Gen. Math. Notes, Vol. 32, No. 2, February 2016, pp.1-14
ISSN 2219-7184; Copyright ©ICSRS Publication, 2016
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On the Classes of β-γ-c-Open Sets and βc-γ-Open Sets in Topological Spaces

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(Received: 25-9-15 / Accepted: 9-1-16)

Abstract

In this paper, we introduce and study the notion of β-γ-c-open sets and βc-γ-open sets in topological spaces and investigate some of their properties. Also, we study the β-γ-continuous functions, β-γ-c-continuous functions and βc-γ-continuous functions and derive some of their properties.

Keywords: *β-open; βc-open; βc-γ-open; β-γ-c-open sets; β-γ-continuous; β-γ-c-continuous; βc-γ-continuous functions.*

1 Introduction

β-Open sets and their properties were studied by Abd El-Monsef [6]. El-Mabhouh and Mizyed [1] introduced the class of βc-open sets which stronger than β-open sets. Also, Mizyed [2] defined a new class of continuous functions called βc-continuous functions. In [5] Ogata defined an operation γ on a topological space and introduced the notion of τ_γ which is the collection of all γ-open sets in a topological space (X, τ).
In this paper, we introduce the notion of β-γ-c-open sets, βc-γ-open sets, β-γ-continuous functions, β-γ-c-continuous functions and βc-γ-continuous functions in topological spaces and investigate some of their fundamental properties.
First, we recall some of the basic definitions and results used in this paper.

2 Preliminaries

Throughout this paper, unless otherwise stated, (X, τ) and (Y, σ) represent topological spaces with no separation axioms are assumed. For a subset $A \subseteq X$, $Cl(A)$ and $Int(A)$ denote the closure of A and the interior of A respectively.
A subset A of a topological space (X, τ) is called β-open [6] if $A \subseteq Cl(Int(Cl(A)))$. The complement of β-open set is β-closed set. The family of all β-open sets of X is denoted by $\beta O(X)$.

Definition 2.1 *[1] A β-open set A of a space X is called βc-open if for each $x \in A$, there exists a closed set F such that $x \in F \subseteq A$.*

Remark 2.2 *[1] A subset B of X is βc-*closed *if and only if $X \backslash B$ is βc-open set. We denote to the families of βc-open sets and βc-closed sets in a topological spaces (X, τ) by $\beta CO(X)$ and $\beta CC(X)$ respectively.*

Definition 2.3 *[6] A function $f : (X, \tau) \to (Y, \sigma)$ is called β-continuous if the inverse image of each open subset of Y is β-open in X.*

Definition 2.4 *[2] Let (X, τ) and (Y, σ) be two topological spaces. The function $f : X \to Y$ is called βc-continuous function at a point $x \in X$ if for each open set V of Y containing $f(x)$, there exists a βc-open set U of X containing x such that $f(U) \subseteq V$. If f is βc-continuous at every point x of X, then it is called βc-continuous.*

Proposition 2.5 *[2] A function $f : X \to Y$ is βc-continuous if and only if the inverse image of every open set in Y is βc-open set in X.*

Corollary 2.6 *[2] Every βc-continuous function is β-continuous function.*

Definition 2.7 *[5] Let (X, τ) be a topological space. An operation $\gamma : \tau \to P(X)$ is a mapping from τ to the power set of X such that $V \subseteq \gamma(V)$ for every $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V.*

Definition 2.8 *[5] A subset A of a topological space (X, τ) is called γ-open if for each $x \in A$ there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. τ_γ denotes the set of all γ-open sets in X.*

Remark 2.9 *[5] For any topological space (X, τ), $\tau_\gamma \subseteq \tau$.*

Definition 2.10 *[4] Let (X, τ) be a topological space and A is a subset of X, then τ_γ-$Int(A) = \bigcup\{U : U$ is a γ-open set and $U \subseteq A\}$.*

Definition 2.11 *[5] A topological space (X, τ) is said to be γ-regular, where γ is an operation on τ, if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U)$ contained in V.*

Proposition 2.12 *[5] If (X, τ) is γ-regular, then $\tau = \tau_\gamma$.*

Definition 2.13 *[3] A function $f : X \to Y$ is said to be γ-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a γ-open set U containing x such that $f(U) \subseteq V$.*

Definition 2.14 *[7] An operation γ on $\beta O(X)$ is a mapping $\gamma : \beta O(X) \to P(X)$ is a mapping from $\beta O(X)$ to the power set $P(X)$ of X such that $V \subseteq \gamma(V)$ for each $V \in \beta O(X)$.*

Remark 2.15 *It is clear that $\gamma(X) = X$ for any operation γ. Also, we assumed that $\gamma(\emptyset) = \emptyset$.*

Definition 2.16 *[7] Let (X, τ) be a topological space and γ an operation on $\beta O(X)$. Then a subset A of X is said to be β-γ-open if for each $x \in A$, there exists a β-open set U such that $x \in U \subseteq \gamma(U) \subseteq A$.*

Remark 2.17 *[7] A subset B of X is called β-γ-closed set if $X \backslash B$ is β-γ-open set. The family of all β-γ-open sets (resp., β-γ-closed sets) of a topological space X is denoted by $\beta O(X)_\gamma$ (resp., $\beta C(X)_\gamma$).*

Definition 2.18 *Let γ be an operation on $\beta O(X)$. Then*

1. *[7] $\beta O(X)_\gamma$-$Cl(A)$ is defined as the intersection of all β-γ-closed sets containing A.*

2. *$\beta O(X)_\gamma$-$Int(A)$ is defined as the union of all β-γ-open sets contained in A.*

Proposition 2.19 *[7] Let γ be an operation on $\beta O(X)$. Then the following statements hold:*

(i) Every γ-open set of (X, τ) is β-γ-open.

(ii) Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of β-γ-open sets in (X, τ). Then, $\bigcup\{A_\alpha : \alpha \in J\}$ is also a β-γ-open set in (X, τ).

3 β-γ-c-Open Sets

In this section, the notion of β-γ-c-open sets is defined and related properties are investigated.

Definition 3.1 *A subset $A \in \beta O(X)_\gamma$ is called β-γ-c-open set if for each $x \in A$, there exists a closed set F such that $X \in F \subseteq A$.*

Remark 3.2 *A subset B of X is called β-γ- c-closed set if $X \backslash B$ is β-γ-c-open set. The family of all β-γ-c-open sets (resp., β-γ-c-closed sets) of a topological space X is denoted by $\beta\gamma CO(X)$ (resp., $\beta\gamma CC(X)$).*

Proposition 3.3 *Let γ be an operation on $\beta O(X)$. Then $\beta\gamma CO(X) \subseteq \beta O(X)_\gamma$, for any space X.*

Directly, from Definition 2.16 and Definition 3.1.

Remark 3.4 *The equality in Proposition 3.3 need be true in general. Consider the following examples.*

Example 3.5 *Consider $X = \{a, b, c\}$ with $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Define an operation γ on $\beta O(X)$ by*

$$\gamma(A) = \begin{cases} A & \text{if } b \in A \\ Cl(A) & \text{if } b \notin A \end{cases}$$

Then,

- $\beta O(X)_\gamma = \{\phi, X, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$
- $\beta\gamma CO(X) = \{\phi, X, \{a, c\}, \{b, c\}\}$

Hence, $\{a, b\} \in \beta O(X)_\gamma$ but $\{a, b\} \notin \beta\gamma CO(X)$.

Proposition 3.6 *Let $\{A_\alpha : \alpha \in \Delta\}$ be any collection of β-γ-c-open sets in a topological space (X, τ). Then, $\bigcup_{\alpha\in\Delta} A_\alpha$ is a β-γ-c-open set.*

Proof. Let $\{A_\alpha : \alpha \in \Delta\}$ be any collection of β-γ-c-open sets in a topological space (X, τ). Then, A_α is β-γ-open set for each $\alpha \in \Delta$. So that, by Part (ii) of Proposition 2.19, $\bigcup_{\alpha\in\Delta} A_\alpha$ is a β-γ-open set. If $x \in \bigcup_{\alpha\in\Delta} A_\alpha$, then there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha_0}$. Since A_{α_0} is β-γ-c-open, there exists a closed set F such that $x \in F \subseteq A_{\alpha_0}$. Therefore, $x \in F \subseteq A_{\alpha_0} \subseteq \bigcup_{\alpha\in\Delta} A_\alpha$. Hence, by Definition 3.1, $\bigcup_{\alpha\in\Delta} A_\alpha$ is β-γ-c-open set.

Remark 3.7 *The intersection of two β-γ-c-open sets need not be β-γ-c-open set. Consider the following example.*

Example 3.8 *In Example 3.5, $\{a, c\} \in \beta\gamma CO(X)$ and $\{b, c\} \in \beta\gamma CO(X)$ while, $\{a, c\} \cap \{b, c\} = \{c\} \notin \beta\gamma CO(X)$.*

Proposition 3.9 *A subset A is β-γ-c-open in a space X if and only if for each $x \in A$, there exists a β-γ-c-open set B such that $x \in B \subseteq A$.*

Proof. Let A be β-γ-c-open set and $x \in A$. Then $B = A$ such that $x \in B \subseteq A$. Conversely, if for each $x \in A$, there exists a β-γ-c-open set B_x such that $x \in B_x \subseteq A$, then $A = \bigcup_{x \in A} B_x$. Hence, by Proposition 3.6, A is β-γ-c-open set.

Proposition 3.10 *Arbitrary intersection of β-γ-c-closed sets is β-γ-c-closed set.*

Proof. Directly by Proposition 3.6 and De Morgan Laws.

Definition 3.11 *[7] An operation γ on $\beta O(X)$ is said to be β-regular if for each $x \in X$ and for every pair of β-open sets U and V containing x, there exists a β-open set W such that $x \in W$ and $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.*

Proposition 3.12 *Let γ be β-regular operation on $\beta O(X)$. If A and B are β-γ-c-open sets, then $A \cap B$ is β-γ-c-open set.*

Proof. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are β-γ-open sets, there exist β-open sets U and V such that $x \in U \subseteq \gamma(U) \subseteq A$ and $x \in V \subseteq \gamma(V) \subseteq B$. Since γ are β-regular, there exists β-open set W such that $x \in W \subseteq \gamma(W) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B$. Therefore, $A \cap B$ is β-γ-open set. Since A and B are β-γ-c- sets, there exist closed sets E and F such that $x \in E \subseteq A$ and $x \in F \subseteq B$. Therefore, $x \in E \cap F \subseteq A \cap B$ where $E \cap F$ is closed set. Hence, by Definition 3.1, $A \cap B$ is β-γ-c-open set.

Corollary 3.13 *Let γ be β-regular operation on $\beta O(X)$. Then, $\beta\gamma CO(X)$ forms a topology on X.*

Proof. Directly, from Proposition 3.6 and Proposition 3.12.

Proposition 3.14 *Let X be T_1 space and γ be an operation on $\beta O(X)$. Then $\beta\gamma CO(X) = \beta O(X)_\gamma$.*

Proof. Let X be T_1 space and γ be an operation on $\beta O(X)$. If $A \in \beta O(X)_\gamma$, then A is β-γ-open set. Since X is T_1 space, then for any $x \in A$, $x \in \{x\} \subseteq A$ where $\{x\}$ is closed. Hence, A is β-γ-c-open and $\beta O(X)_\gamma \subseteq \beta\gamma CO(X)$. Conversely, by Proposition 3.3, $\beta\gamma CO(X) \subseteq \beta O(X)_\gamma$. Therefore, $\beta\gamma CO(X) = \beta O(X)_\gamma$.

Corollary 3.15 *Let X be T_1 space and γ be an operation on $\beta O(X)$. Then every γ-open set is β-γ-c-open set.*

Proof. Directly, by Part (i) of Proposition 2.19 and Proposition 3.14.

Proposition 3.16 *Let X be a locally indiscrete space. Then, every γ-open is β-γ-c-open set.*

Proof. Let A be γ-open set, then by Part (i) of Proposition 2.19, A is β-γ-open. Since A is γ-open, then A is open. But X is locally indiscrete which implies, A is closed. Hence, by Definition 3.1, A is β-γ-c-open set.

Proposition 3.17 *Let X be a regular space. Then, every γ-open is β-γ-c-open set.*

Proof. Let A be γ-open set, then by Part (i) of Proposition 2.19, A is β-γ-open. Since X is regular, then for any $x \in A$, there exists an open set G such that $x \in G \subseteq Cl(G) \subseteq A$. Hence, by Definition 3.1, A is β-γ-c-open set.

Corollary 3.18 *Let X be both regular and γ-regular space. Then, every open is β-γ-c-open set.*

Proof. Let X be regular space. Then, by Proposition 3.17, every γ-open is β-γ-c-open. Since X is γ-regular, then by Proposition 2.12, γ-open and open sets are the same. Hence, every open is β-γ-c-open set.

Definition 3.19 *Let (X, τ) be a topological space with an operation γ on $\beta O(X)$ and $A \subseteq X$.*

1. *The union of all β-γ-c-open sets contained in A is called the β-γ-c-*interior *of A and denoted by $\beta\gamma c$-$Int(A)$.*
2. *The intersection of all β-γ-c-closed sets containing A is called the β-γ-c-*closure *of A and denoted by $\beta\gamma c$-$Cl(A)$.*

Now, we state the following propositions without proofs.

Proposition 3.20 *For subsets A and B of X with an operation γ on $\beta O(X)$. The following statements hold.*

1. $A \subseteq \beta\gamma c\text{-}Cl(A)$.
2. $\beta\gamma c\text{-}Cl(\phi) = \phi$ *and* $\beta\gamma c\text{-}Cl(X) = X$.
3. *A is β-γ-c-closed if and only if* $\beta\gamma c\text{-}Cl(A) = A$.
4. *If $A \subseteq B$, then* $\beta\gamma c\text{-}Cl(A) \subseteq \beta\gamma c\text{-}Cl(B)$.
5. $\beta\gamma c\text{-}Cl(A) \cup \beta\gamma c\text{-}Cl(B) \subseteq \beta\gamma c\text{-}Cl(A \cup B)$.
6. $\beta\gamma c\text{-}Cl(A \cap B) \subseteq \beta\gamma c\text{-}Cl(A) \cap \beta\gamma c\text{-}Cl(B)$.
7. $x \in \beta\gamma c\text{-}Cl(A)$ *if and only if $V \cap A \neq \phi$ for every β-γ-c-open set V containing x.*

Proposition 3.21 *For subsets A and B of X with an operation γ on $\beta O(X)$. The following statements hold.*

1. $\beta\gamma c\text{-}Int(A) \subseteq A$.
2. $\beta\gamma c\text{-}Int(\phi) = \phi$ *and* $\beta\gamma c\text{-}Int(X) = X$.
3. *A is β-γ-c-open if and only if* $\beta\gamma c\text{-}Int(A) = A$.
4. *If $A \subseteq B$, then* $\beta\gamma c\text{-}Int(A) \subseteq \beta\gamma c\text{-}Int(B)$.
5. $\beta\gamma c\text{-}Int(A) \cup \beta\gamma c\text{-}Int(B) \subseteq \beta\gamma c\text{-}Int(A \cup B)$.
6. $\beta\gamma c\text{-}Int(A \cap B) \subseteq \beta\gamma c\text{-}Int(A) \cap \beta\gamma c\text{-}Int(B)$.
7. $x \in \beta\gamma c\text{-}Int(A)$ *if and only if there exists β-γ-c-open set V such that* $x \in V \subseteq A$.

4 βc-γ-Open Sets

Now, we study the class of βc-γ-open sets and we investigate some of the related properties.

Definition 4.1 *Let (X, τ) be a topological space and γ an operation on $\beta O(X)$. Then a subset A of X is said to be βc-γ-open if for each $x \in A$, there exists a βc-open set U such that $x \in U \subseteq \gamma(U) \subseteq A$.*

Remark 4.2 *A subset B of X is called βc-γ-closed set if $X \backslash B$ is βc-γ-open set. The family of all βc-γ-open sets (resp., βc-γ-closed sets) of a topological space X is denoted by $\beta CO(X)_\gamma$ (resp., $\beta CC(X)_\gamma$).*

Proposition 4.3 *Let γ be an operation on $\beta O(X)$. Then $\beta CO(X)_\gamma \subseteq \beta O(X)_\gamma$, for any space X.*

Proof. Let γ be an operation on $\beta O(X)$ and A be a βc-γ-open set. Then, for any $x \in A$, there exists a βc-open set U such that $x \in U \subseteq \gamma(U) \subseteq A$. Since every βc-open is β-open, U is β-open. Therefore, by Definition 2.16, A is β-γ-open set.

Proposition 4.4 *Let γ be an operation on $\beta O(X)$. Then $\beta CO(X)_\gamma \subseteq \beta\gamma CO(X)$, for any space X.*

Proof. Let A be βc-γ-open set. Then for any $x \in A$, there exists βc-open set U such that $x \in U \subseteq \gamma(U) \subseteq A$. Since U is βc-open, then U is β-open which implies, $A \in \beta O(X)_\gamma$. Since U is βc-open and $x \in U$, there exists a closed set F such that $x \in F \subseteq U \subseteq A$. Hence, A is β-γ-c-open set

Proposition 4.5 *Let X be T_1 space and γ be an operation on $\beta O(X)$. Then $\beta CO(X)_\gamma = \beta\gamma CO(X)$.*

Proof. Let X be T_1 space with an operation γ on $\beta O(X)$ and A be a β-γ-c-open set. Then, for any $x \in A$, there is β-open set U such that $x \in U \subseteq \gamma(U) \subseteq A$. Since for each $x \in U$, $x \in \{x\} \subseteq U$ where $\{x\}$ is a closed set in T_1 space. Hence, by Definition 2.1, U is βc-open set. Therefore, by Definition 4.1, A is βc-γ-open set and so, $\beta\gamma CO(X) \subseteq \beta CO(X)_\gamma$. On the other hand, by Proposition 4.4, $\beta CO(X)_\gamma \subseteq \beta\gamma CO(X)$. Hence, $\beta CO(X)_\gamma = \beta\gamma CO(X)$.

Proposition 4.6 *Let X be a regular space and γ be an operation on $\beta O(X)$. Then, $\beta CO(X)_\gamma = \beta\gamma CO(X)$.*

Proof. Let X be a regular space with an operation γ on $\beta O(X)$ and A be a β-γ-c-open set. Then, for any $x \in A$, there is β-open set U such that $x \in U \subseteq \gamma(U) \subseteq A$. Since for each $x \in U$, there exists an open set G such that $x \in G \subseteq Cl(G) \subseteq U$. Hence, by Definition 2.1, U is βc-open set. Therefore, by Definition 4.1, A is βc-γ-open set and so, $\beta\gamma CO(X) \subseteq \beta CO(X)_\gamma$. On the other hand, by Proposition 4.4, $\beta CO(X)_\gamma \subseteq \beta\gamma CO(X)$. Hence, $\beta CO(X)_\gamma = \beta\gamma CO(X)$.

Definition 4.7 *Let (X, τ) be a topological space with an operation γ on $\beta O(X)$ and $A \subseteq X$.*

1. *The union of all βc-γ-open sets contained in A is called the βc-γ-*interior *of A and denoted by $\beta c\gamma$-$Int(A)$.*

2. *The intersection of all βc-γ-closed sets containing A is called the βc-γ-*closure *of A and denoted by $\beta c\gamma$-$Cl(A)$.*

Now, we state the following propositions without proofs.

Proposition 4.8 *For subsets A and B of X with an operation γ on $\beta O(X)$. The following statements hold.*

1. $A \subseteq \beta c\gamma$-$Cl(A)$.
2. $\beta c\gamma$-$Cl(\phi) = \phi$ *and* $\beta c\gamma$-$Cl(X) = X$.
3. *A is βc-γ-closed if and only if* $\beta c\gamma$-$Cl(A) = A$.
4. *If $A \subseteq B$, then* $\beta c\gamma$-$Cl(A) \subseteq \beta c\gamma$-$Cl(B)$.
5. $\beta c\gamma$-$Cl(A) \cup \beta c\gamma$-$Cl(B) \subseteq \beta c\gamma$-$Cl(A \cup B)$.
6. $\beta c\gamma$-$Cl(A \cap B) \subseteq \beta c\gamma$-$Cl(A) \cap \beta c\gamma$-$Cl(B)$.
7. $x \in \beta c\gamma$-$Cl(A)$ *if and only if $V \cap A \neq \phi$ for every βc-γ-open set V containing x.*

Proposition 4.9 *For subsets A and B of X with an operation γ on $\beta O(X)$. The following statements hold.*

1. $\beta c\gamma$-$Int(A) \subseteq A$.
2. $\beta c\gamma$-$Int(\phi) = \phi$ *and* $\beta c\gamma$-$Int(X) = X$.
3. *A is βc-γ-open if and only if* $\beta c\gamma$-$Int(A) = A$.
4. *If $A \subseteq B$, then* $\beta c\gamma$-$Int(A) \subseteq \beta c\gamma$-$Int(B)$.
5. $\beta c\gamma$-$Int(A) \cup \beta c\gamma$-$Int(B) \subseteq \beta c\gamma$-$Int(A \cup B)$.
6. $\beta c\gamma$-$Int(A \cap B) \subseteq \beta c\gamma$-$Int(A) \cap \beta c\gamma$-$Int(B)$.
7. $x \in \beta c\gamma$-$Int(A)$ *if and only if there exists βc-γ-open set V such that* $x \in V \subseteq A$.

5 β-γ-Continuous Functions, β-γ-c-Continuous Functions and βc-γ-Continuous Functions

Definition 5.1 *Let (X, τ) and (Y, σ) be two topological spaces with an operation γ on $\beta O(X)$. Then $f : (X, \tau) \longrightarrow (Y, \sigma)$ is called,*

1. *β-γ-continuous if for each $x \in X$ and for each open set V of Y containing $f(x)$, there exists a β-γ-open set U of X containing x such that $f(U) \subseteq V$.*

2. *β-γ-c-continuous if for each $x \in X$ and for each open set V of Y containing $f(x)$, there exists a β-γ-c-open set U of X containing x such that $f(U) \subseteq V$.*

3. *βc-γ-continuous if for each $x \in X$ and for each open set V of Y containing $f(x)$, there exists a βc-γ-open set U of X containing x such that $f(U) \subseteq V$.*

Corollary 5.2 *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function with an operation γ on $\beta O(X)$. Then,*

1. *f is β-γ-continuous if and only if the inverse image of every open set in Y is a β-γ-open set in X.*

2. *f is β-γ-c-continuous if and only if the inverse image of every open set in Y is a β-γ-c-open set in X.*

3. *f is βc-γ-continuous if and only if the inverse image of every open set in Y is a βc-γ-open set in X.*

Corollary 5.3 *Let $f : (X, \tau) \longrightarrow (Y, \sigma)$ be a function with an operation γ on $\beta O(X)$. Then,*

1. *Every βc-γ-continuous is β-γ-c-continuous function.*

2. *Every β-γ-c-continuous is βc-continuous function.*

3. *Every β-γ-c-continuous is β-γ-continuous function.*

4. *Every β-γ-continuous is β-continuous function.*

Remark 5.4 *From Corollary 2.6 and Corollary 5.3, we obtain the following diagram of implications:*

$$\begin{array}{ccccc}
\beta c\text{-}\gamma\text{-con.} & \longrightarrow & \beta\text{-}\gamma\text{-}c\text{-con.} & \longrightarrow & \beta c\text{-con.} \\
 & & \downarrow & & \downarrow \\
\gamma\text{-con.} & \longrightarrow & \beta\text{-}\gamma\text{-con.} & \longrightarrow & \beta\text{-con.}
\end{array}$$

Where con. = continuous.
In the sequel, we shall show that none of the implications that concerning β-γ-continuity and β-γ-c-continuity is reversible.

Example 5.5 *Consider $X = \{a, b, c\}$ with the topology $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Define an operation γ on $\beta O(X)$ by*

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a\} \\ A \cup \{b\} & \text{if } A \neq \{a\} \end{cases}$$

Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a & \text{if } x = a \\ a & \text{if } x = b \\ c & \text{if } x = c \end{cases}$$

Then f is β-γ-continuous but not γ-continuous at b because $\{a, b\}$ is open set in (X, σ) containing $f(b) = a$ but there is no γ-open set U in (X, σ) containing b such that $f(U) \subseteq \{a, b\}$.

Example 5.6 *Consider $X = \{a, b, c\}$ with the topology $\tau = \sigma = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. Define an operation γ on $\beta O(X)$ by*

$$\gamma(A) = \begin{cases} A & \text{if } A = \{a, c\} \\ X & \text{if } A \neq \{a, c\} \end{cases}$$

Define a function $f : (X, \tau) \to (Y, \sigma)$ as follows:

$$f(x) = \begin{cases} a & \text{if } x = a \\ c & \text{if } x = b \\ b & \text{if } x = c \end{cases}$$

Then f is β-continuous but not β-γ-continuous at b because $\{a\}$ is open in (X, σ) and $f^{-1}(\{a\}) = \{a\}$ is not β-γ-open in (X, τ) because there is no β-open set U such that $a \in U \subseteq \gamma(U) \subseteq \{a, c\}$.

Example 5.7 *Let $X = \{a, b, c\}$ and define the topology $\tau = \sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Define an operation γ on $\beta O(X)$ by $\gamma(A) = A$ and define the function $f : (X, \tau) \to (X, \sigma)$ as follows*

$$f(x) = \begin{cases} a & \text{if } x = a \\ b & \text{if } x = b \\ c & \text{if } x = c \end{cases}$$

Then, f is β-γ-continuous function but not β-γ-c-continuous function because $\{a\}$ is an open set in (X,σ) and $f^{-1}(\{a\}) = \{a\}$ is not β-γ-c-open in (X,τ) because there is no closed set F such that $a \in F \subseteq \{a\}$.

Example 5.8 *Let $X = \{a,b,c\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}\}$ and $Y = \{1,2,3,4\}$ with the topology $\sigma = \{\phi, Y, \{1\}, \{1,2\}, \{1,2,3\}\}$. Define an operation γ on $\beta O(X)$ by*

$$\gamma(A) = \begin{cases} A & \text{if } A \neq \{a,c\} \\ X & \text{if } A = \{a,c\} \end{cases}$$

Define a function $f : (X,\tau) \to (Y,\sigma)$ as follows:

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 3 & \text{if } x = b \\ 1 & \text{if } x = c \end{cases}$$

Then, f is βc-continuous but not β-γ-c-continuous because $\{1\}$ is open in (Y,σ) and $f^{-1}(\{1\}) = \{a,c\}$ is not β-γ-open because there is no β-open set U such that $c \in U \subseteq \gamma(U) \subseteq \{a,c\}$. Hence, $\{a,c\}$ is not β-γ-c-open set in (X,τ).

Proposition 5.9 *A function $f : (X,\tau) \to (Y,\sigma)$ with an operation γ on $\beta O(X)$ is β-γ-continuous if and only if f is β-continuous and for each $x \in X$ and each open set V of Y containing $f(x)$, there exists a β-open set G of X containing x such that $f(\gamma(G)) \subseteq V$.*

Proof. Let f be β-γ-continuous such that $x \in X$ and V be any open set containing $f(x)$. By hypothesis, there exists a β-γ-open set U of X containing x such that $f(U) \subseteq V$. Since U is a β-γ-open set, then for each $x \in U$, there exists a β-open set G of X such that $x \in G \subseteq \gamma(G) \subseteq U$. Therefore, we have $f(\gamma(G)) \subseteq V$. Also, β-γ-continuous always implies β-continuous. Conversely, let V be any open set of Y. Since f is β-continuous, then $f^{-1}(V)$ is a β-open set in X. Let $x \in f^{-1}(V)$. Then $f(x) \in V$. By hypothesis, there exists a β-open set G of X containing x such that $f(\gamma(G)) \in V$. Which implies that, $x \in \gamma(G) \subseteq f^{-1}(V)$. Therefore, $f^{-1}(V)$ is a β-γ-open set in X. Hence, by Corollary 5.2, f is β-γ-continuous.

Proposition 5.10 *For a function $f : (X,\tau) \to (Y,\sigma)$ with γ-operation on $\beta O(X)$. The following are equivalent:*

1. *f is γ-continuous.*
2. *f is β-γ-continuous and for each open set V of Y, τ_γ-$Int(f^{-1}(V)) = \beta O(X)_\gamma$-$Int(f^{-1}(V))$.*

Proof. ($1 \Rightarrow 2$) Let f be γ-continuous and let V be any open set in Y, then $f^{-1}(V)$ is γ-open in X which implies by Proposition 2.19, $f^{-1}(V)$ is β-γ-open set and so, f is β-γ-continuous. Also, $\tau_\gamma\text{-}Int(f^{-1}(V)) = f^{-1}(V) = \beta O(X)_\gamma\text{-}Int(f^{-1}(V))$.
($2 \Rightarrow 1$) let V be any open set of Y. Since f is β-γ-continuous, then $f^{-1}(V)$ is β-γ-open set in X. So $f^{-1}(V) = \beta O(X)_\gamma\text{-}Int(f^{-1}(V)) = \tau_\gamma\text{-}Int(f^{-1}(V))$. Thus, $f^{-1}(V)$ is γ-open and hence, f is γ-continuous.

Proposition 5.11 *For a function $f : (X, \tau) \to (Y, \sigma)$ with γ-operation on $\beta O(X)$ and for each open set V of Y, $Int(f^{-1}(V)) = \beta O(X)_\gamma\text{-}Int(f^{-1}(V))$. The following are equivalent:*

1. *f is continuous.*
2. *f is β-γ-continuous.*

Proof. ($1 \Rightarrow 2$) Let V be any open set in Y. Since f is continuous, $f^{-1}(V)$ is open in X. Hence, $f^{-1}(V) = Int(f^{-1}(V)) = \beta O(X)_\gamma\text{-}Int(f^{-1}(V)) \in \beta O(X)_\gamma$. Therefore, f is β-γ-continuous.
($2 \Rightarrow 1$) Let V be any open set in Y. $f^{-1}(V)$ is a β-γ-open set in X. So $f^{-1}(V) = \beta O(X)_\gamma\text{-}Int(f^{-1}(V)) = Int(f^{-1}(V))$. Hence, $f^{-1}(V)$ is open in X. Therefore, f is continuous.
Now, we state the following two propositions without proofs.

Proposition 5.12 *Let γ be an operation on $\beta O(X)$. The following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$.*

1. *f is β-γ-c-continuous.*
2. *The inverse image of every closed set in Y is β-γ-c-closed set in X.*
3. *$f(\beta\gamma c\text{-}Cl(A)) \subseteq Cl(f(A))$, for every subset A of X.*
4. *$\beta\gamma c\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for every subset B of Y.*

Proposition 5.13 *Let γ be an operation on $\beta O(X)$. The following are equivalent for a function $f : (X, \tau) \to (Y, \sigma)$.*

1. *f is βc-γ-continuous.*
2. *The inverse image of every closed set in Y is βc-γ-closed set in X.*
3. *$f(\beta c\gamma\text{-}Cl(A)) \subseteq Cl(f(A))$, for every subset A of X.*
4. *$\beta c\gamma\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, for every subset B of Y.*

Proposition 5.14 *Let γ be an operation on $\beta O(X)$ and (X, τ) is a T_1 space. The following functions $f : (X, \tau) \to (Y, \sigma)$ are equivalent.*

1. *f is β-γ-c-continuous.*
2. *f is βc-γ-continuous.*
3. *f is β-γ-continuous.*

Proof. Directly, by Proposition 3.14, Proposition 4.5.

Acknowledgement: The author would like to thank the referees for their critical comments, suggestions and corrections towards the development of the paper.

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