Gen. Math. Notes, Vol. 23, No. 2, August 2014, pp.28-42
ISSN 2219-7184;
www.i-csrs.org
Available free online at http://www.geman.in

Almost Generalized Derivations in Prime Rings

Hasret Yazarli[1] and Mehmet Ali Öztürk[2]

[1]Cumhuriyet University, Faculty of Sciences
Department of Mathematics, 58140 Sivas, Turkey
E-mail: hyazarli@cumhuriyet.edu.tr
[2]Adıyaman University, Faculty of Arts and Sciences
Department of Mathematics, 02040 Adıyaman, Turkey
E-mail: maozturk@adiyaman.edu.tr

(Received: 31-1-14 / Accepted: 26-5-14)

Abstract

In this paper, we prove some results about that what happens if we take trace of symmetric bi-derivation or permuting tri-derivation instead of derivation in definition of generalized derivation. Also we apply these results to very well-known results.

Keywords: *Ring, Prime ring, Derivation, Symmetric bi-derivation, Generalized derivation, Permuting tri-derivation*

1 Introduction

Throughout R will be a ring and $Z(R)$ will be its center. A ring R is prime, if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. $xy - yx$ is denoted by $[x, y]$.

It is very interesting and important that the similar properties of derivation which is the one of the basic theory in analysis and applied mathematics are also satisfied in the ring theory. The commutativity of prime rings with derivations was introduced by Posner in [24]. An additive map $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. Recently, a lot of work has been done on commutativity of prime rings with derivation (see [3], [4], [5], [6], ...).

In [7], Bresar defined concept of generalized derivation. An additive map $d : R \to R$ is called generalized derivation if there exists a derivation α of R such that $d(xy) = d(x)y + x\alpha(y)$ for all $x, y \in R$. Thus the concept of generalized derivation contains both the concepts of a derivation and of a left multiplier (i.e., additive maps satisfying $f(xy) = f(x)y$ for all $x, y \in R$). Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \mapsto ax + xb$ for some $a, b \in R$). In [7], Bresar showed that if R has the property that $Rx = \{0\}$ implies $x = 0$ and $h : R \to R$ is any function, $d : R \to R$ is any additive map satisfying $d(xy) = d(x)y + xh(y)$ for all $x, y \in R$, then d is uniquely determined by h and moreover h must be derivation.

In [15] and [16], Maksa defined bi-derivation in ring theory mutually to partial derivations and examined some properties of this derivation. A map $D(.,.) : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A map $d : R \to R$ defined by $d(x) = D(x, x)$ is called the trace of $D(.,.)$ where $D(.,.) : R \times R \to R$ is a symmetric map. It is clear that if $D(.,.)$ is bi-additive (i.e., additive in all arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x + y) = d(x)+d(y)+2D(x, y)$ for all $x, y \in R$. A symmetric bi-additive map $D(.,.) : R \times R \to R$ is called symmetric bi-derivation if $D(xz, y) = D(x, y)z + xD(z, y)$ for all $x, y, z \in R$. For any $y \in R$, the map $x \mapsto D(x, y)$ is a derivation. Let $D(.,.)$ is a symmetric bi-additive map on R. $D(0, y) = 0$ for all $y \in R$ and $D(-x, y) = -D(x, y)$ for all $x, y \in R$. The trace of $D(.,.)$ is an even function.

A map $D(.,.,.) : R \times R \times R \to R$ is called permuting if $D(x, y, z) = D(x, z, y) = D(z, x, y) = D(z, y, x) = D(y, z, x) = D(y, x, z)$ hold for all $x, y, z \in R$. A map $d : R \to R$ defined by $d(x) = D(x, x, x)$ is called trace of $D(.,.,.)$, where $D(.,.,.) : R \times R \times R \to R$ is a permuting map. It is obvious that, if $D(.,.,.) : R \times R \times R \to R$ is permuting tri-additive (i.e., additive in all three arguments), then the trace of $D(.,.,.)$ satisfies the relation $d(x + y) = d(x) + d(y) + 3D(x, x, y) + 3D(x, y, y)$ for all $x, y \in R$. A permuting tri-additive map $D(.,.,.) : R \times R \times R \to R$ is called permuting tri-derivation if $D(xw, y, z) = D(x, y, z)w + xD(w, y, z)$ for all $x, y, z, w \in R$. The trace of $D(.,.,.)$ is an odd function. Let $D(.,.,.)$ be a permuting tri-derivation of R. In this case, for any fixed $a \in R$ and for all $x, y \in R$, a map $D_1(.,.) : R \times R \to R$ defined by $D_1(x, y) = D(a, x, y)$ and a map $d_2 : R \to R$ defined by $d_2(x) = D(a, a, x)$ are a symmetric bi-derivation (in this meaning, permuting 2-derivation is a symmetric bi-derivation) and a derivation, respectively.

In this paper, we will take ring R as a prime ring with right and symmetric Martindale ring of quotients $Q_r(R)$ and $Q_s(R)$, extended centroid C and central closure $R_C = RC$. Let us review some important facts about these rings (see [2], [17] and [23] for details).

The ring $Q_r(R)$ can be characterized by the following four properties:

(i) $R \subseteq Q_r(R)$,
(ii) for $q \in Q_r(R)$ there exists a non-zero ideal I of R such that $qI \subseteq R$,
(iii) if $q \in Q_r(R)$ and $qI = \{0\}$ for some non-zero ideal I of R, then $q = 0$,
(iv) if I is a non-zero ideal of R and $\varphi : I \to R$ is a right $R-$module map, then there exists $q \in Q_r(R)$ such that $\varphi(x) = qx$ for all $x \in I$.

The ring $Q_s(R)$ consists of those $q \in Q_r(R)$ for which $Iq \subseteq R$ for some non-zero ideal I of R. The extended centroid C is a field and it is the center of both $Q_r(R)$ and $Q_s(R)$. Thus, one can view the ring R as a subring of algebras R_C, $Q_r(R)$ and $Q_s(R)$ over C. The extended centroid of R_C is equal to C, whence R_C is equal to its central closure.

In [13], Hvala gave a relation, using generalized derivation defined by Bresar, between prime rings and its extended centroid in ring theory. Many authors have investigated comparable results on prime or semi-prime rings with generalized derivations (see [1], [12], [18], [19],...)

In this paper, we prove some results about that what happens if we take trace of symmetric bi-derivation or permuting tri-derivation instead of derivation in definition of generalized derivation. Also we apply these results to very well-known results.

2 Generalized Derivation Determined By Trace of Symmetric Bi-Derivation

Definition 2.1 *Let R be a ring, $D(.,.) : R \times R \to R$ be symmetric bi-derivation and d be trace of $D(.,.)$. An additive map $f : R \to R$ is called right generalized derivation determined by d, if $f(xy) = f(x)y + xd(y)$ holds for all $x, y \in R$ and denoted by $(f-d)_r$.*

Example 2.2 *Let $R = \left\{ \left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \middle| a, b \in I \right\}$ ring where I is the ring of integers, a map $D(.,.) : R \times R \to R$, defined by $D\left(\left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right), \left(\begin{array}{cc} c & 0 \\ d & 0 \end{array} \right) \right) =$*

$\begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix}$ *and a map* $f : R \to R$ *defined by* $f\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) = \begin{pmatrix} a+b & 0 \\ 0 & 0 \end{pmatrix}$. $D(.,.)$ *is a symmetric bi-derivation. A map* $d : R \to R$, $d(x) = D(x,x)$ *is defined by* $d\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) = D\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix}$ *is the trace of* $D(.,.)$. f *is an additive map and* $f(xy) = f(x)y + xd(y)$ *holds for all* $x, y \in R$. *Thus,* f *is a right generalized derivation determined by* d. *But,* f *isn't a derivation.*

Lemma 2.3 *[13, Lemma 2] Let* $f : R \to R_C$ *be an additive map satisfying* $f(xy) = f(x)y$ *for all* $x, y \in R$. *Then there exists* $q \in Q_r(R_C)$ *such that* $f(x) = qx$ *for all* $x \in R$.

If we consider the definition of generalized derivation introduced by us in the Definition 2.1 and Lemma 2.3, it is important to give the following remark.

Remark 2.4 *Let* R *be a prime ring with* $charR \neq 2$, $D(.,.) : R \times R \to R$ *be a symmetric bi-derivation,* d *be a trace of* $D(.,.)$ *and* $(f-d)_r$ *be a right generalized derivation of* R. *Replacing* y *by* $-y$ *in Definition 2.1, we get* $xd(y) = 0$. *If* $d = 0$, *then* $f(xy) = f(x)y$ *holds for all* $x, y \in R$. *From Lemma 2.3, there exists* $q \in Q_r(R_C)$ *such that* $f(x) = qx$ *for all* $x \in R$. *If* $d \neq 0$, *then* $R = \{0\}$. *So that* $(f-d)_r$ *right generalized derivation has not got any meaning in prime ring.*

We can generalize above definition as follows:

Definition 2.5 *Let* R *be a ring,* $D(.,.) : R \times R \to R$ *be a symmetric bi-derivation,* d *be trace of* $D(.,.)$. *An additive map* $f : R \to R$ *is called right generalized* $\alpha-$*derivation determined by* d, *if there exists a function* $\alpha : R \to R$ *such that* $f(xy) = f(x)\alpha(y) + xd(y)$ *for all* $x, y \in R$ *and denoted by* $(f-\alpha-d)_r$.

Example 2.6 *Let* $R = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \middle| a, b \in I_2 \right\}$ *ring where* I_2 *is the ring of integers modulo* 2, *a map* $D(.,.) : R \times R \to R$, *defined by* $D\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} c & 0 \\ d & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ ac & 0 \end{pmatrix}$, *a map* $f : R \to R$ *defined by* $f\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) = \begin{pmatrix} a^2 & 0 \\ 0 & 0 \end{pmatrix}$ *and a map* $\alpha : R \to R$ *defined by* $\alpha\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) = \begin{pmatrix} a^2 & 0 \\ b^2 & 0 \end{pmatrix}$. $D(.,.)$ *is a symmetric bi-derivation. A map* $d : R \to R$, $d(x) = D(x,x)$ *is defined by* $d\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) = D\left(\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix}$ *is the trace of* $D(.,.)$. f *is an additive map and* $f(xy) = f(x)\alpha(y) + xd(y)$ *for all* $x, y \in R$. *But* f *is not derivation.*

Remark 2.7 *Let R be a prime ring with* $\mathrm{char} R \neq 2$, $D(.,.) : R \times R \to R$ *be a symmetric bi-derivation, d be a trace of $D(.,.)$ and $(f-\alpha-d)_r$ be a right generalized α-derivation of R. Suppose that α is an odd function. Replacing y by $-y$ in Definition 2.5, we get $xd(y) = 0$. Since R is a prime ring, $d = 0$ or $x = 0$. If $d = 0$, then $f(xy) = f(x)\alpha(y)$ for all $x, y \in R$. If $d \neq 0$, then $R = \{0\}$.*

Definition 2.8 *Let R be a ring, $D(.,.) : R \times R \to R$ be a symmetric bi-derivation, d be trace of $D(.,.)$. An additive map $f : R \to R$ is called right generalized $(\alpha,\beta)-$derivation determined by d, if there exist functions $\alpha : R \to R$ and $\beta : R \to R$ such that $f(xy) = f(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$.*

Example 2.9 *Let $R = \left\{ \left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \middle| a, b \in I_2 \right\}$ ring where I_2 is the ring of integers modulo 2, a map $D(.,.) : R \times R \to R$, defined by $D\left(\left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right), \left(\begin{array}{cc} c & 0 \\ d & 0 \end{array} \right) \right) = \left(\begin{array}{cc} 0 & 0 \\ ac & 0 \end{array} \right)$, a map $f : R \to R$ defined by $f\left(\left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \right) = \left(\begin{array}{cc} a^2 & 0 \\ 0 & 0 \end{array} \right)$, a map $\alpha : R \to R$ defined by $\alpha\left(\left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \right) = \left(\begin{array}{cc} a^2 & 0 \\ b^2 & 0 \end{array} \right)$ and a map $\beta : R \to R$ defined by $\beta\left(\left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \right) = \left(\begin{array}{cc} b^2 & 0 \\ a^2 & 0 \end{array} \right)$. $D(.,.)$ is a symmetric bi-derivation. A map $d : R \to R$, $d(x) = D(x,x)$ is defined by $d\left(\left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \right) = D\left(\left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right), \left(\begin{array}{cc} a & 0 \\ b & 0 \end{array} \right) \right) = \left(\begin{array}{cc} 0 & 0 \\ a^2 & 0 \end{array} \right)$ is the trace of $D(.,.)$. f is right generalized $(\alpha,\beta)-$derivation determined by d. But, f isn't a derivation.*

Remark 2.10 *We can also give same Remark 2.7 in place.*

Definition 2.11 *Let R be a ring, $D(.,.) : R \times R \to R$ be a symmetric bi-derivation, d be trace of $D(.,.)$. An even function $f : R \to R$ is called almost right generalized $(\alpha,\beta)-$derivation determined by d, if there exist even functions $\alpha : R \to R$ and $\beta : R \to R$ such that $f(xy) = f(x)\alpha(y) + \beta(x)d(y)$ for all $x, y \in R$ and denoted by $f - (\alpha,\beta)_r - d$.*

Now, let A be any of the rings R, R_C, $R_C + C$, $Q_r(R)$, $Q_s(R)$, $Q_r(R_C)$ and $Q_s(R_C)$. We shall give make an extensive use of the following result.

Lemma 2.12 *[10, Lemma 1] If $a_i, b_i \in A$ satisfy $\sum a_i x b_i = 0$ for all $x \in R$, then the a_i's as well as b_i's are C-dependent, unless all $a_i = 0$ or all $b_i = 0$.*

Lemma 2.13 *[25, Lemma 3.1] Let R be a prime ring with $\operatorname{char} R \neq 2$ and let d_1 and d_2 be traces of symmetric bi-derivations $D_1(.,.)$ and $D_2(.,.)$, respectively. If the identity*

$$d_1(x)\, d_2(y) = d_2(x)\, d_1(y)\,,\ \forall x, y \in R$$

holds and $d_1 \neq 0$, then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ for all $x \in R$.

Proposition 2.14 *Let R be a prime ring with $\operatorname{char} R \neq 2$, $D_1(.,.)$, $D_2(.,.)$, $D_3(.,.)$ and $D_4(.,.)$ be symmetric bi-derivations of R, $0 \neq d_1, 0 \neq d_2, 0 \neq d_3$ and $0 \neq d_4$ be traces of $D_1(.,.)$, $D_2(.,.)$, $D_3(.,.)$ and $D_4(.,.)$, respectively, $f_1 - (\alpha,\beta)_r - d_1$, $f_2 - (\alpha,\beta)_r - d_2$, $f_3 - (\alpha,\beta)_r - d_3$ and $f_4 - (\alpha,\beta)_r - d_4$ be right almost generalized derivations of R. If the identity*

$$f_1(x)\, f_2(y) = f_3(x)\, f_4(y)\,,\ \forall x, y \in R \tag{1}$$

holds, $0 \neq f_1$ and β is surjective , then there exists $\lambda \in C$ such that $f_3(x) = \lambda f_1(x)$ for all $x \in R$.

Proof Let $x, y, z \in R$. Replacing y by yz in (1), we get

$$f_1(x)\, f_2(y)\, \alpha(z) + f_1(x)\, \beta(y)\, d_2(z) = f_3(x)\, f_4(y)\, \alpha(z) + f_3(x)\, \beta(y)\, d_4(z)\,.$$

From (1), we have

$$f_1(x)\, \beta(y)\, d_2(z) = f_3(x)\, \beta(y)\, d_4(z)\,,\ \forall x, y, z \in R. \tag{2}$$

Replacing $\beta(y)$ by $\beta(y)\, d_4(v)$ in (2), we get

$$f_1(x)\, \beta(y)\, d_4(v)\, d_2(z) = f_3(x)\, \beta(y)\, d_4(v)\, d_4(z)\,.$$

From (2),

$$f_1(x)\, \beta(y)\, d_4(v)\, d_2(z) = f_1(x)\, \beta(y)\, d_2(v)\, d_4(z)\,.$$

Hence

$$f_1(x)\, \beta(y)\, (d_4(v)\, d_2(z) - d_2(v)\, d_4(z)) = 0.$$

Since $f_1 \neq 0$, β is surjective and R is a prime ring, we get $d_4(v)\, d_2(z) = d_2(v)\, d_4(z)$ for all $v, z \in R$. From Lemma 2.13, since $d_4 \neq 0$, there exists $\lambda \in C$ such that $d_2(z) = \lambda d_4(z)$ for all $z \in R$. Using last relation in (2), we get $f_1(x)\, \beta(y)\, \lambda d_4(z) = f_3(x)\, \beta(y)\, d_4(z)$ for all $x, y, z \in R$. That is, $(\lambda f_1(x) - f_3(x))\, \beta(y)\, d_4(z) = 0$. Since $d_4 \neq 0$, β is surjective and R is a prime ring, $f_3(x) = \lambda f_1(x)$ for all $x \in R$ and for $\lambda \in C$.

Corollary 2.15 *Let R be a prime ring with char$R \neq 2$, $D_1(.,.)$ and $D_2(.,.)$ be symmetric bi-derivations of R, $0 \neq d_1$ and $0 \neq d_2$ be traces of $D_1(.,.)$ and $D_2(.,.)$, respectively, $f_1 - (\alpha, \beta)_r - d_1$ and $f_2 - (\alpha, \beta)_r - d_2$ be right almost generalized derivations of R. If the identity*

$$f_1(x) f_2(y) = f_2(x) f_1(y), \ \forall x, y \in R$$

holds, $0 \neq f_1$ and β is surjective , then there exists $\lambda \in C$ such that $f_2(x) = \lambda f_1(x)$ for all $x \in R$.

Lemma 2.16 *Let R be a prime ring with char$R \neq 2$, $D(.,.)$ be a symmetric bi-derivation of R, $0 \neq d$ be trace of $D(.,.)$, $f - (\alpha, \beta)_r - d$ be right almost generalized derivation of R, β is surjective and $a \in R$. If $af(x) = 0$ for all $x \in R$, then $a = 0$.*

Proof Let $af(x) = 0$ for all $x \in R$. Replacing x by xy, we get $af(x)\alpha(y) + a\beta(x)d(y) = 0$. Using hypothesis, we have $a\beta(x)d(y) = 0$. Since R is a prime ring, we have $a = 0$.

3 Generalized Derivation Determined By Trace of Permuting Tri-Derivation

Definition 3.1 *Let R be a ring, $D(.,.,.) : R \times R \times R \to R$ be a permuting tri-derivation and d be trace of $D(.,.,.)$. An additive map $f : R \to R$ is called right generalized derivation determined by d, if $f(xy) = f(x)y + xd(y)$ holds for all $x, y \in R$. An additive map $f : R \to R$ is called left generalized derivation determined by d, if $f(xy) = xf(y) + d(x)y$ holds for all $x, y \in R$. Also, an additive map $f : R \to R$ is called generalized derivation determined by d, if it is both a right generalized and a left generalized derivation.*

Example 3.2 *Let* $R = \left\{ \left. \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \right| a, b, c \in I \right\}$ *ring where I is the ring of integers, a map $D(.,.,.) : R \times R \times R \to R$, defined by*

$$D\left(\begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & 0 & 0 \\ b_2 & 0 & 0 \\ c_2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & 0 & 0 \\ b_3 & 0 & 0 \\ c_3 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 a_2 a_3 & 0 & 0 \end{pmatrix}$$

and a map $f : R \to R$ defined by

$$f\left(\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

D is a permuting tri-derivation. A map $d : R \to R$, $d(x) = D(x,x,x)$ is defined by $d\left(\left(\begin{array}{ccc} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{array}\right)\right) = D\left(\left(\begin{array}{ccc} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{array}\right), \left(\begin{array}{ccc} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{array}\right), \left(\begin{array}{ccc} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{array}\right)\right) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a^3 & 0 & 0 \end{array}\right)$ *is the trace of $D(.,.,.)$. f is an additive map and $f(xy) = f(x)y + xd(y)$ holds for all $x, y \in R$. But, f is not a derivation.*

Remark 3.3 *Let R be a prime ring with $charR \neq 2,3$, $D(.,.,.) : R \times R \times R \to R$ be a permuting tri-derivation, d be a trace of $D(.,.,.)$ and f be a right generalized derivation determined by d. Therefore $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. Replacing y by $y + z$, $z \in R$ in this relation, we get*

$$\begin{aligned} f(x(y+z)) &= f(x)y + f(x)z + xd(y) + xd(z) \\ &+ 3xD(y,y,z) + 3xD(y,z,z) \end{aligned}$$

and

$$f(x(y+z)) = f(xy) + f(xz) = f(x)y + f(x)z + xd(y) + xd(z).$$

Comparing these relations, we get

$$xD(y,y,z) + xD(y,z,z) = 0$$

since $charR \neq 3$. Replacing z by y, we get $2xd(y) = 0$ for all $x, y \in R$. Since R is a prime ring and $charR \neq 2$, $x = 0$ or $d(y) = 0$ for all $y \in R$. If $d = 0$, then $f(xy) = f(x)y$ holds for all $x, y \in R$. From Lemma 2.3, there exists $q \in Q_r(R_C)$ such that $f(x) = qx$ for all $x \in R$. In this case, we must take f be not additive when R is a prime $charR \neq 2,3$.

We can give definition of generalized derivation according to above the definition as follows:

Definition 3.4 *Let R be a ring, $D(.,.,.) : R \times R \times R \to R$ be a permuting tri-derivation and d be trace of $D(.,.,.)$. A map $f : R \to R$ is called right almost generalized derivation determined by d, if $f(xy) = f(x)y + xd(y)$ holds for all $x, y \in R$ and denoted by $(f-d)_r$. A map $f : R \to R$ is called left almost generalized derivation determined by d, if $f(xy) = xf(y) + d(x)y$ holds for all $x, y \in R$ and denoted by $(f-d)_l$. Also, a map $f : R \to R$ is called almost generalized derivation determined by d, if it is both a right almost generalized and a left almost generalized derivation and denoted by $f - d$.*

Lemma 3.5 *[29, Lemma 4.2] Let R be a prime ring with $\mathrm{char} R \neq 2,3$ and let d_1 and d_2 be traces of permuting tri-derivations $D_1(.,.,.)$ and $D_2(.,.,.)$, respectively. If the identity $d_1(x)\, d_2(y) = d_2(x)\, d_1(y)$ holds for all $x, y \in R$ and $d_1 \neq 0$, then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$.*

Proposition 3.6 *Let R be a prime ring with $\mathrm{char} R \neq 2,3$, $D_1(.,.,.)$, $D_2(.,.,.)$, $D_3(.,.,.)$ and $D_4(.,.,.)$ be permuting tri-derivations of R, $0 \neq d_1, 0 \neq d_2, 0 \neq d_3$ and $0 \neq d_4$ be traces of $D_1(.,.,.)$, $D_2(.,.,.)$, $D_3(.,.,.)$ and $D_4(.,.,.)$, respectively, $(f_1 - d_1)_r$, $(f_2 - d_2)_r$, $(f_3 - d_3)_r$ and $(f_4 - d_4)_r$ be right almost generalized derivations of R. If the identity*

$$f_1(x)\, f_2(y) = f_3(x)\, f_4(y)\,,\ \forall x, y \in R \tag{3}$$

holds and $0 \neq f_1$, then there exists $\lambda \in C$ such that $f_3(x) = \lambda f_1(x)$ for all $x \in R$.

Proof Let $x, y, z \in R$. Replacing y by yz in (3) and using (3), we get

$$f_1(x)\, y d_2(z) = f_3(x)\, y d_4(z) \tag{4}$$

Replacing y by $yd_4(v)$ in (4), we have

$$f_1(x)\, y d_4(v)\, d_2(z) = f_3(x)\, y d_4(v)\, d_4(z)$$

and so from (4),

$$f_1(x)\, y d_4(v)\, d_2(z) = f_1(x)\, y d_2(v)\, d_4(z)\,.$$

Thus $f_1(x)\, y\,(d_4(v)\, d_2(z) - d_2(v)\, d_4(z)) = 0$. Since $f_1 \neq 0$ and R is a prime ring, we get $d_4(v)\, d_2(z) = d_2(v)\, d_4(z)$ for all $v, z \in R$. Since $d_4 \neq 0$, there exists $\lambda \in C$ such that $d_2(z) = \lambda d_4(z)$ for all $y \in R$ by Lemma 3.5. Writing last relation in (4), we get $f_1(x)\, y \lambda d_4(z) = f_3(x)\, y d_4(z)$ for all $x, y, z \in R$. That is, $(\lambda f_1(x) - f_3(x))\, y d_4(z) = 0$. Since $d_4 \neq 0$ and R is a prime ring, we get $f_3(x) = \lambda f_1(x)$ for all $x \in R$ and for $\lambda \in C$.

Corollary 3.7 *Let R be a prime ring with $\mathrm{char} R \neq 2,3$, $D_1(.,.,.)$ and $D_2(.,.,.)$ be permuting tri-derivations of R, $0 \neq d_1$ and $0 \neq d_2$ be traces of $D_1(.,.,.)$ and $D_2(.,.,.)$, respectively, $(f_1 - d_1)_r$ and $(f_2 - d_2)_r$ be right almost generalized derivations of R. If the identity*

$$f_1(x)\, f_2(y) = f_2(x)\, f_1(y)\,,\ \forall x, y \in R$$

holds and $0 \neq f_1$, then there exists $\lambda \in C$ such that $f_2(x) = \lambda f_1(x)$ for all $x \in R$.

Lemma 3.8 *Let R be a prime ring, $D(.,.,.)$ be a permuting tri-derivation of R, $0 \neq d$ be trace of $D(.,.,.)$, $(f-d)_r$ be right almost generalized derivation of R and $a \in R$. Then*

(i) If $af(x) = 0$ for all $x \in R$, then $a = 0$.
(ii) If $[a, f(x)] = 0$ for all $x \in R$ and $charR \neq 2,3$, then $a \in Z(R)$.

Proof (i) Let $af(x) = 0$ for all $x \in R$. Replacing x by xy, we get $0 = af(xy) = af(x)y + axd(y)$ for all $x, y \in R$. Hence $axd(y) = 0$. Since R is prime ring and $d \neq 0$, we get $a = 0$.

(ii) Let $[a, f(x)] = 0$ for all $x \in R$. Replacing x by xy, $y \in R$ and using hypothesis, we get

$$f(x)[a,y] + [a,x]d(y) + x[a,d(y)] = 0 \tag{5}$$

Replacing y by $y+z$ in (5) and using (5),

$$0 = [a,x]D(y,y,z) + [a,x]D(y,z,z) + x[a,D(y,y,z)] + x[a,D(y,z,z)] \tag{6}$$

since $charR \neq 3$.

In (6), replacing z by $-z$ and comparing with (6) we get

$$[a,x]D(y,y,z) + x[a,D(y,y,z)] = 0$$

since $charR \neq 2$. Therefore

$$axD(y,y,z) = xD(y,y,z)a$$

That is, $axd(y) = xd(y)a$ for all $x, y \in R$. Replacing x by xr for all $r \in R$ in this relation, we get $axrd(y) = xrd(y)a = xard(y)$. Hence $[a,x]rd(y) = 0$. Since R is prime ring and $d \neq 0$, we have $a \in Z(R)$.

Lemma 3.9 *Let R be a prime ring with $charR \neq 2,3$, $D(.,.,.)$ be a permuing tri-derivation of R, $0 \neq d$ be trace of $D(.,.,.)$, $(f-d)_r$ be right almost generalized derivation of R. If $[f(x), f(y)] = 0$ for all $x, y \in R$, then R is commutative ring.*

Proof From Lemma 3.8(ii), we get $f(R) \subseteq Z(R)$. So that $[f(x), r] = 0$ for all $x, r \in R$. Replacing x by xy, $y \in R$ and using hypothesis, we get

$$0 = f(x)[y,r] + [x,r]d(y) + x[d(y),r] \tag{7}$$

Replacing y by $y+z$, $z \in R$ and using (7) and $charR \neq 2,3$, we have

$$[x,r]d(y) + x[d(y),r] = 0.$$

Therefore $xd(y)r = rxd(y)$ for all $x, y, r \in R$. In this relation, replacing x by xz, $z \in R$ and using this relation, we get $[x,r]zd(y) = 0$. Since R is a prime ring and $d \neq 0$, R is a commutative ring.

Theorem 3.10 *[24, Theorem 1] Let R be prime ring with $charR \neq 2$ and d_1, d_2 derivations of R such that iterate $d_1 d_2$ is also derivation; then one at least of d_1, d_2 is zero.*

Lemma 3.11 *[22, Corollary 10] Let R be a prime ring with $charR \neq 2$ and $3-torsionfree$. Let D be a permuting tri-derivation of R and d be the trace of D. If $D\left(d\left(x\right), x, x\right) = 0$ for all $x \in R$, then $D = 0$.*

Theorem 3.12 *Let R be a non-commutative prime ring with $charR \neq 2, 3$, $D\left(.,.,.\right)$ be a nonzero permuting tri-derivation of R, $0 \neq d$ be trace of $D\left(.,.,.\right)$, $\left(f-d\right)_r$ be right almost generalized derivation of R and $a \in R$. If $\left[a, f\left(R\right)\right] \subseteq Z\left(R\right)$, then $a \in Z\left(R\right)$.*

Proof Let $\left[a, f\left(x\right)\right] \in Z\left(R\right)$ for all $x \in R$. Replacing x by xy, $y \in R$, we get

$$\left[a, f\left(x\right)\right] y + f\left(x\right)\left[a, y\right] + \left[a, x\right] d\left(y\right) + x\left[a, d\left(y\right)\right] \in Z\left(R\right)$$

In this relation, replacing y by $y+z$, $z \in R$ and using this relation and $charR \neq 2, 3$, we get

$$\left[a, x\right] D\left(y, y, z\right) + \left[a, x\right] D\left(y, z, z\right) + x\left[a, D\left(y, y, z\right)\right] + x\left[a, D\left(y, z, z\right)\right] \in Z\left(R\right) \tag{8}$$

In (8), replacing z by $-z$ and comparing with (8), we get

$$\left[a, x\right] D\left(y, y, z\right) + x\left[a, D\left(y, y, z\right)\right] \in Z\left(R\right)$$

since $charR \neq 2, 3$. Replacing x by c, $c \in Z\left(R\right)$ in this relation, we have $c\left[a, D\left(y, y, z\right)\right] \in Z\left(R\right)$. Since R is a prime ring, we get $c = 0$ or $\left[a, D\left(y, y, z\right)\right] \in Z\left(R\right)$. If $c = 0$, then $\left[a, f\left(x\right)\right] = 0$ for all $x \in R$. From Lemma 3.8(ii), $a \in Z\left(R\right)$. Let $\left[a, D\left(y, y, z\right)\right] \in Z\left(R\right)$. Replacing z by a^2, we get $a\left[a, D\left(y, y, a\right)\right] \in Z\left(R\right)$. Since $a\left[a, D\left(y, y, a\right)\right] \in Z\left(R\right)$, we have $\left[a, D\left(y, y, a\right)\right] = 0$ or $a \in Z\left(R\right)$. In both cases $\left[a, D\left(y, y, a\right)\right] = 0$.

Since $\left[a, D\left(y, y, z\right)\right] \in Z\left(R\right)$ for all $z \in R$, $\left[a, D\left(y, y, \left[a, x\right]\right)\right] \in Z\left(R\right)$ for all $x \in R$. Therefore we get

$$\begin{aligned}\left[a, D\left(y, y, \left[a, x\right]\right)\right] &= \left[a, \left[a, D\left(y, y, x\right)\right]\right] + \left[a, \left[D\left(y, y, a\right), x\right]\right] \\ &= \left[a, \left[D\left(y, y, a\right), x\right]\right] \in Z\left(R\right).\end{aligned}$$

Replacing x by ax in this relation, we have $a\left[a, \left[D\left(y, y, a\right), x\right]\right] \in Z\left(R\right)$. Since R is prime ring and $\left[a, \left[D\left(y, y, a\right), x\right]\right] \in Z\left(R\right)$, we get $\left[a, \left[D\left(y, y, a\right), x\right]\right] = 0$ or $a \in Z\left(R\right)$. If $\left[a, \left[D\left(y, y, a\right), x\right]\right] = 0$ for all $x \in R$, $\left(I_a I_{D(y,y,a)}\right)\left(x\right) = 0$ where I_a and $I_{D(y,y,a)}$ are inner-derivations determined by a and $D\left(y, y, a\right)$, respectively. From Theorem 3.10, we get $a \in Z\left(R\right)$ or $D\left(y, y, a\right) \in Z\left(R\right)$.

Suppose that $D(y,y,a) \in Z(R)$. Since $[a, D(y,y,ax)] \in Z(R)$ for all $x \in R$, we get

$$[a, D(y,y,ax)] = D(y,y,a)[a,x] + a[a, D(y,y,x)] \in Z(R). \tag{9}$$

Therefore we have

$$[a, D(y,y,a)[a,x] + a[a, D(y,y,x)]] = 0$$

That is, $D(y,y,a)[a,[a,x]] = 0$. If $D(y,y,a) \neq 0$, $a \in Z(R)$ for all $x \in R$. If $D(y,y,a) = 0$, we get $[a, D(y,y,x)] \in Z(R)$ from (9). Hence $a \in Z(R)$ or $[a, D(y,y,x)] = 0$.

Suppose that $[a, D(y,y,x)] = 0$ for all $x, y \in R$. Replacing x by xz, we get

$$0 = [a, D(y,y,xz)] = D(y,y,x)[a,x] + [a,x] D(y,y,z) \tag{10}$$

If z commutes with a, then $[a,z] = 0$. From (10), $[a,x] D(y,y,z) = 0$. If $a \notin Z(R)$, then $D(y,y,z) = 0$, since R is prime ring. That is, $D(y,y,z) = 0$ on set $C_R(a) = \{z \in R |\, az = za\}$. But since $D(y,y,x) \in C_R(a)$ for all $x \in R$, $D(y,y,D(y,y,z)) = 0$ for all $y, z \in R$. Therefore $D(d(x), x, x) = 0$ for all $x \in R$. From Lemma 3.11, we get $D = 0$. This is a contradiction. Thus $a \in Z(R)$.

Proposition 3.13 *Let R be a non-commutative prime ring with char$R \neq 2, 3$, $D(.,.,.)$ be permuting tri-derivation of R, d be trace of $D(.,.,.)$ and $(f-d)_r$ be right almost generalized derivation of R. If $f([x,y]) = 0$ for all $x, y \in R$, then $d = 0$.*

Proof Let

$$f([x,y]) = 0, \ \forall x, y \in R. \tag{11}$$

Replacing y by yx in (11), we get $[x,y] d(x) = 0$. Replacing y by yr, $r \in R$ in this relation, we get $[x,y] r d(x) = 0$. Since R is prime ring, for any $x \notin Z(R)$, we get $d(x) = 0$. Let $x \in Z(R)$, $y \notin Z(R)$. Then $x + y \notin Z(R)$ and $-y \notin Z(R)$. Thus $0 = d(x+y) = d(x) + 3D(x,x,y) + 3D(x,y,y)$ and $0 = d(x + (-y)) = d(x) - 3D(x,x,y) + 3D(x,y,y)$ which imply that $d(x) = 0$. Therefore we have proved that $d(x) = 0$ for all $x \in R$.

Theorem 3.14 *Let R be a prime ring with char$R \neq 2, 3$, $D(.,..)$ be permuting tri-derivation of R, d be trace of $D(.,.,.)$ and $(f-d)_r$ be right almost generalized derivation of R. If $f([x,y]) = \mp [x,y]$ for all $x, y \in R$, then $d = 0$.*

Proof Let

$$f([x,y]) = \mp [x,y], \ \forall x, y \in R. \tag{12}$$

Replacing y by yx in (12), we get $[x,y] d(x) = 0$. Using same method in proof of Proposition 3.13, we get $d(x) = 0$ for all $x \in R$.

Theorem 3.15 *Let R be a prime ring with $charR \neq 2,3$, $D_1(.,.,.)$ and $D_2(.,.,.)$ be permuting tri-derivations of R, $0 \neq d_1$ and $0 \neq d_2$ be traces of $D_1(.,.,.)$ and $D_2(.,.,.)$, respectively, $(f_1-d_1)_r$ and $(f_2-d_2)_r$ be right almost generalized derivations of R and $a \in R$. If $af_1(x) = f_2(x)a$ for all $x \in R$, then $a \in Z(R)$.*

Proof Let

$$af_1(x) = f_2(x)a, \ \forall x \in R. \tag{13}$$

Replacing x by xy in (13), we get

$$af_1(x)y + axd_1(y) = f_2(x)ya + xd_2(y)a, \ \forall x,y \in R. \tag{14}$$

Replacing y by $y+z$ in (14), from (14) we get

$$axD_1(y,y,z) + axD_1(y,z,z) = xD_2(y,y,z)a + xD_2(y,z,z)a, \qquad \forall x,y,z \in R \tag{15}$$

since $charR \neq 3$. Replacing z by $-z$ in (15) and comparing with (15), since $charR \neq 2$, we obtain that $axd_1(y) = xd_2(y)a$, for all $x,y \in R$. Replacing x by xr in last relation, we get $[a,x]rd_1(y) = 0$, for all $x,y,r \in R$. Since $d_1 \neq 0$ and R is prime ring, we get $[a,x] = 0$ for all $x \in R$. That is, $a \in Z(R)$.

References

[1] N. Argaç and E. Albaş, Generalized derivations of prime rings, *Algebra Coll.*, 11(3) (2004), 399-410.

[2] K.I. Beidar, W.S. Martindale and A.V. Mikhalev, Rings with generalized identities, *Marcel Dekker, Monographs and Text-Books in Pure and Applied Mathematics*, New York, (1995), 195.

[3] K.I. Beidar and M.A. Chebotar, On Lie derivations of Lie ideals of prime rings, *Israel Math.*, 123(2001), 131-148.

[4] H.E. Bell and W.S. Martindale, Semiderivations and commutativity in prime rings, *Canad. Math. Bull.*, 31(4) (1988), 500-508.

[5] H.E. Bell and M.N. Daif, On derivations and commutativity in prime rings, *Acta Math. Hungary*, 66(4) (1995), 337-343.

[6] M. Bresar, A note on derivations, *Math. J. Okayama Univ.*, 32(1990), 83-88.

[7] M. Bresar, On the distance of the compositions of two derivations to generalized derivations, *Glasgow Math. J.*, 33(1991), 89-93.

[8] M. Bresar, Centralizing mappings and derivations in prime rings, *J. Algebra*, 156(2) (1993), 385-394.

[9] M. Bresar, Commuting trace of bi-additive mappings, commutativity-preserving mappings and Lie mappings, *Trans. Amer. Math. Soc.*, 335(1993), 525-546.

[10] M. Bresar, Functional identities of degree two, *J. Algebra*, 172(3) (1995), 690-720.

[11] M. Bresar, On generalized bi-derivations and related maps, *J. Algebra*, 172(3) (1995), 764-786.

[12] M.N. Daif and H.E. Bell, Remarks on derivations on semi-prime rings, *Internat. J. Math. and Math. Sci.*, 15(1) (1992), 205-206.

[13] B. Hvala, Generalized derivation in rings, *Comm. Algebra*, 26(4) (1998), 1147-1166.

[14] V.K. Kharchenko, Automorphisms and derivations of associative rings, *Mathematics and Its Applications*, Kluwer Academic Publishers, Dordrecht, (1991).

[15] Gy. Maksa, A remark on symmetric bi-additive functions having non-negative diagonalization, *Glasnik Mat.-III Ser.*, 15(2) (1980), 279-282.

[16] Gy. Maksa, On the trace of symmetric bi-derivations, *C. R. Math. Rep. Acad. Sci. Canada*, 9(1987), 1303-1307.

[17] W.S. Martindale, Prime rings satisfying a generalized polynomial identity, *J. Algebra*, 12(1969), 576-584.

[18] J. Mayne, Centralizing mappings of prime rings, *Canad. Math. Bull.*, 27(1984), 122-126.

[19] M.A. Öztürk and M. Sapancı, Orthogonal symmetric bi-derivation on semi-prime gamma rings, *Hacettepe Bull. Nat. Sci. Eng.*, 26(1997), 31-46.

[20] M.A. Öztürk and M. Sapancı, On generalized symmetric bi-derivations in prime rings, *East Asian Math. J.*, 15(2) (1999), 165-176.

[21] M.A. Öztürk, Y.B. Jun and K.H. Kim, Orthogonal traces on semi-prime gamma rings, *Sci. Math. Jpn.*, 53(3) (2001), 495-501.

[22] M.A. Öztürk, Permuting tri-derivations in prime and semi-prime rings, *East Asian Math. J.*, 15(2) (1999), 177-190.

[23] D. Passman, *Infinite Crossed Products*, Academic Press, San Diego, (1989).

[24] E.C. Posner, Derivations in prime rings, *Proc. Amer. Math. Soc.*, 8(1957), 1093-1100.

[25] M. Sapancı, M.A. Öztürk and Y.B. Jun, Symmetric bi-derivations on prime rings, *East Asian Math. J.*, 15(1) (1999), 105-109.

[26] M. Uçkun and M.A. Öztürk, On trace of symmetric bi-gamma-derivations in gamma-near-rings, *Houston J. Math.*, 33(2) (2007), 323-339.

[27] J. Vukman, Symmetric bi-derivations on prime and semi-prime rings, *Aequationes Math.*, 38(1989), 245-254.

[28] J. Vukman, Commuting and centralizing mappings in prime rings, *Proc. Amer. Math. Soc.*, 109(1990), 47-52.

[29] H. Yazarlı, M.A. Öztürk and Y.B. Jun, Tri-additive maps and permuting tri-derivations, *Commun. Fac. Sci. Univ. Ank. Series*, A1 54(1) (2005), 1-8.