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Degree of Approximation of Continuous Functions by (E, q) (C, δ) Means

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Abstract

In this paper, we obtain a theorem on the degree of approximation of function belonging to the Lipschitz class by (E, q) (C, δ) product means of its Fourier series. Our theorem provides the Jackson order as the degree of approximation.

Keywords: *Cesàro matrix, Euler matrix, degree of approximation.*

1 Definition and Notations

Let f be 2π – periodic and L- integrable over $[-\pi, \pi]$. The Fourier series of f at a point x is given by

$$(1.1) \qquad f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

A function $f \in Lip\,\alpha$ $(0 < \alpha \leq 1)$ if

(1.2) $$f(x+t)-f(x)=O(|t|^{\alpha}).$$

It may be observe that such functions are necessarily continuous.
The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial t_n of order *n* is defined by Zygmund [12, p-114],

(1.3) $$\|t_n - f\| = sup\{|t_n(x) - f(x)|: x \in R\},$$

Let $\sum_{n=0}^{\infty} a_n$ be given infinite series with the sequence (s_n) of partial sums of its first *(n+1)*-terms. The Euler means of the sequence (s_n) are defined by

$$(E,q) = E_n^q = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k, \quad (q \geq 0),$$

where E_n^0 is defined to be s_n. If $t_n \to s$; as $n \to \infty$, we say that (s_n) or $\sum_{n=0}^{\infty} a_n$ is summable (E,q) $(q > 0)$ to s or symbolically we write $(s_n) \in s(E,q)$, for $q > 0$. See Hardy [8, p-180] and for real and complex values of $q \neq -1$, see Chandra [5].

The sequence (s_n) is said to be summable (C,δ) $(\delta > -1)$ to limit *s* if

$$(A_n^{\delta})^{-1} \sum_{k=0}^{n} A_{n-k}^{\delta-1} s_k \to s \text{ as } n \to \infty.$$

where A_n^{δ} are the binomial coefficients. See Zygmund [12, p-76].

The (E,q) transform of the (C,δ) transform defines the $(E,q)(C,\delta)$ transform of the partial sums s_n of the series $\sum_{n=0}^{\infty} a_n$.

The transform $(E,q)(C,\delta)$ reduces to (E,q) and (C,δ) respectively for $\delta = 0$ and $q = 0$.

Thus if

$$\left(E_q C_\delta\right)_n = (1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} \left(A_v^{\delta}\right)^{-1} \sum_{k=0}^{n} A_{v-k}^{\delta-1} s_k \to s \text{ as } n \to \infty.$$

Then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable by $(E,q)(C,\delta)$ means or simply summable $(E,q)(C,\delta)$ to s.

Let $s_n(f\,;x)$ be the n^{th} partial sum of the series (1.1). Then $(E,q)(C,\delta)$ mean of $(s_n(f\,;x))$, where $q > 0$ and $\delta > -1$, is given by

(1.4) $$\left(E_q C_\delta\right)_n (f;x) = (1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} \left(A_v^\delta\right)^{-1} \sum_{k=0}^{n} A_{v-k}^{\delta-1} s_k(f;x).$$

We shall use the following notations for each $x \in R$:

(1.5) $$\emptyset_x(t) = f(x+t) + f(x-t) - 2f(x).$$

(1.6) $$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin(2n+1)(t/2)}{2\sin(t/2)}.$$

(1.7) $$K_v^\delta(t) = \left(A_v^\delta\right)^{-1} \sum_{k=0}^{v} A_{v-k}^{\delta-1} D_k(t)\,.$$

(1.8) $$A_v^\delta = \binom{v+\delta}{v} \qquad (\delta \geq 0).$$

2 Introduction

The degree of approximation of functions belonging to $Lip\,\alpha$ $(0 < \alpha \leq 1)$, by Cesàro means and Nörlund means have been discussed by a number of researchers like Lebesgue [9], Alexits [1] and Chandra [6].

In 1910, Lebesgue [9] proved the following :

Theorem A: *If* $f \in C_{2\pi} \cap Lip\alpha$ $(0 < \alpha \leq 1)$, *then*

(2.1) $$\|s_n(f) - f\| = O\{n^{-\alpha} \log n\}.$$

In 1961, Alexits [1, p-301] proved the following along with other results.

Theorem B: *If* $f \in C_{2\pi} \cap Lip\alpha$ $(0 < \alpha \leq 1)$, *then*

(2.2) $$\max_{0 \leq x \leq 2\pi} |f(x) - \sigma_n^r(f;x)| = O\{n^{-\alpha}\}.$$

where $0 < \alpha < r \leq 1$ *and* $\sigma_n^r(f;x)$ *is* (C,r)*-mean of* $s_n(f;x)$.
The case $\alpha = r = 1$ *was proved by Bernstein [3].*

In 1981, Chandra [6] proved the following :

Theorem C: *If* $f \in C_{2\pi} \cap Lip\alpha$ $(0 < \alpha \leq 1)$, *then*

(2.3) $$\|E_n^q(f) - f\| = O\{n^{-\alpha/2}\} \quad (q > 0).$$

The estimate in (2.3) was improved by Chandra [7].

In 2010, Nigam [10] obtained the following result on product summability method:

Theorem D: *If* $f \in C_{2\pi} \cap Lip\alpha \ (0 < \alpha < 1)$, *then*

(2.4) $$\|(EC)_n^1 - f\| = O\{(n+1)^{-\alpha}\}.$$

and, Tiwari and Bariwal [11] proved the following for $(E,1)(C,1)$ and $(E,q)(C,1)$ means of its Fourier series.

Theorem E: *If* $f \in C_{2\pi} \cap Lip\alpha \ (0 < \alpha < 1)$, *then*

(2.5) $$\left\|(EC)_n^q - f\right\| = O\{(n+1)^{-\alpha}\}.$$

In this paper we obtain a theorem on the degree of approximation of continuous functions by $(E,q)(C,\delta)$ means of its Fourier series. This generalizes the result for $(E,1)(C,1)$ and $(E,q)(C,1)$ means.

Theorem: *If* $f \in C_{2\pi} \cap Lip\alpha \ (0 < \alpha \le 1)$ *then*

(2.6) $$\left\|\left(E_q C_\delta\right)_n (f;x) - f(x)\right\| = \begin{cases} O\{(n+1)^{-\alpha}\}, & (0 < \alpha < \delta \le 1)\ (0 < \alpha \le 1,\ \delta > 1) \\ O\{(n+1)^{-\alpha}\log(n+1)\}, & (0 < \alpha \le \delta \le 1). \end{cases}$$

3 Lemmas

We shall use the following lemmas in the proof of the theorems:

Lemma 1[12, p-94]: *For* $(0 < \delta \le 1)$, $n = 1,2,3 - --,\ 0 < t \le \pi$,

(3.1) $$\left|K_v^\delta(t)\right| \le A_\delta v^{-\delta} t^{-(\delta+1)},$$

where A_δ *depending on* δ *only.*

Lemma 2[4]: *For* $q > 0$,

(3.2) $$\sum_{v=0}^{n} \binom{n}{v} q^{n-v}(v+1)^{-1} = O\left\{(1+q)^{n+1} \big/ (n+1)\right\}.$$

Lemma 3: *For* $\delta > 1$,

(3.3) $$\left|K_v^\delta(t)\right| = O(1)\left(\frac{\delta}{(v+1)t^2}\right).$$

Proof: By (1.8), we have

$$\left|K_v^\delta(t)\right| \le \left|\left(A_v^\delta\right)^{-1} \sum_{k=0}^{v} A_{v-k}^{\delta-1} D_k(t)\right|$$

$$\le \frac{1}{2\sin(t/2)}\left(A_n^{\delta}\right)^{-1}\left|\sum_{k=0}^{v} A_{v-k}^{\delta-1}\sin(2k+1)(t/2)\right|$$

where $A_{v-k}^{\delta-1}$ is monotonic decreasing then it gives maximum value at $k=0$, by Abel's lemma

$$\le \frac{1}{2(t/\pi)}\left(A_n^{\delta}\right)^{-1} A_v^{\delta-1} \max_{0\le k'\le v}\left|\sum_{k=k'}^{v}\sin(2k+1)(t/2)\right|$$

$$\le \frac{\delta}{(v+1)t^2}$$

This completes the proof of the Lemma.

4 Proof of the Theorem

The *nth* partial sum of the series (1.1) (see Zygmund [12, p-50]) is,

$$s_n(f;x) = f(x) + \frac{1}{\pi}\int_0^{\pi} \emptyset_x(t)\, D_n(t)\, dt\,.$$

Then

$$\left(E_q C_\delta\right)_n(f;x) - f(x) = \frac{1}{\pi}\int_0^{\pi} \emptyset_x(t)\,(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\left(A_v^{\delta}\right)^{-1}\sum_{k=0}^{n} A_{v-k}^{\delta-1} D_k(t)\, dt$$

$$\left|\left(E_q C_\delta\right)_n(f;x) - f(x)\right| \le \frac{1}{\pi}\int_0^{\pi} |\emptyset_x(t)|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\left(A_v^{\delta}\right)^{-1}\sum_{k=0}^{n} A_{v-k}^{\delta-1} D_k(t)\right| dt$$

$$\le \frac{1}{\pi}\left\{\int_0^{\frac{1}{(n+1)}} + \int_{\frac{1}{(n+1)}}^{\pi}\right\}$$

$$\le |I_1| + |I_2|, \text{ say.}$$

Now, for $0 \le t \le \frac{1}{(n+1)}$, $\sin nt \le n \sin t$, see Zygmund [12, p-91],

$$|I_1| \le \frac{1}{\pi}\int_0^{\frac{1}{(n+1)}} |\emptyset_x(t)|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\left(A_v^{\delta}\right)^{-1}\sum_{k=0}^{n} A_{v-k}^{\delta-1} D_k(t)\right| dt$$

$$\le \frac{1}{2\pi}\int_0^{\frac{1}{(n+1)}} |\emptyset_x(t)|\left|(1+q)^{-n}\sum_{v=0}^{n}\binom{n}{v}q^{n-v}\left(A_v^{\delta}\right)^{-1}\sum_{k=0}^{n} A_{v-k}^{\delta-1}(2k+1)\right| dt\,,$$

We have by Boos [2, p-104],

$$|I_1| \leq \frac{1}{2\pi} \int_0^{\frac{1}{(n+1)}} |\emptyset_x(t)|\,(1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v}(2v+1)dt\,,$$

$$\leq \frac{1}{2\pi} \int_0^{\frac{1}{(n+1)}} |\emptyset_x(t)|\,(2n+1)dt\,,$$

by (1.2), we have

$$= O(n+1) \int_0^{\frac{1}{(n+1)}} t^{\alpha}\, dt\,,$$

(4.1) $$= O(n+1)^{-\alpha}.$$

by (1.8), we have

$$|I_2| \leq \frac{1}{\pi} \int_{\frac{1}{(n+1)}}^{\pi} |\emptyset_x(t)|(1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} \left|K_v^{\delta}(t)\right| dt$$

Case-I: for $\delta \leq 1$, by Lemma 1, we have

$$|I_2| \leq \frac{1}{\pi} \int_{\frac{1}{(n+1)}}^{\pi} |\emptyset_x(t)|\,(1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} A_{\delta}\, v^{-\delta} t^{-(\delta+1)}\, dt$$

$$\leq \frac{A_{\delta}}{\pi} \int_{\frac{1}{(n+1)}}^{\pi} |\emptyset_x(t)|\, t^{-(\delta+1)}\,(1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v}(v+1)^{-\delta}\, dt$$

by Lemma 2 and (1.2), we get

$$|I_2| = O\left((n+1)^{-\delta}\right) \int_{\frac{1}{(n+1)}}^{\pi} t^{\alpha-(\delta+1)} dt$$

Condition I: when $\alpha = \delta$, then

$$|I_2| = O((n+1)^{-\alpha}) \int_{\frac{1}{(n+1)}}^{\pi} t^{-1} dt$$

(4.2) $$= O((n+1)^{-\alpha}) \log(n+1).$$

Condition II: when $\alpha < \delta$, then

$$|I_2| = O\left((n+1)^{-\delta}\right)\left(t^{\alpha-\delta}\right)^{\pi}_{\frac{1}{(n+1)}}$$

(4.3) $$= O\left((n+1)^{-\alpha}\right).$$

Combining (4.2) and (4.3), we have

(4.4) $$|I_2| = \begin{cases} O\{(n+1)^{-\alpha}\}, & (0 < \alpha < \delta \le 1) \\ O\{(n+1)^{-\alpha}\log(n+1)\}, & (0 < \alpha \le \delta \le 1). \end{cases}$$

Case-II: for $\delta > 1$, by Lemma 3, we have

$$|I_2| \le \frac{1}{\pi} \int_{\frac{1}{(n+1)}}^{\pi} |\phi_x(t)|\,(1+q)^{-n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} \frac{\delta}{(v+1)t^2}\, dt$$

By Lemma 3 and (1.2), we have

$$= O\left((n+1)^{-1}\right) \int_{\frac{1}{(n+1)}}^{\pi} t^{\alpha-2}\,dt$$

(4.5) $$= O\left((n+1)^{-\alpha}\right).$$

Now, collecting the estimate (4.1), (4.4) and (4.5) we get required result (2.6).

References

[1] G. Alexits, Über die annäherung einer statigen function druch die cesàro-schen mittel ihrer fourier-reihe, *Math. Ann.*, 100(1928), 264-277.

[2] J. Boos, *Classical and Modern Methods in Summability*, Oxford Univ. Press, (2000).

[3] S.N. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné, *Mém. Royale Acad. Belgique*, 2(4) (1912), 1-103.

[4] P. Chandra, On the absolute Euler summability of conjugate series of a fourier series, *Periodica Math. Hungarica*, 3(3-4) (1973), 189-194.

[5] P. Chandra, On some summability methods, *Bollettino della Unione Mathematica Italiana*, 4(3)(12) (1975), 211-224.

[6] P. Chandra, On the degree of approximation of continuous functions, *Comm. Fac. Sci. Univ. Ankara*, Sér A, 30(1981), 7-16.

[7] P. Chandra, Degree of approximation of continuous functions, *Riv. Mat. Univ. Parma*, 4(14) (1988), 61-70.

[8] G.H. Hardy, *Divergent Series*, Oxford, (1949).

[9] H. Lebesgue, Sur la representation trigonométrique approchée des fonctions satisfaisant à une condition de Lipschitz, *Bull. Soc. Math. France*, 38 (1910), 184-210.

[10] H.K. Nigam, Degree of approximation of functions belonging to $Lip\alpha$ class and weighted $(L^r, \varepsilon(t))$ class by product summability method, *Surveys in Mathematics and its Applications*, 5 (2010), 113-122.

[11] S.K. Tiwari and C. Bariwal, Degree of approximation of function belonging to the Lipschitz class by almost (E, q)(C, 1) means of its fourier series, *Int. J. Math. Archive*, (1)(1) (2010), 2-4.

[12] A. Zygmund, *Trigonometric Series*, *Vol. I (Second Edition*), Cambridge University Press, London/New York, (1968).