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On the Square Submodule of a Mixed Module

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Abstract

The notion of the square submodule of a module M over an arbitrary commutative ring R, which is denoted by $\Box_R M$, was introduced by Aghdam and Najafizadeh in [3]. In fact, $\Box_R M$ is the $R-$submodule of M generated by the images of all bilinear maps on M. Furthermore, given a submodule N of an $R-$module M, we say that M is nil modulo N if $\mu(M \times M) \leq N$ for all bilinear maps μ on M. The main question about the square submodule is that whether the quotient module $M/\Box M$ is a nil module? In this paper, we investigate the square submodules of some classes of modules over commutative domains. Then, we have some results related to splitting modules which we need in our discussions. Finally, we give some examples of mixed Abelian groups A such that the quotient groups $A/\Box A$ are not nil.

Keywords: *Mixed module, Nil module, Square submodule.*

1 Introduction

Given an Abelian group A, we call R a ring over A if the additive group $R^+ = A$. In this situation we write $R = (A, *)$, where $*$ denotes the ring multiplication. The multiplication is not assumed to be associative. Every group can be provided with a ring structure in a trivial way, by defining all products to be 0; such a ring is called a zero-ring. In general, we call a group A a nil group if there is no ring on A other than the zero-ring. A generalization of the notion of a nil group was considered by Feigelstock [5]. Given a subgroup B of A, the group A is called to be nil modulo B if $A * A \subseteq B$ for every ring $(A, *)$ on A. The term square was used for the first time for an Abelian group by Stratton and Webb [10]. In fact, given an Abelian group A, the square subgroup of A denoted by $\Box A$, is defined as:

$$\Box A = \cap\{B \subseteq A \mid A \ is \ nil \ modulo \ B\}.$$

The basic question about the square subgroup is whether $A/\Box A$ is a nil group? Aghdam [1] showed that if A is any arbitrary group, then

$$A/\Box A \cong (D/T) \oplus (N/\Box N)\ ,\ \ \Box D \leq T \leq D,$$

for some subgroup T of D, where D and N are the maximal divisible subgroup and reduced part of A respectively. Moreover, in the case that A is a reduced torsion group then $A = \Box A$ and if A is non-torsion, then

$$A/\Box A \cong N/\Box N\ .$$

Aghdam and Najafizadeh [2] continued the investigation for torsion-free groups of rank two using their typesets. Then, they introduced the concept of the square submodule of a module and a nil module in [3] . They studied different properties of the square submodule for torsion and torsion-free modules over commutative domains. In this paper, the square submodules of some classes of modules over commutative domains are studied. Moreover, we have some results related to splitting modules. Finally, the square submodule of a mixed module is studied. It is shown by some examples that if M is a mixed module, then $M/\Box M$ is not a nil module in general.

2 Notations and Preliminaries

Throughout this paper, unless otherwise stated, R means a commutative domain with identity and Q its field of quotients. An $R-$submodule I of Q for which there exist a non-zero element r in R such that $rI \leq R$ is called a fractional ideal of R. The set $F(R)$ of the non-zero fractional ideals of R is a multiplicative monoid by the multiplication $I.J = \{\sum_{i=1}^{i=n} a_i b_i \mid a_i \in I, b_i \in J, n \in \mathbb{N}\}$. A non-zero fractional ideal I of R is called invertible if it is invertible as an element of $F(R)$. Moreover, we use the symbol $I : J$ to the residual which is defined as $I : J = \{q \in Q \mid qJ \leq I\}$. An $R-$module H is called $h-$ divisible if it is an epic image of a direct sum of copies of Q. The trace of a submodule N of an $R-$module M, which is denoted by $Tr_M(N)$, is the $R-$submodule of M generated by the images of all $R-$homomorphisms in $Hom_R(N, M)$. Finally, an $R-$module M splits if its torsion part, tM, is a summand of M.

Definition 2.1. *Let M be an $R-$module over a commutative ring R. A bilinear map on M is a function $\mu : M \times M \longrightarrow M$ such that for all $m, n, m_i, n_i \in M$ and $r \in R$:*

(1) $\mu(m_1 + m_2, n) = \mu(m_1, n) + \mu(m_2, n)$;

(2) $\mu(m, n_1 + n_2) = \mu(m, n_1) + \mu(m, n_2)$;

(3) $\mu(rm, n) = \mu(m, rn) = r\mu(m, n)$.

Definition 2.2. *Let μ and ν be bilinear maps on M and $r \in R$. We define*

$$(\mu + \nu)(m, n) = \mu(m, n) + \nu(m, n)$$

$$(r\mu)(m, n) = r \cdot \mu(m, n).$$

The set of all bilinear maps on M forms an $R-$module which is denoted by $Mult_R(M)$. We call any element of $Mult_R(M)$ a multiplication on the $R-$module M.

Definition 2.3. *Let N be a submodule of an $R-$module M. Then, we say that M is nil modulo N if $\mu(M \times M) \leq N$ for all $\mu \in Mult_R(M)$.*

Definition 2.4. *Let M be a module over a commutative ring R. The square submodule of M is denoted by $\Box_R M$ and is defined as:*

$$\Box_R M = \sum \{Im(\varphi) \mid \varphi \in Mult_R(M)\}.$$

We use the symbol $\Box M$ if no ambiguity arises.

Definition 2.5. *Let M be a module over a commutative ring R. Then M is called a nil module if $\Box M = 0$.*

Clearly, M is a nil $R-$module if and only if M is nil modulo 0. Moreover, $\Box M$ is the intersection of all submodules N of M such that M is nil modulo N, i.e., the smallest $R-$submodule N of M such that M is nil modulo N.

Theorem 2.6. *Let M be a module over a commutative ring R with $S = End_R(M)$. Then*

(1) $Mult_R(M) \cong Hom_R(M \otimes_R M, M) \cong Hom_R(M, End_R(M))$,

(2) $\Box M = Tr_M(M \otimes_R M) = Tr_S(M)M$.

Proof. 1) Straightforward.
2) We prove that $Tr_M(M \otimes_R M) = Tr_S(M)M$. To do this, we observe that for any $\theta \in Hom_R(M, S)$ and for any $x, y \in M$, the map

$$f : M \times M \to M$$

$$f(x, y) = \theta(x)(y)$$

is a bilinear map which induces the $R-$homomorphism,

$$\varphi : M \otimes_R M \to M$$

$$\varphi(x \otimes y) = \theta(x)(y).$$

But M is nil modulo $\Box M$, hence $\theta(n)(m) = \varphi(m \otimes n) \in \Box M$ for all $\theta \in Hom_R(M, S)$ and $m, n \in M$, thus $Tr_S(M)M \subseteq \Box M$. Conversely, for any $\varphi \in Hom_R(M \otimes_R M, M)$ and any $m \in M$, the map $\theta : M \to S$ such that,

$$\theta(m) : M \to M$$

$$\theta(m)(n) = \varphi(m \otimes n),$$

is an $R-$homomorphism which satisfies $\theta(m)(n) \in Tr_S(M)M$, hence $\varphi(m \otimes n) \in Tr_S(M)M$ which means M is nil modulo $Tr_S(M)M$. Therefore $\Box M \subseteq Tr_S(M)M$ and consequently $\Box M = Tr_S(M)M$. □

Proposition 2.7. *Let I be an ideal of commutative domain R with quotient field Q and $S = End_R(I)$. Then*

(1) $\Box I = (S : I)I^2$. In particular, $\Box I = 0$ if and only if $I = 0$.

(2) $\Box I = I$ for every ideal I of R exactly if R is a Clifford regular domain.

Proof. 1) By Theorem 2.6 we have $\Box I = Tr_S(I)I$. Hence, it suffices to show that $Tr_S(I) = (S : I)I$. This follows from the fact that $S = I : I$ and every $R-$homomorphism $\theta : I \to S$ can be considered as an element of $S : I$. Now the last assertion is clear.
2) An appeal to [4], shows that $(S : I)I = (I : I^2)I^2$. Therefore, $\Box I = I$ exactly if the isomorphy class $[I]$ of I is a regular element of the semigroup $S(R)$ of fractional ideals of R. □

Proposition 2.8. *Let M be a reduced torsion module over a Dedekind domain R. Then $\Box M = M$.*

Proof. See [3, Proposition 5.2]. □

Theorem 2.9. *Let R be a commutative domain. Then,*

(1) If R is a Prüfer domain, then relative divisibility and purity are equivalent.

(2) The torsion submodule of all mixed $R-$modules are pure if and only R is Prüfer.

Proof. See [8, Theorem 8.11 and Theorem 8.12 page 47]. □

Theorem 2.10. *Let R be a commutative domain. Then, the torsion submodule of an $h-$divisible module is a summand.*

Proof. See [8, Lemma 2.2 page 251]. □

Theorem 2.11. *Let A and C be Abelian groups. Then, there are isomorphisms*

$$t(A \otimes C) \cong [tA \otimes tC] \oplus [tA \otimes C/tC] \oplus [A/tA \otimes tC],$$
$$(A \otimes C)/t(A \otimes C) \cong A/tA \otimes C/tC.$$

Proof. See [7, Theorem 61.5]. □

3 Non-nil Quotients Modulo Square Submodule

In this section, we investigate about the torsion part of any module over a commutative ring. Moreover, we have have some results related to splitting modules which we need in our discussions.

Lemma 3.1. *Let $M = \bigoplus_{i=1}^{i=n} M_i$ be a module over a commutative domain R. Then M splits if and only if each M_i is splitting for all $i = 1, 2, ..., n$.*

Proof. It is sufficient to prove for the case $n = 2$.

$\Rightarrow$) Suppose that M_1 and M_2 splits, hence there exist submodules N_1 of M_1 and N_2 of M_2 such that $M_1 = tM_1 \oplus N_1$ and $M_2 = tM_2 \oplus N_2$. Therefore

$$M = M_1 \oplus M_2 = (tM_1 \oplus N_1) \oplus (tM_2 \oplus N_2) = t(M_1 \oplus M_2) \oplus (N_1 \oplus N_2).$$

$\Leftarrow$) Suppose that M splits, then the sequence

$$0 \to \ tM_1 \oplus tM_2 \xrightarrow{\alpha} M_1 \oplus M_2$$

is an splitting exact sequence, in which $\alpha(x) = x$. This means that there exists a homomorphism β as

$$0 \to \ M_1 \oplus M_2 \xrightarrow{\beta} tM_1 \oplus tM_2 \ ,$$

such that $\beta\alpha$ is identity on $tM_1 \oplus tM_2$. Now we show that the exact sequence

$$0 \to \ tM_1 \oplus tM_2 \xrightarrow{\alpha_1} M_1 \oplus tM_2$$

where $\alpha_1(x) = \alpha(x)$ for all $x \in tM_1 \oplus tM_2$, is an splitting exact. To do this, we define $\beta_2 : M_1 \oplus tM_2 \to tM_1 \oplus tM_2$ by $\beta_2(x) = \beta(x)$ for all $x \in M_1 \oplus tM_2$. Therefore,

$$\beta_2\alpha_2(x) = \beta_2(\alpha_2(x)) = \beta_2(\alpha_1(x)) = \beta_1(\alpha_1(x)) = x.$$

Consequently, $\beta_2\alpha_2$ is identity on $tM_1 \oplus tM_2$, which yields $M_1 \oplus tM_2$ is splitting. Thus, M_1 splits. By the same manner we deduce that M_2 is splitting. □

Lemma 3.2. *Let M be a mixed module over a Prüfer domain R. If M/tM is divisible, then $t(M \otimes M) = t(M) \otimes t(M)$.*

Proof. We observe that the following exact sequence of $R-$modules is pure;

$$0 \to t(M) \to M \to M/t(M) \to 0.$$

In fact, in view of Theorem 2.9, tM is a pure submodule of M. Now tensoring this sequence successively with $t(M), M$ and $M/t(M)$ yields the following pure exact sequences,

$$0 \to t(M) \otimes t(M) \to M \otimes t(M) \to M/t(M) \otimes t(M) \to 0$$

$$0 \to t(M) \otimes M \to M \otimes M \to M/t(M) \otimes M \to 0$$

$$0 \to t(M) \otimes M/t(M) \to M \otimes M/t(M) \to M/t(M) \otimes M/t(M) \to 0.$$

Now in view of the fact that the tensor product of a torsion module with a divisible module is the zero module, we reach the pure exact sequence,

$$0 \to t(M) \otimes t(M) \to M \otimes M \to M/t(M) \otimes M/t(M) \to 0.$$

Consequently,

$$t(M \otimes M) = t(M) \otimes t(M).$$

□

In the case of Abelian groups, we get the following corollary.

Corollary 3.3. *Let A be a mixed group such that $A/t(A)$ is $p-$divisible and $t(A)$ is a $p-$group. Then*

$$t(A \otimes A) = t(A) \otimes t(A).$$

Proof. Follows from Theorem 2.11 and the fact that the tensor product of a $p-$group with a $p-$divisible group is zero. □

Proposition 3.4. *Let M and N be $h-$divisible modules over a commutative domain R. Then $M \otimes_R N$ is splitting.*

Proof. First we observe that $M \otimes_R N$ is $h-$divisible. Now our assertion follows from Theorem 2.10. □

In the case of Abelian groups we have the following theorem of I. M. Irwin and his colleagues.

Corollary 3.5. *Let X, Y be groups such that $t(X), t(Y)$ are $p-$primary and $X/t(X)$, $Y/t(Y)$ are $p-$divisible. If $\{x_i\}_{i \in I}$ is a maximal torsion-free linear independent subset of X such that x_i is of infinite $p-$height, then $X \otimes Y$ splits.*

Proof. See [9, Theorem 3.2] □

4 Examples

Now, we are in the position to give examples of rank one and rank n mixed Abelian groups A in which $A/\Box A$ are not nil. Let p be a prime number. Let

$$M = \bigoplus_{i=1}^{i=\infty} \mathbb{Z}(x_i) \oplus \mathbb{Q},$$

in which $o(x_i) = p^i$. Take $A = \mathbb{Z}(x_i + p^{-i}\ ;\ i = 1, 2, 3, ...)$. Clearly, A is of rank one. On the other hand $p^i(x_i + p^{-i}) = 1$, so 1 is of infinite $p-$height. Moreover, $t(A) = t(M) \cap A$ is a $p-$group. Now we have

$$< 1 >\subseteq p^{\infty}A \subseteq A \cap p^{\infty}\mathbb{Q} =< 1 >,$$

which implies that A is a $p-$reduced and non-splitting group. But 1 has infinite order and infinite $p-$height in A, hence by Corollary 3.5, $A \otimes A$ splits and by Corollary 3.3, $A \otimes A = (t(A \otimes A)) \oplus D$ for some subgroup D of $A \otimes A$. Now using the fact that $Hom_{\mathbb{Z}}(D, A) = 0$, we have

$$\begin{aligned} Mult_{\mathbb{Z}}(A) &\cong Hom_{\mathbb{Z}}(A \otimes_{\mathbb{Z}} A, A) \\ &\cong Hom_{\mathbb{Z}}((t(A) \otimes_{\mathbb{Z}} t(A)) \oplus D, A) \\ &\cong Hom_{\mathbb{Z}}(t(A) \otimes_{\mathbb{Z}} t(A), A) \\ &\cong Hom_{\mathbb{Z}}(t(A) \otimes_{\mathbb{Z}} t(A), t(A)). \end{aligned}$$

Therefore, by (2) of Theorem 2.6 and Proposition 2.8, $\Box A = t(A)$. Consequently,

$$A/\Box A = A/t(A) \cong< p^{-i}; i = 1, 2, 3, ... >= \mathbb{Q}^p$$

is not a nil group. A is an example of a rank one mixed group with $A/\Box A$ non-nil.

For a rank n mixed group, take $B = \oplus_{i=1}^{i=n} A_i$ where $A_i \cong A$. In view of Lemma 3.1, B is not a splitting group. On the other hand by Corollary 3.5, $B \otimes B$ splits hence $B \otimes B = (t(B) \otimes t(B)) \oplus D'$ for some subgroup D' of $B \otimes B$. Now as in the case of rank one group A we conclude that $B = t(B)$ and $B/\Box B = B/t(B)$ is not a nill group.

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