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Sum of Powers Applied In Cartesian Plane

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Abstract

This paper proposes a demonstration that if the sum of powers where A^x and B^y have a common prime factor, and generates a compound number C^z, then Beal's conjecture involving $A^x + B^y = C^z$ is true for all A, B, C, x, y, z positive integers and $x, y, z > 2$, as well as showing types where the sum $A^x + B^y$ generates no prime factor among A, B and C.

Keywords: *Beal's Conjecture, Compound Number, Prime Factor, Sum of Powers.*

1 Introduction

In 1993, Andrew Beal [4] began to investigate new hypotheses by Fermat's theorem, and one of them was the idea that a sum of powers with the form $A^x + B^y = C^z$ could have at least one prime factor in common among A, B and C, if A, B, C, x, y and z are positive integers with $A, B, C > 1$ and $x, y, z > 2$. Since then, many mathematicians around the world have investigated this assumption and formulated several theories as a proof of Beal‘s conjecture. However, no method has shown that at all possibilities of this sum, there is a common factor in A, B and C. Perhaps, the problem of not finding a mathematical demonstration really effective was the fact that mathematicians only work with a sum of powers when generating a compound number with equal factors in form of C^z. The sum of powers $A^x + B^y$ will generate in the set of positive integers two types of results: prime numbers and compounds including C^z. Nowadays, there is any mathematical tool to find a relation only among all numbers of form C^z when generated by this sum $A^x + B^y$. It will be shown that if a sum $A^x + B^y$ has the greatest common divisor $gcd(A, B) \neq 1$, with A and B belonging to the positive integers greater than 1, the result of sum only generate compound numbers includind powers with equal factors in the form C^m, and always will have a factor called *y'* in common among A, B and C. It also will be shown that if a sum $A^x + B^y$ has the greatest divisor $gcd = 1$, with A and B belonging to the positive integers greater than 1, the result of this sum can generate prime and compound numbers, but not of form C^m, and will never generate any integer positive factor among A, B and the result of sum. The theorems and proofs will comprove and exemplify simple methods with concepts of Number Theory, as Cartesian graphs, the greatest divisor between two numbers, constant functions, graph with curves [3], and notions of set and subjets [1]. Some algebra are also used to demonstrate equations when factoring numbers and incognites need to be simplyfied. All graphs in this paper were designed using Geogebra [2].

2 Preliminaries

In the mathematical field of Number Theory, there is the possibility of working with sums of form $A^x + B^y = C^z$ in a Cartesian plane determined by axes oriented from a point of intersection O with x and y perpendicular to each other, generating a plane α. From now on, $A^x + B^y = C^z$ will be written as $A^k + B^l = C^m$, to facilitate understanding. In this plan α you can transform a sum $A^k + B^l = C^m$ in a graph with curves in form of $x^k + y^l = C^m$, where $x^k = A^k$, and $y^l = B^l$.

Definition 2.1 *Consider the sum $A^x + B^y = C^z$. It can be rewritten as*

$$A^k + B^l = C^m$$

with A, B, C, k, l, m belonging to positive integers, $A, B, C > 1$ and $k, l, m > 2$.

Now consider a plan α. In this plan, you can transform a sum $A^k + B^l = C^m$ in a graph with curves in form of $x^k + y^l = C^m$, where $x^k = A^k$, and $y^l = B^l$. The graph of this function generates infinite points to the Cartesian plane, and always passes through a point P(x,y) in the first quadrant where $x = A$ and $y = B$ belongs to positive integers, and $x, y > 1$. C^m is generated by the sum of powers, being C^m a compound number.

Definition 2.2 *The graph $x^k + y^l = C^m$ always passes through a point P(x,y) , where $x = A$ and $y = B$, with A, B, C, k, l, m belonging to positive integers, $A, B, C > 1$ and $k, l, m > 2$.*

It will be shown that if the greatest common divisor $gdc(x, y) \neq 1$ with $x, y > 1$, the sum $A^k + B^l$ will always generate compound numbers, its graph always passes through the point P(x,y) which has coodinates $x = A$ and $y = B$, and for this same point passes a function $f(x) = y$, where y is a constant $c = B$ and oriented perpendiculary to the y-axis, generating into this axis a point P' with coordinates $P' = (0, y')$, and y' will be into a interval $2 \leq y' \leq y$, being y' the common factor among A, B and the compose number generated by the sum, and if the greatest common divisor $gdc(x, y) = 1$ with $x, y > 1$, the number generated by this sum can be prime or compose, and the point P' will be out of this interval, with coordinates $P' = (0, 1)$, so $y' = 1$, implying y' not being a integer common factor among x, y and the number generated by this sum.

The set of positive integers can be divided into two major subsets: the prime numbers and composites. As the integers in the form C^m, with $C > 1$ and $m > 2$ are compounds, then the numbers of the form C^m are subset of compound numbers. C^m is contained in the set of compound numbers. The union the set of prime numbers and composite numbers generate the set of all positive integers. When a proof of a theorem is true for al compound numbers, it implies be true for all numbers of form C^m, being C^m a product of equal factors.

Definition 2.3 *If numbers of form C^m are contained on subset of compound numbers, then a propherty apllied to all compound numbers implies be true also for all C^m.*

The figure below shows a diagram of integers set:

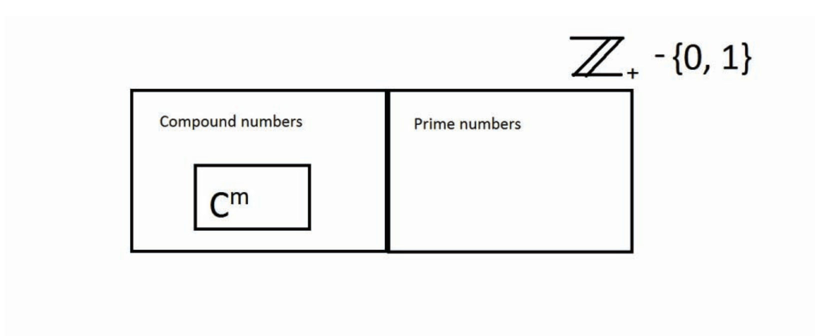

Figure 1: The set of positive integers less {0,1} divided in two subsets: prime and compound numbers.

3 Functions of the Form $x^k + y^l = Z$

Consider x, y, Z, k, l belonging to positive integers with $x, y, Z > 1$ and $k, l, > 2$. The sum $A^k + B^l = Z$ can be written as a function of form $x^k + y^l = Z$, where $A^k = x^k$, and $B^l = y^l$.

Definition 3.1 *Consider the sum $A^x + B^y = Z$. It can be rewritten as*

$$A^k + B^l = Z$$

with A, B, Z, k, l, belonging to positive integers, $A, B, Z > 1$ and $k, l > 2$.

Consider now a constant function $f(x) = c$ perpendicular to the y-axis which intersects the graph $x^k + y^l = Z$, written in the form $A^k + y^l = Z \Rightarrow f(x) = \sqrt[l]{Z - A^k} = c$ where $A^k = x^k$. The intersection point determined by the graph $x^k + y^l = Z$ and $f(x) = \sqrt[l]{Z - A^k} = y = B = constant$ is called a point P of coordinates (x, y). The abscissa x of P is discovered by applying the constant value c in the function $x^k + y^l = Z$, generating the equation $x^k + Z - A^k = Z \Rightarrow x = A$. The gcd (greatest common divisor) between the coordinates of P(x, y) = P(A,B) give an ordinate y' which is a common factor among A, B and Z, if the greatest common divisor $(x, y) \neq 1$. The straight line $f(x) = y = B = constant$ will generate a closed interval on y-axis between 2 and y. The point y' will be in this interval, if $gcd(x, y) \neq 1$, and give the solution of the largest prime factor among A, B and Z. The graph of all functions of the form $x^k + y^l = Z$ will generate a point P(x, y) = P(A,B) that intersects the function. If the sum of $x^k + y^l$ has $gcd(x, y) \neq 1$, the result will always generate compound numbers with y' being the prime factor in common with x, y and Z. If the sum $x^k + y^l$ has $gcd(x, y) = 1$, the result can generate prime and compound numbers not of form C^m, and there is still a point P(x, y) = P(A,B) that intersects the graph of $x^k + y^l = Z$, without a prime factor in common with x, y and Z. The figure below exemplifies the graphs on a Cartesian plane:

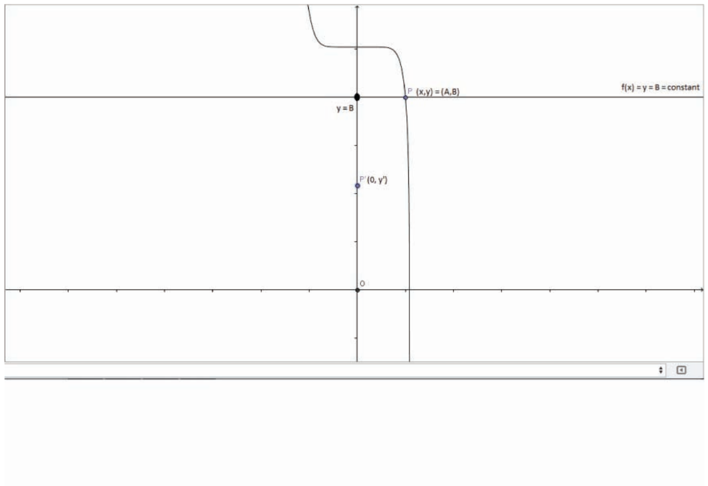

Figure 2: Cartesian representation of $x^k + y^l = Z$, $f(x) = B$ and the ordinate y' of P'(0,y')
.

4 Cases in which There is a Common Factor in the Sum of $A^k + B^l = Z$

The Sum of type $A^k + B^l = Z$ with, $A = B$ and $k \neq l$ and the greatest common divisor $gcd(A, B) \neq 1$ always have a prime factor among A, B and the result generated by its sum.

Theorem 4.1 *When a sum of powers of type*

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z \tag{1}$$

has $A = B > 1$, $k \neq l > 2$ and the greatest common divisor $gcd(A, B) \neq 1$, the result always is a compound number, the point P(x, y) intersection between the constant function $f(x) = c$ and graphic $x^k + y^l = Z$ have coordinates (A, B) and there are at least a factor common to A, B, and Z, given by the greatest common divisor $(A, B) =$ y' where y' is the ordinate of point P'(0, y) , with y' in y-axis interval $2 \leq y' \leq y$.

Proof. Consider A, B, k, l, Z belonging to the positive integers, $k, l > 2$, $x = A$, $y = B$, with $A, B, Z > 1$ and the greatest common divisor $gdc(x, y) \neq 1$. From Definition 2.2 and 3.1:

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z$$
$$f(x) = y = \sqrt[l]{Z - A^k} = constant.$$

But $A^k = x^k$ and $Z = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k+y^l = Z$. How $x^k + y^l = Z$ and $y = \sqrt[l]{Z - A^k}$, then:

$$x^k + Z - A^k = Z \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = Z$, so $(y'.q)^k + (y'.r)^l = Z$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = Z \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = Z.$$

Notice that Z is a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and Z .

Lemma 4.2 *This proof implies be true when a sum $A^k + B^l$ results in a compound number of kind C^m and the greatest common divisor $gcd(A, B) = A = y$' or the greatest common divisor $gcd(A, B) = B = y$', also generating a common factor among all those terms.*

Proof. Consider A, B, C, k, l, m belonging to positive integers, $A, B, C > 1$, $k, l, m > 2$ and the greatest common divisor $gcd(x, y) \neq 1$. From Definition 2.1 and 2.2:

$$A^k + B^l = C^m \Leftrightarrow x^k + y^l = C^m$$
$$f(x) = y = \sqrt[l]{C^m - A^k} = constant.$$

But $A^k = x^k$ and $C^m = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = C^m$. How $x^k + y^l = C^m$ and $y = \sqrt[l]{C^m - A^k}$, then:

$$x^k + C^m - A^k = C^m \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = C^m$, so $(y'.q)^k + (y'.r)^l = C^m$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = C^m \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m.$$

$$y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m$$

$$(y'^{k-1}.q^k + y'^{l-1}.r^l) = \frac{C^m}{y'}$$

$$q^k r^l.(\frac{y'^{k-1}}{r^l} + \frac{y'^{l-1}}{q^k}) = \frac{C^m}{y'}$$

$$q^k r^l.(\frac{y'^{k-1}.q^k + y^{l-1}.r^l}{q^k r^l}) = \frac{C^m}{y'}$$

$$y'^{k-1}.q^k + y^{l-1}.r^l = \frac{C^m}{y'}$$

$$y'.(y'^{k-1}.q^k + y^{l-1}.r^l) = C^m$$

$$y'^k.q^k + y^l.r^l = C^m$$

$$A^k + B^l = C^m$$

Notice that C^m is also a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and C^m .

Example 4.3 $7^6 + 7^7 = 98^3$

This sum generates a graphic of a function $x^6 + y^7 = 98^3$. Let's find the value of a constant function which intercepts the graph of $x^6 + y^7 = 98^3$, given by $f(x) = y = \sqrt[l]{C^m - A^k}$:

$f(x) = y = \sqrt[7]{98^3 - 7^6}$
$f(x) = y = \sqrt[7]{823543}$
$f(x) = y = 7$

Now, applying the value of $y = 7$ in $x^6 + y^7 = 98^3$, we will obtain the value of abscissa x from P(x,7) :

$x^6 + y^7 = 98^3$
$x^6 + 7^7 = 98^3$
$x = \sqrt[6]{98^3 - 7^6}$
$x = \sqrt[6]{823543}$
$x = -7$ *(rejected) or* $x = 7$ *(acepted)*

The point $P(x,y)$ *which intercepts* $x^6 + y^7 = 98^3$ *with f(x) = 7 is P(7,7) and the greatest common divisor* $gcd(7,7) = 7$. *Let's see the common factor among* 7^6, 7^7 *and* 98^3*:*

$7^6 + 7^7 = 98^3$
$7.(7^5 + 7^6) = 98^3$.

Corollary 4.4 *Notice that* 98^3 *is a product of two factors: 7 and* $(7^5 + 7^6)$ *In fact,* $134456.7 = 98^3$, *and 7 is a factor in common with* 7^6 *and* 7^7.

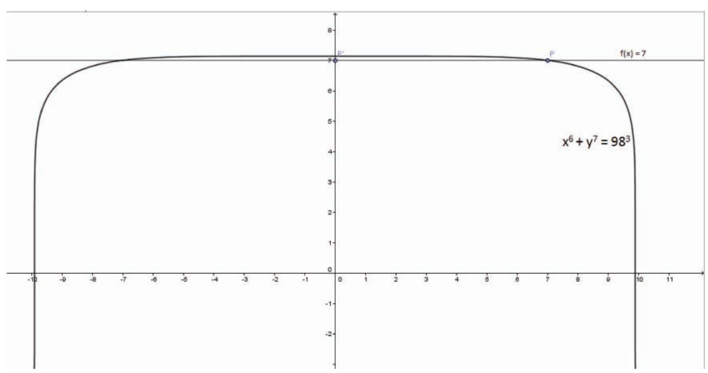

Figure 3: Graph of $x^6 + y^7 = 98^3$ and the ordinate $y' = 7$ in P'.

Example 4.5 $5^3 + 5^5 = 3250$

This sum generates a graphic of a function $x^3 + y^5 = 3250$. *Let's find the value of a constant function which intercepts the graph of* $x^3 + y^5 = 3250$ *, given by* $f(x) = y = \sqrt[l]{Z - A^k}$ *:*

$f(x) = y = \sqrt[5]{3250 - 5^3}$
$f(x) = y = \sqrt[5]{3125}$
$f(x) = y = 5$

Now, applying the value of $y = 5$ *in* $x^3 + y^5 = 3250$ *, we will obtain the value of abscissa x from P(x,5) :*

$x^3 + y^5 = 3250$
$x^3 + 5^5 = 3250$
$x = \sqrt[3]{3250 - 5^5}$
$x = \sqrt[3]{125}$
$x = 5$

The point $P(x, y)$ *which intercepts* $x^3 + y^5 = 3250$ *with f(x) = 5 is P(5,5) and the greatest common divisor* $gcd(5, 5) = 5$. *Let's see the common factor among* 5^3, 5^5 *and* 3250*:*

$5^3 + 5^5 = 3250$
$5.(5^2 + 5^4) = 3250.$

Corollary 4.6 *Notice that* 3250 *is a product of two factors:* 5 *and* (5^2+5^4). *In fact,* $650.5 = 3250$*, and* 5 *is a factor in common with* 5^3 *and* 5^5.

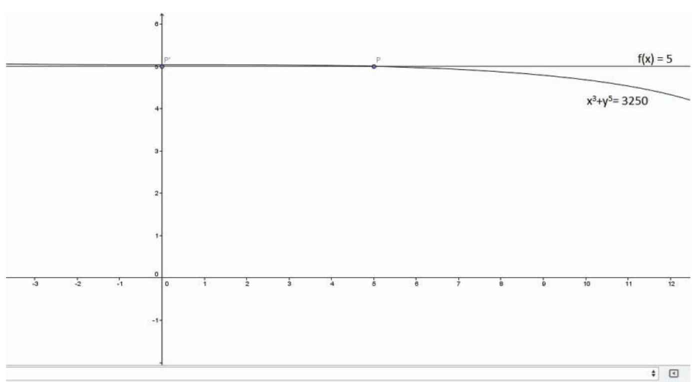

Figure 4: Graph of $x^3 + y^5 = 3250$ and the ordinate $y' = 5$ in P'.

Sum of type $A^k + B^l$ with $A = B$, $k = l$ and the greatest common divisor $gcd(A, B) \neq 1$ always have a prime factor among A, B, and the result generated by its sum.

Theorem 4.7 *When a sum of powers of type*

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z \tag{2}$$

has $A = B > 1$, $k = l > 2$ *and the greatest common divisor* $gcd(A, B) \neq 1$*, the result always is a compound number, the point P(x, y) intersection between the constant function* $f(x) = c$ *and graphic* $x^k + y^l = Z$ *have coordinates (A, B) and there are at least a factor common to* A*,* B*, and* Z*, given by the greatest common divisor* $(A, B) =$ *y' where y' is the ordinate of point P'(0, y) , with y' in y-axis interval* $2 \leq y' \leq y$.

Proof. Consider A, B, k, l, Z belonging to the positive integers, $k, l > 2$, $x = A$, $y = B$, with $A, B, Z > 1$ and the greatest common divisor $gdc(x, y) \neq 1$. From Definition 2.2 and 3.1:

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z$$
$$f(x) = y = \sqrt[l]{Z - A^k} = constant.$$

But $A^k = x^k$ and $Z = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \;\Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = Z$. How $x^k + y^l = Z$ and $y = \sqrt[l]{Z - A^k}$, then:

$$x^k + Z - A^k = Z \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = Z$, so $(y'.q)^k + (y'.r)^l = Z$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = Z \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = Z.$$

Notice that Z is a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and Z .

Lemma 4.8 *This proof implies be true when a sum $A^k + B^l$ results in a compound number of kind C^m and the greatest common divisor $gcd(A, B) = A = y$' or the greatest common divisor $gcd(A, B) = B = y$', also generating a common factor among all those terms.*

Proof. Consider A, B, C, k, l, m belonging to positive integers, $A, B, C > 1$, $k, l, m > 2$ and the greatest common divisor $gcd(x, y) \neq 1$. From Definition 2.1 and 2.2:

$$A^k + B^l = C^m \Leftrightarrow x^k + y^l = C^m$$
$$f(x) = y = \sqrt[l]{C^m - A^k} = constant.$$

But $A^k = x^k$ and $C^m = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \;\Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = C^m$. How $x^k + y^l = C^m$ and $y = \sqrt[l]{C^m - A^k}$, then:

$$x^k + C^m - A^k = C^m \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{ P(A,B) }.$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = C^m$, so $(y'.q)^k + (y'.r)^l = C^m$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = C^m \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m.$$

$$y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m$$

$$(y'^{k-1}.q^k + y'^{l-1}.r^l) = \frac{C^m}{y'}$$

$$q^k r^l.(\frac{y'^{k-1}}{r^l} \ + \frac{y'^{l-1}}{q^k}) \ = \frac{C^m}{y'}$$

$$q^k r^l.(\frac{y'^{k-1}.q^k + y^{l-1}.r^l}{q^k r^l}) \ = \frac{C^m}{y'}$$

$$y'^{k-1}.q^k + y^{l-1}.r^l = \frac{C^m}{y'}$$

$$y'.(y'^{k-1}.q^k + y^{l-1}.r^l) = C^m$$

$$y'^k.q^k + y^l.r^l = C^m$$

$$A^k + B^l = C^m$$

Notice that C^m is also a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and C^m .

Example 4.9 $2^3 + 2^3 = 2^4$

This sum generates a graphic of a function $x^3 + y^3 = 2^4$. Let's find the value of a constant function which intercepts the graph of $x^3 + y^3 = 2^4$, given by $f(x) = y = \sqrt[l]{C^m - A^k}$:

$f(x) = y = \sqrt[3]{2^4 - 2^3}$
$f(x) = y = \sqrt[3]{8}$
$f(x) = y = 2$

Now, applying the value of $y = 2$ in $x^3 + y^3 = 2^4$, we will obtain the value of abscissa x from P(x,2) :

$x^3 + y^3 = 2^4$
$x^3 + 2^3 = 2^4$
$x = \sqrt[3]{2^4 - 2^3}$
$x = \sqrt[3]{8}$
$x = 2$

The point $P(x, y)$ which intercepts $x^3 + y^3 = 2^4$ with f(x) = 2 is P(2,2) and the greatest common divisor gcd$(2, 2) = 2$. Let's see the common factor among 2^3, 2^3 and 2^4:

$2^3 + 2^3 = 2^4$
$2.(2^2 + 2^2) = 2^4$.

Corollary 4.10 *Notice that 2^4 is a product of two factors: 2 and (2^2+2^2). In fact, $2.8 = 16$, and 2 is a factor in common with 2^3.*

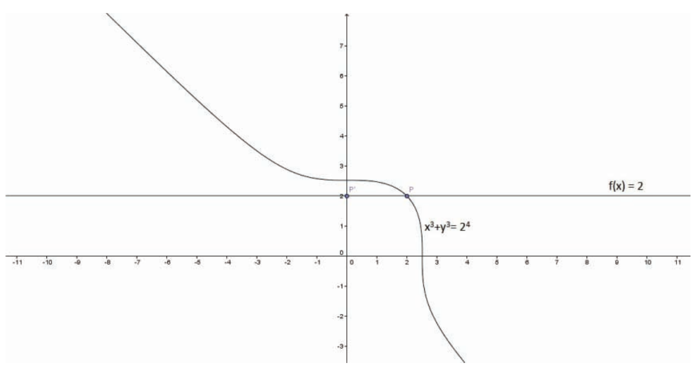

Figure 5: Graph of $x^3 + y^3 = 2^4$ and the ordinate $y' = 2$ in P'.

Example 4.11 $5^5 + 5^5 = 6250$

This sum generates a graphic of a function $x^5 + y^5 = 6250$. Let's find the value of a constant function which intercepts the graph of $x^5 + y^5 = 6250$, given by $f(x) = y = \sqrt[l]{Z - A^k}$:

$f(x) = y = \sqrt[5]{6250 - 5^5}$
$f(x) = y = \sqrt[5]{3125}$
$f(x) = y = 5$

Now, applying the value of $y = 5$ in $x^5 + y^5 = 6250$, we will obtain the *value of abscissa x from P(x,5) :*

$x^5 + y^5 = 6250$
$x^5 + 5^5 = 6250$
$x = \sqrt[5]{6250 - 5^5}$
$x = \sqrt[5]{3125}$
$x = 5$

The point $P(x, y)$ *which intercepts* $x^5 + y^5 = 6250$ *with f(x) = 5 is P(5,5) and the greatest common divisor gcd*$(5, 5) = 5$*. Let's see the common factor among* 5^3*,* 5^5 *and* 6250*:*

$5^5 + 5^5 = 6250$
$5.(5^4 + 5^4) = 6250.$

Corollary 4.12 *Notice that* 6250 *is a product of two factors:* 5 *and* $(5^4 + 5^4)$*. In fact,* $5.1250 = 6250$*, and* 5 *is a factor in common with* 5^4 *and* 5^5*.*

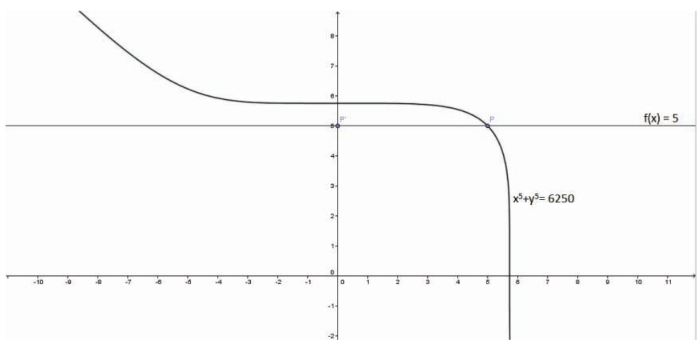

Figure 6: Graph of function $x^5 + y^5 = 6250$ and the ordinate $y' = 5$.

Sum of the type $A^k + B^l$, with $A \neq B$, $k = l$ and the greatest common divisor $gcd(x, y) \neq 1$ always have a prime factor among A, B, and the result generated by its sum.

Theorem 4.13 *When a sum of powers of type*

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z \tag{3}$$

has $A \neq B > 1$*,* $k = l > 2$ *and the greatest common divisor gcd*$(A, B) \neq 1$*, the result always is a compound number, the point P(x, y) intersection between the constant function* $f(x) = c$ *and graphic* $x^k + y^l = Z$ *have coordinates (A, B) and there are at least a factor common to* A*,* B*, and* Z*, given by the greatest common divisor* $(A, B) = y$*' where y' is the ordinate of point P'(0, y) , with y' in y-axis interval* $2 \leq y' \leq y$*.*

Proof. Consider A, B, k, l, Z belonging to the positive integers, $k, l > 2$, $x = A$, $y = B$, with $A, B, Z > 1$ and the greatest common divisor $gdc(x, y) \neq 1$. From Definition 2.2 and 3.1:

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z$$
$$f(x) = y = \sqrt[l]{Z - A^k} = constant.$$

But $A^k = x^k$ and $Z = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = Z$. How $x^k + y^l = Z$ and $y = \sqrt[l]{Z - A^k}$, then:

$$x^k + Z - A^k = Z \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = Z$, so $(y'.q)^k + (y'.r)^l = Z$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = Z \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = Z.$$

Notice that Z is a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and Z .

Lemma 4.14 *This proof implies be sure when a sum $A^k + B^l$ results in a compound number of kind C^m and the greatest common divisor $gcd(A, B) = A = y$' or the greatest common divisor $gcd(A, B) = B = y$', also generating a common factor among all those terms.*

Proof. Consider A, B, C, k, l, m belonging to positive integers, $A, B, C > 1$, $k, l, m > 2$ and the greatest common divisor $gcd(x, y) \neq 1$. From Definition 2.1 and 2.2:

$$A^k + B^l = C^m \Leftrightarrow x^k + y^l = C^m$$
$$f(x) = y = \sqrt[l]{C^m - A^k} = constant.$$

But $A^k = x^k$ and $C^m = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = C^m$. How $x^k + y^l = C^m$ and $y = \sqrt[l]{C^m - A^k}$, then:

$$x^k + C^m - A^k = C^m \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = C^m$, so $(y'.q)^k + (y'.r)^l = C^m$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = C^m \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m.$$

$$y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m$$

$$(y'^{k-1}.q^k + y'^{l-1}.r^l) = \tfrac{C^m}{y'}$$

$$q^k r^l.(\tfrac{y'^{k-1}}{r^l} + \tfrac{y'^{l-1}}{q^k}) = \tfrac{C^m}{y'}$$

$$q^k r^l.(\tfrac{y'^{k-1}.q^k + y^{l-1}.r^l}{q^k r^l}) = \tfrac{C^m}{y'}$$

$$y'^{k-1}.q^k + y^{l-1}.r^l = \tfrac{C^m}{y'}$$

$$y'.(y'^{k-1}.q^k + y^{l-1}.r^l) = C^m$$

$$y'^k.q^k + y^l.r^l = C^m$$

$$A^k + B^l = C^m$$

Notice that C^m is also a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and C^m .

Example 4.15 $3^3 + 6^3 = 3^5$

This sum generates a graphic of a function $x^3 + y^3 = 3^5$. Let's find the value of a constant function which intercepts the graph of $x^3 + y^3 = 3^5$, given by $f(x) = y = \sqrt[l]{C^m - A^k}$:

$f(x) = y = \sqrt[3]{3^5 - 3^3}$
$f(x) = y = \sqrt[3]{216}$
$f(x) = y = 6$

Now, applying the value of $y = 6$ *in* $x^3 + y^3 = 3^5$ *, we will obtain the value of abscissa* x *from P(x,6) :*

$x^3 + y^3 = 3^5$
$x^3 + 6^3 = 3^5$
$x = \sqrt[3]{3^5 - 6^3}$
$x = \sqrt[3]{27}$
$x = 3$

The point $P(x, y)$ *which intercepts* $x^3 + y^3 = 3^5$ *with f(x) = 5 is P(3,6) and the greatest common divisor* $gcd(3, 6) = 3$. *Let's see the common factor among* 3^3, 6^3 *and* 3^5*:*

$3^3 + 6^3 = 3^5$
$3.(3^2 + 2^3.3^2) = 3^5$

Corollary 4.16 *Notice that* 3^5 *is a product of two factors:* 3 *and* $(3^2 + 2^3.3^2)$ *In fact,* $81.3 = 3^5$, *and* 3 *is a factor in common with* 3^3 *and* 6^3.

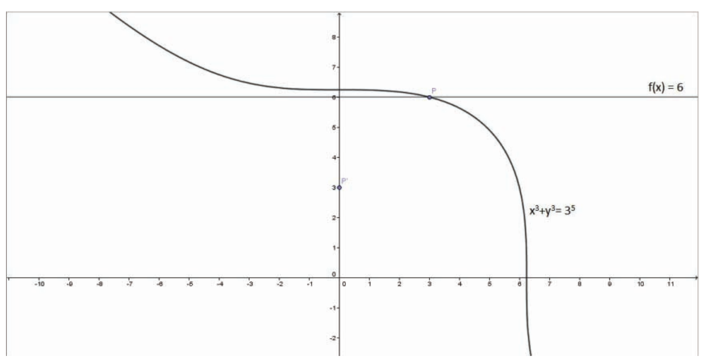

Figure 7: Graph of $x^3 + y^3 = 3^5$ and the ordinate $y' = 6$.

Example 4.17 $5^4 + 10^4 = 10625$

This sum generates a graphic of a function $x^4 + y^4 = 10625$. *Let's find the value of a constant function which intercepts the graph of* $x^4 + y^4 = 10625$ *, given by* $f(x) = y = \sqrt[l]{Z - A^k}$:

$f(x) = y = \sqrt[4]{10625 - 5^4}$
$f(x) = y = \sqrt[4]{10000}$
$f(x) = y = 10$

Now, applying the value of $y = 10$ *in* $x^4 + y^4 = 10625$ *, we will obtain the value of abscissa x from P(x,10) :*

$x^4 + y^4 = 10625$
$x^4 + 10^4 = 10625$
$x = \sqrt[4]{10625 - 10^4}$
$x = \sqrt[4]{625}$
$x = 5$

The point $P(x, y)$ *which intercepts* $x^4 + y^4 = 10625$ *with f(x) = 5 is P(5,10) and the greatest common divisor* $\gcd(5, 10) = 5$*. Let's see the common factor among* 5^4*,* 10^4 *and* 10625*:*

$5^4 + 10^4 = 10625$
$5.(5^3 + 2^4.5^3) = 10625$

Corollary 4.18 *Notice that* 10625 *is a product of two factors:* 5 *and* $(5^3 + 2^4.5^3)$ *In fact,* $2125.5 = 10625$*, and* 5 *is a factor in common with* 5^4 *and* 10^4*.*

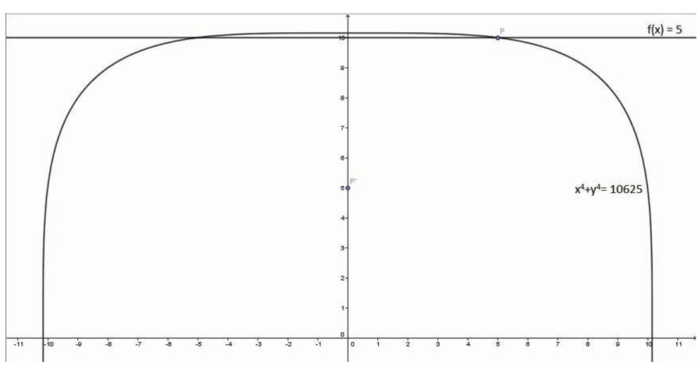

Figure 8: Graph of $x^4 + y^4 = 10625$ and the ordinate $y' = 10$ in P'.

Sum of type $A^k + B^l$, with $A \neq B$, $k \neq l$ and the greatest divisor $gcd(x, y) \neq 1$ always have a prime factor among A, B, and the result generated by its sum.

Theorem 4.19 *When a sum of powers of type*

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z \tag{4}$$

has $A \neq B > 1$*,* $k \neq l > 2$ *and the greatest common divisor* $gcd(A, B) \neq 1$*, the result always is a compound number, the point* $P(x, y)$ *intersection between the constant function* $f(x) = c$ *and graphic* $x^k + y^l = Z$ *have coordinates* (A, B)

and there are at least a factor common to A, B, and Z, given by the greatest common divisor $(A, B) = y$' where y' is the ordinate of point P'(0, y) , with y' in y-axis interval $2 \leq y' \leq y$.

Proof. Consider A, B, k, l, Z belonging to the positive integers, $k, l > 2$, $x = A$, $y = B$, with $A, B, Z > 1$ and the greatest common divisor $gdc(x, y) \neq 1$. From Definition 2.2 and 3.1:

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z$$
$$f(x) = y = \sqrt[l]{Z - A^k} = constant.$$

But $A^k = x^k$ and $Z = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = Z$. How $x^k + y^l = Z$ and $y = \sqrt[l]{Z - A^k}$, then:

$$x^k + Z - A^k = Z \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = Z$, so $(y'.q)^k + (y'.r)^l = Z$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = Z \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = Z.$$

Notice that Z is a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and Z .

Lemma 4.20 *This proof implies be sure when a sum $A^k + B^l$ results in a compound number of kind C^m and the greatest common divisor $gcd(A, B) = A = y$' or the greatest common divisor $gcd(A, B) = B = y$', also generating a common factor among all those terms.*

Proof. Consider A, B, C, k, l, m belonging to positive integers, $A, B, C > 1$, $k, l, m > 2$ and the greatest common divisor $gcd(x, y) \neq 1$. From Definition 2.1 and 2.2:

$$A^k + B^l = C^m \Leftrightarrow x^k + y^l = C^m$$
$$f(x) = y = \sqrt[l]{C^m - A^k} = constant.$$

But $A^k = x^k$ and $C^m = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = C^m$. How $x^k + y^l = C^m$ and $y = \sqrt[l]{C^m - A^k}$, then:

$$x^k + C^m - A^k = C^m \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = y'$. Consider $A = y'.q$ and $B = y'.r$, where q and r is any factor positive integer non-zero of A and B respectively, and $q, r > 1$. How $A^k + B^l = C^m$, so $(y'.q)^k + (y'.r)^l = C^m$. By this way

$$(y'^k.q^k) + (y'^l.r^l) = C^m \Rightarrow y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m.$$

$$y'.(y'^{k-1}.q^k + y'^{l-1}.r^l) = C^m$$

$$(y'^{k-1}.q^k + y'^{l-1}.r^l) = \frac{C^m}{y'}$$

$$q^k r^l.(\frac{y'^{k-1}}{r^l} + \frac{y'^{l-1}}{q^k}) = \frac{C^m}{y'}$$

$$q^k r^l.(\frac{y'^{k-1}.q^k + y^{l-1}.r^l}{q^k r^l}) = \frac{C^m}{y'}$$

$$y'^{k-1}.q^k + y^{l-1}.r^l = \frac{C^m}{y'}$$

$$y'.(y'^{k-1}.q^k + y^{l-1}.r^l) = C^m$$

$$y'^k.q^k + y^l.r^l = C^m$$

$$A^k + B^l = C^m$$

Notice that C^m is also a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $A = y'.q$ and $B = y'.r$ So, y' is the common factor among A, B, and C^m .

Example 4.21 $2^9 + 8^3 = 4^5$

This sum generates a graphic of a function $x^9 + y^3 = 4^5$. Let's find the value of a constant function which intercepts the graph of $x^9 + y^3 = 4^5$, given by $f(x) = y = \sqrt[l]{C^m - A^k}$:

$f(x) = y = \sqrt[3]{4^5 - 2^9}$
$f(x) = y = \sqrt[3]{512}$
$f(x) = y = 8$

Now, applying the value of $y = 8$ *in* $x^9 + y^3 = 4^5$ *, we will obtain the value of abscissa x from P(x,8) :*

$x^9 + y^3 = 4^5$
$x^9 + 8^3 = 4^5$
$x = \sqrt[9]{4^5 - 8^3}$
$x = \sqrt[9]{512}$
$x = 2$

The point $P(x, y)$ *which intercepts* $x^9 + y^3 = 4^5$ *with f(x) = 5 is P(2,8) and the greatest common divisor gcd*$(2, 8) = 2$. *Let's see the common factor among* 2^9, 8^3 *and* 4^5*:*

$2^9 + 8^3 = 4^5$
$2.(2^8 + 2^8) = 4^5.$

Corollary 4.22 *Notice that* 4^5 *is a product of two factors:* 2 *and* $(2^8 + 2^8)$*. In fact,* $512.2 = 1024$*, and* 2 *is a factor in common with* 2^9 *and* 8^3*.*

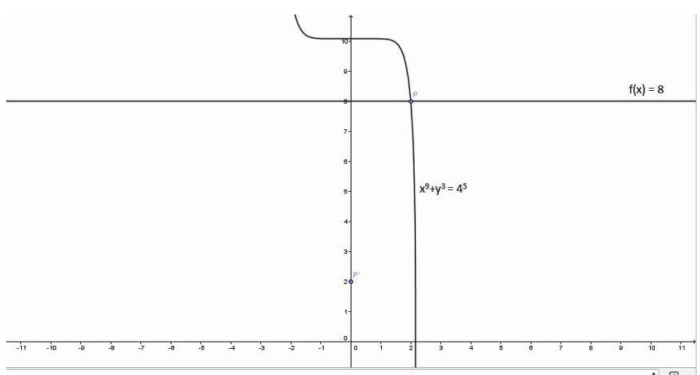

Figure 9: Graph of $x^9 + y^3 = 4^5$ and the ordinate $y' = 8$ of P'.

Example 4.23 $3^4 + 9^5 = 59130$

This sum generates a graphic of a function $x^4 + y^5 = 59130$*. Let's find the value of a constant function which intercepts the graph of* $x^4 + y^5 = 10625$ *, given by* $f(x) = y = \sqrt[l]{Z - A^k}$ *:*

$f(x) = y = \sqrt[5]{59130 - 3^4}$
$f(x) = y = \sqrt[5]{59049}$
$f(x) = y = 9$

Now, applying the value of $y = 9$ *in* $x^4 + y^5 = 59130$ *, we will obtain the value of abscissa* x *from P(x,9) :*

$x^4 + y^4 = 59130$
$x^4 + 10^4 = 59130$
$x = \sqrt[4]{59130 - 9^5}$
$x = \sqrt[4]{81}$
$x = -3$ *(rejected) or* $x = 3(acepted)$

The point $P(x, y)$ *which intercepts* $x^4 + y^5 = 59130$ *with f(x) = 9 is P(3,9) and the greatest common divisor* $gcd(3, 9) = 3$*. Let's see the common factor among* 3^4, 9^5 *and* 59130*:*

$3^4 + 9^5 = 59130$
$3.(3^3 + 3^9) = 59130.$

Corollary 4.24 *Notice that* 59130 *is a product of two factors:* 3 *and* $(3^3 + 3^8)$*. In fact,* $19710.3 = 59130$*, and* 3 *is a factor in common with* 3^4 *and* 9^5*.*

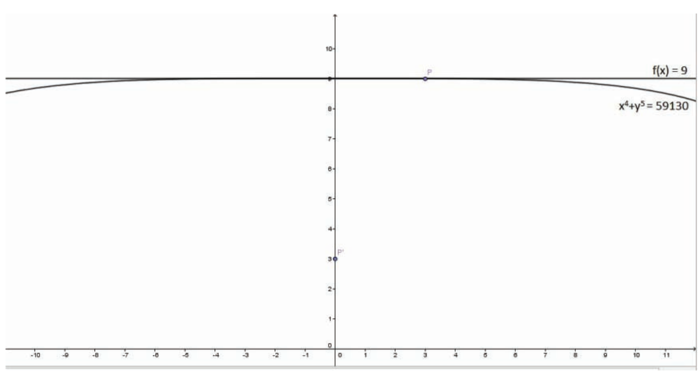

Figure 10: Graph of $x^4 + y^5 = 59130$ and the ordinate $y' = 9$.

5 Case in which There is no Common Factor in Sums of Form $A^k + B^l = Z$

If A, B, Z, k, l are any positive integers non-zero, $A, B > 1$, $k, l > 2$, A and B are co-primes, the greatest common divisor $gcd(A, B) = 1$, then there is no prime factor among A, B, and the result generated by its sum.

Theorem 5.1 *When a sum of powers of type*

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z \tag{5}$$

where A, B, Z, k, l are any positive integers non-zero, $A, B > 1$, $k, l > 2$, A and B are co-primes, the point P(x, y) intersection between the constant function $f(x) = c$ and graphic $x^k + y^l = Z$ have coordinates (x, y) = (A,B) the result Z can be prime or compound not not of form C^m and there is no a prime common factor among A, B, and Z, the ordinate of point P'(0, y') is out of range $2 \leq y' \leq y$, being P' a point with coodinates $(0, 1)$, and Z can be prime or compound.

Proof. Consider A, B, k, l, Z belonging to the positive integers, $k, l > 2$, $x = A$, $y = B$, with $A, B, C > 1$ and the greatest common divisor $gdc(x, y) = 1$. From definiton 2.2 and 3.1:

$$A^k + B^l = Z \Leftrightarrow x^k + y^l = Z$$
$$f(x) = y = \sqrt[l]{Z - A^k} = constant.$$

But $A^k = x^k$ and $Z = x^k + y^l$. Then

$$f(x) = c = \sqrt[l]{x^k + y^l - x^k} \Rightarrow f(x) = y = B = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^k + y^l = Z$. How $x^k + y^l = Z$ and $y = \sqrt[l]{Z - A^k}$, then:

$$x^k + Z - A^k = Z \Rightarrow x^k = A^k \Rightarrow x = A.$$

The coordinates of $P(x, y)$ are

$$\text{P(x,y)} = \text{P(A,B)} .$$

The greatest common divisor $gcd(A, B) = 1$. Consider $A = 1.A$ and $B = 1.B$. How $A^k + B^l = Z$, so $(1.B)^k + (1.B)^l = Z$. By this way

$$(1^k.A^k) + (1^l.B^l) = Z \Rightarrow 1.(1^{k-1}.A^k + 1^{l-1}.B^l) = Z.$$

Notice that Z is a product of a integer factor with the number 1. It implies

$$1.Z = Z$$

being Z prime or a compound number. However, A, B and Z have not any integer factor in common for every cases.

Lemma 5.2 *It is not possible a sum of kind $A^k + B^l$ with $gcd(A, B) = 1$, $A, B > 1$ and $k, l > 2$ generate compound numbers of form C^m*

Proof. Consier $1.Z = Z$. As show before on theorem 5.1, when $gcd(A, B) = 1$, the result Z can be prime or compound. Supose a sum $A^k + B^l$ with $gcd(A, B) = 1$, $A, B > 1$ and $k, l > 2$ generate a number of form C^m, then there is only a possibility for the equality $1.Z = Z$ be true:

$$1.C^m = C^m$$

$$1.(A^k + B^l)^m = C^m$$

$$1.[AB.(\tfrac{A^{k-1}}{B} + \tfrac{B^{l-1}}{A})\ ]^m = C^m$$

$$1.(AB)^m.(\tfrac{A^{k-1}}{B} + \tfrac{B^{l-1}}{A})\ ^m = C^m$$

$$1.AB.\tfrac{A^k+B^l}{AB}\ = C$$

$$1.(A^k + B^l) = C$$

By this way, $A^k + B^l$ with $gcd(A, B) = 1$, $A, B > 1$ and $k, l > 2$ cannot generate a power with equal factors.

Example 5.3 $3^5 + 5^3 = 368$

This sum generates a graphic of a function $x^5 + y^3 = 368$. *Let's find the value of a constant function which intercepts the graph of* $x^5 + y^3 = 368$, *given by* $f(x) = y = \sqrt[l]{Z - A^k}$:

$f(x) = y = \sqrt[3]{368 - 3^5}$
$f(x) = y = \sqrt[3]{125}$
$f(x) = y = 5$

Now, applying the value of $y = 5$ *in* $x^5 + y^3 = 368$, *we will obtain the value of abscissa* x *from P(x,5) :*

$x^5 + y^3 = 368$
$x^5 + 5^3 = 368$
$x = \sqrt[5]{368 - 5^3}$
$x = \sqrt[5]{243}$
$x = 3$

The point $P(x, y)$ *which intercepts* $x^5 + y^3 = 368$ *with f(x) = 9 is P(3,5) and the greatest common divisor* $gcd(3, 5) = 1$. 3 *and* 5 *are co-primes, then there is no prime common factor between 3 and 5. It means there is no integer common factor among* 3^5, 5^3 *and* 368.

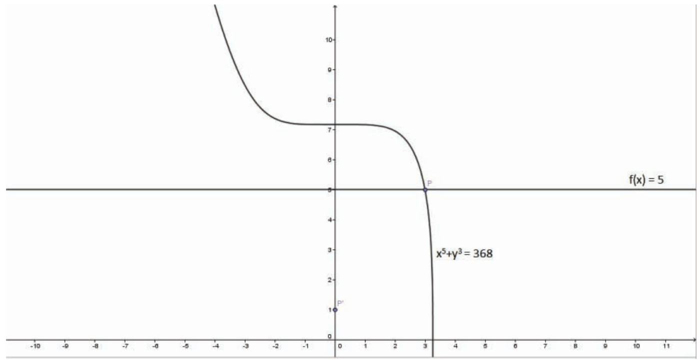

Figure 11: Graph of $x^5 + y^3 = 368$ and the ordinate $y' = 1$ in P'.

Example 5.4 $5^3 + 6^3 = 341$

This sum generates a graphic of a function $x^3 + y^3 = 341$. *Let's find the value of a constant function which intercepts the graph of* $x^3 + y^3 = 341$, *given by* $f(x) = y = \sqrt[l]{Z - A^k}$:

$f(x) = y = \sqrt[3]{341 - 5^3}$
$f(x) = y = \sqrt[3]{216}$
$f(x) = y = 6$

Now, applying the value of $y = 6$ *in* $x^3 + y^3 = 341$, *we will obtain the value of abscissa x from P(x,6) :*

$x^3 + 6^3 = 341$
$x^3 + 6^3 = 341$
$x = \sqrt[3]{341 - 6^3}$
$x = \sqrt[3]{125}$
$x = 3$

The point $P(x, y)$ *which intercepts* $x^3 + y^3 = 341$ *with f(x) = 6 is P(5,6) and the greatest common divisor* $gcd(5, 6) = 1$. 5 *and* 6 *are co-primes, then there is no common factor between 5 and 6. It means there is no prime common factor among* 5^3, 6^3 *and* 341.

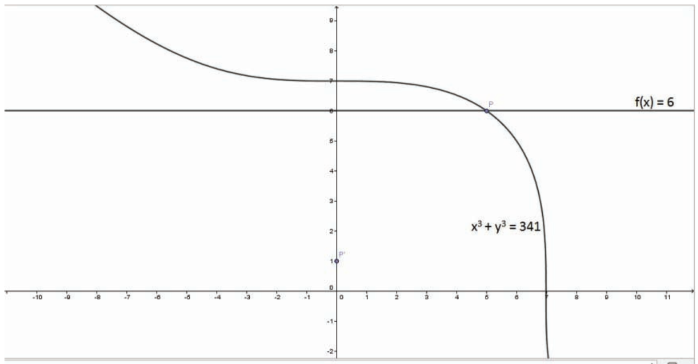

Figure 12: Graph of $x^3 + y^3 = 341$ and the ordinate $y' = 1$ of P'.

6 Graphic Implications when the Order of Factors is Changed

In any kind of sum, when the order of the factors is changed, the product remains the same. In case of sum $A^k + B^l = Z$, the graph produced is of form $x^k + y^l = Z$. If the order is changed to the sum $B^l + A^k = Z$, the generated graph will have the form $x^l + y^k = Z$. It shows that the exponents on the graph $x^l + y^k = Z$ changed their order and the point P generated at the intersection of the graph $x^l + y^k = Z$ with the constant function $f(x) = c = \sqrt[k]{Z - B^l}$ have coordinates P(x, y) = (B, A) but the ordinate y' which is a common factor among A, B and Z of point P'(0, y') still remain within the interval $2 \leq y' \leq y$. This is applied in the cases of Section 4. In the case of Section 5, the point P' is still outside the range $2 \leq y' \leq y$, as shown before.

Proof. Consider A,B, k, l, Z belonging to the positive integers, with $A, B, Z > 1$, $k, l > 2$, $B = x$, $A = y$, the greatest common divisor $gcd(x, y) = y$'.

$$B^l + A^k = Z \Leftrightarrow x^l + y^k = Z \tag{6}$$

with $x = B$ and $y = A$.

$$f(x) = \sqrt[k]{Z - B^l} \ = c$$

As $B^l = x^l$ and $Z = x^l + y^k$, so:

$$f(x) = c = \sqrt[k]{x^l + y^k - x^l} \ \Rightarrow f(x) = y = c \Rightarrow f(x) = A$$

The ordinate $y = A$. Let's find the value of abcissa x:

$$Z = x^l + y^k \text{ and } y = \sqrt[k]{Z - B^l}$$

So $x^l + Z - B^l = Z \Rightarrow x^l = B^l \Rightarrow x = B$. The coordinates P(x,y) = P(B,A) . The greatest common divisor $gcd(B, A) = y$'. Consider $B = y'.r$ and $A = y'.q$, where r and q are any factor positive integer non-zero of B and A, respectively. If $B^l + A^k = Z$, so

$$(y'.r)^l + (y'.q)^k = Z$$

$$(y'^l.r^l) + (y'^k.q^k) = Z$$

$$y'(y'^{l-1}.r^l + y'^{k-1}.q^k) = Z$$

Notice that Z is a product of two factors: y' and $(y'^{l-1}.r^l + y'^{k-1}.q^k)$. y' is the same factor in common with $B = y'.r$ and $A = y'.q$. So, y' is the common factor among A, B, and Z .

Lemma 6.1 *This proof implies be sure when a sum $B^l + A^k$ results in a compound number of kind C^m and the greatest common divisor $gcd(B, A) = B = y$' or the greatest common divisor $gcd(B, A) = A = y$', also generating a common factor among all those terms.*

Proof. Consider A, B, C, k, l, m belonging to positive integers, $A, B, C > 1$, $k, l, m > 2$ and the greatest common divisor $gcd(x, y) \neq 1$:

$$B^l + A^k = C^m \Leftrightarrow x^l + y^k = C^m$$
$$f(x) = y = \sqrt[k]{C^m - B^l} = constant.$$

But $B^l = x^l$ and $C^m = x^l + y^k$. Then

$$f(x) = c = \sqrt[k]{x^l + y^k - x^l} \Rightarrow f(x) = y = A = constant.$$

Let's find the value of x , apllying the value of function $f(x) = y$ in $x^l + y^k = C^m$. How $x^l + y^k = C^m$ and $y = \sqrt[k]{C^m - B^l}$, then:

$$x^l + C^m - B^l = C^m \Rightarrow x^l = B^l \Rightarrow x = B.$$

The coordinates of $P(x, y)$ are

P(x,y) = P(B,A) .

The greatest common divisor $gcd(B, A) = y'$. Consider $B = y'.q$ and $A = y'.r$, where q and r is any factor positive integer non-zero of B and A respectively, and $q, r > 1$. How $B^l + A^k = C^m$, so $(y'.q)^k + (y'.r)^l = C^m$. By this way

$$(y'^l.q^l) + (y'^k.r^k) = C^m \Rightarrow y'.(y'^{l-1}.q^l + y'^{k-1}.r^k) = C^m.$$

$$y'.(y'^{l-1}.q^l + y'^{k-1}.r^k) = C^m$$
$$(y'^{l-1}.q^l + y'^{k-1}.r^k) = \frac{C^m}{y'}$$
$$q^l r^k.(\frac{y'^{l-1}}{r^k} \ + \frac{y'^{k-1}}{q^l}) \ = \frac{C^m}{y'}$$
$$q^l r^k.(\frac{y'^{l-1}.q^l + y^{k-1}.r^k}{q^l r^k}) \ = \frac{C^m}{y'}$$
$$y'^{l-1}.q^l + y^{k-1}.r^k = \frac{C^m}{y'}$$
$$y'.(y'^{l-1}.q^l + y^{k-1}.r^k) = C^m$$
$$y'^l.q^l + y^k.r^k = C^m$$
$$B^l + A^k = C^m$$

Corollary 6.2 *Notice that C^m is also a product of two factors: y' and $(y^{k-1'}.q^k + y^{l-1'}.r^l)$. y' is the same factor in common with $B = y'.q$ and $A = y'.r$ So, y' is the common factor among A, B, and C^m .*

Example 6.3 $2^9 + 8^3 = 4^5$ *and* $8^3 + 2^9 = 4^5$

As shown on Example 4.22, this sum generates the graphic of a function $x^9 + y^3 = 4^5$. This one intercepts a point with $f(x) = y = \sqrt[l]{Z - A^k}$. Let's see it on example of this sum $8^3 + 2^9 = 4^5$:

$f(x) = y = \sqrt[9]{4^5 - 8^3}$
$f(x) = y = \sqrt[9]{512}$
$f(x) = y = 2$

Now, applying the value of $y = 2$ in $x^3 + y^9 = 4^5$, we will obtain the value of abscissa x from P(x,2) :

$x^3 + y^9 = 4^5$
$x^3 + 2^9 = 4^5$
$x = \sqrt[3]{4^5 - 2^9}$
$x = \sqrt[3]{512}$
$x = 8$

The point $P(x, y)$ which intercepts $x^3 + y^2 = 4^5$ with f(x) = 2 is P(8,2) and the greatest common divisor $gcd(8, 2) = 2$. Let's see the common factor among 2^9, 8^3 and 4^5:

$8^3 + 2^9 = 4^5$
$2.(2^8 + 2^8) = 4^5$

Corollary 6.4 *Notice that* 4^5 *is a product of two factors: 2 and* $(2^8 + 2^8)$*. In fact,* $512.2 = 1024$*, and 2 is a factor in common with* 2^9 *and* 8^3*. See the graph below:*

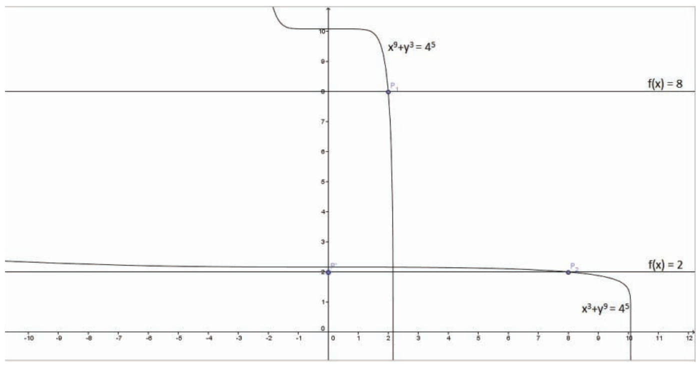

Figure 13: Graph of $x^9 + y^3 = 4^5$ and $x^3 + y^9 = 4^5$ on the same plan, with differents points P(A,B) and P(B,A) .

Example 6.5 $3^3 + 6^3 = 3^5$ *and* $6^3 + 3^3 = 3^5$

As shown on Example 4.16, this sum generates the graphic of a function $x^3 + y^3 = 3^5$*. This one intercepts a point with* $f(x) = y = \sqrt[l]{Z - A^k}$ *. Let's see it on example of this sum* $6^3 + 3^3 = 3^5$*:*

$f(x) = y = \sqrt[3]{3^5 - 6^3}$
$f(x) = y = \sqrt[3]{27}$
$f(x) = y = 3$

Now, applying the value of $y = 3$ *in* $x^3 + y^3 = 3^5$ *, we will obtain the value of abscissa* x *from P(x,3) :*

$x^3 + y^3 = 3^5$
$x^3 + 3^3 = 3^5$
$x = \sqrt[3]{3^5 - 3^3}$
$x = \sqrt[3]{216}$
$x = 6$

The point $P(x, y)$ *which intercepts* $x^3 + y^3 = 3^5$ *with f(x) = 2 is P(6,3) and the greatest common divisor* $gcd(6, 3) = 3$*. Let's see the common factor among* 6^3*,* 3^3 *and* 3^5*:*

$6^3 + 3^3 = 3^5$
$3.(2^3.3^2 + 3^2) = 3^5$

Corollary 6.6 *Notice that* 3^5 *is a product of two factors:* 3 *and* $(2^3.3^2{+}3^2)$*. In fact,* $81.3 = 3^5$*, and* 3 *is a factor in common with* 6^3 *and* 3^3*. See the Figure 13, and notice when a the graph* $x^k + y^l = Z$ *and* $x^l + y^k = Z$ *has* $k = l$*, the graph will be the same for these two cases, and the points P(A,B) and P(B,A) belongs to the same function:*

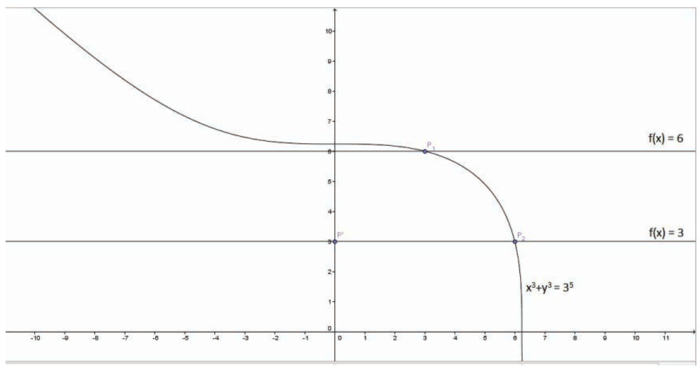

Figure 14: Graph of $x^3 + y^3 = 3^5$, the points P(A,B) and P(B,A) on the same function.

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