal{H})$, or $\phi(E) = UE^tU^*$ for all E projection in $\mathscr{B}(\mathcal{H})$. Suppose first that $\phi(E) = UEU^*$ for all E projection in $\mathscr{B}(\mathcal{H})$. K. Matsumoto in [11], show that for any operator A there exists a sequence $p_1, \ldots, p_{10}$ orthogonal projections and complex scalars $\lambda_1, \ldots \lambda_{10}$ such that $A = \sum_{i=1}^{10} \lambda_i p_i$ hence

$$\begin{aligned}
\phi(A) &= \phi(\sum_{i=1}^{10} \lambda_i p_i) \\
&= \sum_{i=1}^{10} \lambda_i \phi(p_i) \\
&= \sum_{i=1}^{10} \lambda_i U p_i U^* \\
&= UAU^*
\end{aligned}$$

for all operator $A \in \mathscr{B}(\mathcal{H})$.
Now suppose $\phi(E) = UE^tU^*$ for all E projection in $\mathscr{B}(\mathcal{H})$. Then by a similar argument, we can show that $\phi(A) = UA^tU^*$ for all operator $A \in \mathscr{B}(\mathcal{H})$, hence the proof is complete.

□

Corollary 2.6. *Let $\mathcal{H}$ be a separable infinite complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a linear surjective map with $\phi(I) = I$. If ϕ preserves unitary operators in both directions, then there exists a unitary operator $U \in \mathscr{B}(\mathcal{H})$ such that*

$$\phi(A) = UAU^* \tag{10}$$

or

$$\phi(A) = UA^tU^* \tag{11}$$

for all $A \in \mathscr{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of $\mathcal{H}$.

Proof. First we shall prove that ϕ is injective. Suppose there exists $A \in \mathscr{B}(\mathcal{H})$ such that $\phi(A) = 0$, then

$$\phi(A + I) = \phi(I) \textit{ and } \phi(A - I) = \phi(-I),$$

since ϕ preserves unitary operators in both directions, then $A + I$ and $A - I$ are unitary operators. Therefore,

$$A^*A + A^* + A = 0 \textit{ and } A^*A - A^* - A = 0.$$

It follows that $A^*A = 0$ consequently $A = 0$. This implies that ϕ is bijective. Now if A is a quasi-unitary operator, then $I - A$ is a unitary operator, since ϕ preserves unitary operators, hence $I - \phi(A)$ is a unitary operator, consequently $\phi(A)$ is a quasi-unitary operator. For reciprocal, i.e., if $\phi(A)$ is a quasi-unitary operator, then A is a quasi-unitary operator. We use similar proof to ϕ^{-1}. Consequently, ϕ preserves quasi-unitary operators in both directions, from theorem 2.5 the result follows. □

Theorem 2.7. *Let $\mathcal{H}$ be a separable infinite complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a linear surjective map. Suppose that for all $A, B \in \mathscr{B}(\mathcal{H})$, AB is a non zero quasi-unitary operator if and only if $\phi(A)\phi(B)$ is a non zero quasi-unitary operator. Then there exists a unitary operator $U \in \mathscr{B}(\mathcal{H})$ such that*

$$\phi(A) = \pm UAU^*, \tag{12}$$

or

$$\phi(A) = \pm UA^tU^* \tag{13}$$

for all $A \in \mathscr{B}(\mathcal{H})$, where A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of $\mathcal{H}$.

Proof. First we prove that ϕ preserves operators pairs whose products are non zero projections in both directions.
Suppose that AB is a non zero projection we consider $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = \lambda + \overline{\lambda}$ then λAB is quasi-unitary , so $\lambda\phi(A)\phi(B)$ is quasi-unitary. We obtain

$$|\lambda|^2(\phi(A)\phi(B))^*(\phi(A)\phi(B)) = \lambda(\phi(A)\phi(B)) + \overline{\lambda}(\phi(A)\phi(B))^*,$$

for $\lambda = 2$, we get

$$2(\phi(A)\phi(B))^*(\phi(A)\phi(B)) = (\phi(A)\phi(B)) + (\phi(A)\phi(B))^*,$$

so we obtain by simple calculus that

$$\frac{\lambda - \overline{\lambda}}{2}(\phi(A)\phi(B))^* = \frac{\lambda - \overline{\lambda}}{2}(\phi(A)\phi(B)).$$

Now if we take $\lambda \in \mathbb{C} - \mathbb{R}$ we get that $(\phi(A)\phi(B))^* = (\phi(A)\phi(B))$, then $(\phi(A)\phi(B))$ is a projection. Now, for ϕ preserves operator pairs whose products are non zero projection in second direction. We use similar proof to ϕ^{-1} which completes the proof. we get that AB is a projection if and only if $(\phi(A)\phi(B))$ is a projection. From Lemma 2.4 in [7], ϕ is injective and $\phi(I) = \pm I$.
Finally, if $\phi(I) = I$, we get that ϕ preserve quasi-unitary, then by Theorem 2.5 we get the result .
If $\phi(I) = -I$ we defined $\psi = -\phi$ and by Theorem 2.5 we get the result. □

Theorem 2.8. *Let $\mathcal{H}$ be a separable infinite complex Hilbert space and let $\phi : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ be a linear surjective map. Suppose that for all $A, B \in \mathscr{B}(\mathcal{H})$ ABA is a non zero quasi-unitary operator if and only if $\phi(A)\phi(B)\phi(A)$ is a non zero quasi-unitary operator. Then there exists a unitary operator $U \in \mathscr{B}(\mathcal{H})$ such that*

$$\phi(A) = \alpha UAU^*, \tag{14}$$

or

$$\phi(A) = \alpha UA^tU^* \tag{15}$$

for all $A \in \mathscr{B}(\mathcal{H})$, with $\alpha^3 = 1$, and A^t is the transpose of A with respect to an arbitrary but fixed orthonormal base of $\mathcal{H}$.

Proof. First we show that ϕ preserve operator pairs whose triple Jordan product are projections.
Suppose that ABA is a projection we consider $\lambda \in \mathbb{C}$ such that $|\lambda|^2 = \lambda + \overline{\lambda}$, then λABA is a quasi-unitary operator, so $\lambda\phi(A)\phi(B)\phi(A)$ is a quasi-unitary operator. We obtain

$$|\lambda|^2(\phi(A)\phi(B)\phi(A))^*(\phi(A)\phi(B)\phi(B)) = \lambda(\phi(A)\phi(B)\phi(A)) + \overline{\lambda}(\phi(A)\phi(B)\phi(A))^*,$$

for $\lambda = 2$, we get

$$2(\phi(A)\phi(B)\phi(A))^*(\phi(A)\phi(B)\phi(A)) = (\phi(A)\phi(B)\phi(A)) + (\phi(A)\phi(B)\phi(A))^*,$$

by simple calculus we get that

$$\frac{\lambda - \overline{\lambda}}{2}(\phi(A)\phi(B)\phi(A))^* = \frac{\lambda - \overline{\lambda}}{2}(\phi(A)\phi(B)\phi(A)).$$

Now if we take $\lambda \in \mathbb{C} - \mathbb{R}$, we get that $(\phi(A)\phi(B)\phi(A))^* = (\phi(A)\phi(B)\phi(A))$, then $(\phi(A)\phi(B)\phi(A))$ is a projection. Now, for ϕ, preserve operators pairs whose triple Jordan products are non zero projection in second direction. We use similar proof to ϕ^{-1} which completes the proof. We get that ABA is a projection if and only if $(\phi(A)\phi(B)\phi(A))$ is a projection. From Lemma 3.2 in [7], ϕ is injective and $\phi(I) = \alpha I, \alpha^3 = 1$.
Finally, if $\phi(I) = I$, we get that ϕ preserve quasi-unitary, then by Theorem 2.5 we get the result .
If $\phi(I) = \alpha I$ we consider $\psi = \overline{\alpha}\phi$, then by using the Theorem 2.5 for ψ, we get the result.

□

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