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Homotopy Analysis Method to Solve Two-Dimensional Fuzzy Fredholm Integral Equation

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Abstract

In this paper, homotopy analysis method is proposed to solve a linear two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE-2). We use parametric form of fuzzy functions and convert a 2D-FFIE-2 to a linear system of Fredholm integral equations of the second kind with three variables in crisp case. We use the homotopy analysis method to find the approximate solution of the converted system, which is the approximate solution for 2D-FFIE-2. The solved problems reveal that the proposed method is effective and simple, and in some cases, it gives the exact solution.

Keywords: *Fuzzy fredholm Integral equation, parametric form of Fuzzy Fredholm Integral equation, Homotopy Analysis method.*

1 Introduction

As we know the Fuzzy differential and integral eguations are one of the important parts of the fuzzy analysis theory that play major role in numerical

analysis. The concept of fuzzy numbers and arithmetic operation on it was introduced by Zadeh [13], which was further enriched by Mizumoto and Tanaka [8]. Dubois and Prade [2], made a significant contribution by introducing the concept of LR fuzzy number and presented a computational formula for operations on fuzzy numbers. Also they [3], introduced the concept of integration of fuzzy function. Later Goetschel and Voxman [5], preferred a Riemann integral type approach, Kaleva [6], chose the defenition of integral of fuzzy function, using the Lebesgue-type concept for integration. One of the first applications of fuzzy integration was given by Wu and Ma who investigated the fuzzy Fredholm integral equation of the second kind. Recently some numerical methods have been introduced to solve linear fuzzy Fredholm integral equation of the second kind in one-dimensional space (FFIE-2) and two-dimensional space (2D-FFIE-2). For example, Effati et al. [4], and Rivaz et al. [12], used homotopy perturbation method for solving FFIE- 2 and 2D-FFIE-2, respectively. The purpose of this paper is to extend the application of the homotopy analysis method (HAM) for solving the two-dimensional fuzzy Fredholm integral equation of the second kind (FFIE-2). In this paper, the basic idea of the HAM is introduced and then the application of the HAM to slove the two-dimensional fuzzy Fredholm integral equation of the second kind (FFIE-2) is extended.
The remainder of this paper is organized as follows: in Section 2, we present the basic notations of fuzzy numbers, fuzzy functions and fuzzy integrals. In Section 3, the 2D-FFIE-2 and its parametric form are discussed. We explain the homotopy analysis method in Section 4. Then we apply the method for sloving 2D-FFIE-2 in Section 5, and compare the results with the exact solutions in Section 6. Conclusions are given in Section 6.

2 Preliminaries

In this section the most basic notations used in fuzzy calculus are introduced. We start with defining a fuzzy number.

Definition 2.1 *A fuzzy number is a fuzzy set* $u : R^1 \to I = [0; 1]$ *which satisfies*
i. u is upper semicontinuous.
ii. $u(x) = 0$ *outside some interval (c,d).*
iii. There are real numbers a,b: $c \le a \le b \le d$ *for which*
1. u(x) is monotonic increasing on (c,a),
2. u(x) is monotonic decreasing on (b,d),
3. $u(x) = 1$, $x \in (a, b)$.

The set of all fuzzy numbers (as given by Definition 1) is denoted by E^1. An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [6].

Definition 2.2 *The parametric form of a fuzzy number u is a pair of functions $(\underline{u}(r), \overline{u}(r))$, $r \in (0,1)$, which satisfies the following requirements:*
1. $\underline{u}(r)$ is a bounded, continuous, monotonic increasing function over [0,1].
2. $\overline{u}(r)$ is a bounded, continuous, monotonic decreasing function over [0,1].
3. $\underline{u}(r) \leq \overline{u}(r), \quad 0 \leq r \leq 1$.
$(\underline{u}(r), \overline{u}(r))$, $(0 \leq r \leq 1)$, are called the r-cut sets of u.

Definition 2.3 *For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$, $0 \leq r \leq 1$ and real number k, we define addition, subtraction, scalar product by k and multiplication are as following:*
subtraction:

$$(u+v)(r) = (\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)),$$
$$(u-v)(r) = (\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)).$$

multiplication:

$$(u.v)(r) = \begin{cases} \underline{uv}(r) = max\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r))\}, \\ \overline{uv}(r) = min\{\underline{u}(r)\underline{v}(r), \underline{u}(r)\overline{v}(r), \overline{u}(r)\underline{v}(r), \overline{u}(r)\overline{v}(r))\}. \end{cases} \quad (1)$$

scalar product:

$$(ku)(r) = \begin{cases} (k\underline{u}(r), k\overline{u}(r)) & k \geq 0, \\ (k\overline{u}(r), k\underline{u}(r)) & k < 0. \end{cases} \quad (2)$$

Definition 2.4 *For arbitrary fuzzy numbers $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r))$, $0 \leq r \leq 1$ the quantity:*

$$D(u,v) = max\{sup|\underline{u}(r) - \underline{v}(r)|, sup|\overline{u}(r) - \overline{v}(r)|\}.$$

This metric is equivalent to the one used by Puri and Ralescu [10], and Kaleva [6]. It is shown [11], that (E^1, D) is a complete metric space.

Definition 2.5 *A function $f : R^2 \to E^1$ is called a fuzzy function in two-dimensional space. f is said to be continuous, if for arbitrary fixed $t_0 \in R^2$ and $\varepsilon > 0$ a $\delta > 0$ exists such that:*

$$|| \, t - t_0 \, || < \delta \Rightarrow D(f(t), f(t_0)) < \varepsilon, \qquad t = (x,y), t_0 = (x_0, y_0).$$

Definition 2.6 *Let $f : [a,b] \times [c,d] \to E^1$ For each partition $p = \{x_1, x_2, ..., x_m\}$ of [a,b] and $q = \{y_1, y_2, ..., y_n\}$ of [c,d], and for arbitrary $\xi_i : x_{i-1} \leq \xi_i \leq x_i, \;\; 2 \leq i \leq m$ and for arbitrary $\eta : y_{j-1} \leq \eta_j \leq y_j, \;\; 2 \leq j \leq n$, let*

$$R_p = \Sigma_{i=2}^{m} \Sigma_{j=2}^{n} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1}).$$

The definite integral of $f(x,y)$ *over* : $[a,b]\times[c,d]$ *is,*

$$\int_a^b\int_c^d f(x,y) = LimR_p,$$

$$(max \mid x_i - x_{i-1} \mid, max \mid y_j - y_{j-1} \mid) \to (0,0),$$

provided that this limit exists in metric D.
If the fuctoion $f(x,y)$ *is continuous in the metric D, its definite integral exists [5].*
Furthermore:

$$\underline{\int_a^b\int_c^d f(x,y,r)} = \int_a^b\int_c^d \underline{f}(x,y,r),$$

$$\overline{\int_a^b\int_c^d f(x,y,r)} = \int_a^b\int_c^d \overline{f}(x,y,r).$$

3 Two-Dimensional Fuzzy Integral Equation

The linear two-dimensional fuzzy Fredholm integral equation of the second kind (2D-FFIE-2) is

$$u(x,y) = f(x,y) + \int_a^b\int_c^d k(x,y,s,t)u(s,t)dsdt, \quad (x,y)\in V \tag{3}$$

where $u(x,y)$ and $f(x,y)$ are fuzzy functions on $V=[a,b]\times[c,d]$ and $k(x,y,s,t)$ is an arbitrary kernel function on $S=[a,b]\times[c,d]\times[a,b]\times[c,d]$, and u is unknown on V.
Now, we introduce parametric form of a 2D-FFIE-2 with respect to Definition 2. Let $(\underline{f}(x,y,r),\overline{f}(x,y,r))$ and $(\underline{u}(x,y,r),\overline{u}(x,y,r))$, $\quad 0\le r\le 1,\ \ (x,y)\in V$, be parametric form of $f(x,y)$ and $u(x,y)$, respectively. Then parametric form of 2D-FFIE-2 is as follows:

$$\begin{array}{l}\underline{u}(x,y,r) = \underline{f}(x,y,r) + \int_a^b\int_c^d \mathrm{v}_1(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))dsdt,\\ \overline{u}(x,y,r) = \overline{f}(x,y,r) + \int_a^b\int_c^d \mathrm{v}_2(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r))dsdt.\end{array} \tag{4}$$

that:

$$\mathrm{v}_1(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r)) = \begin{cases} k(x,y,s,t)\underline{u}(s,t,r) & k(x,y,s,t)\ge 0,\\ k(x,y,s,t)\overline{u}(s,t,r), & k(x,y,s,t)<0.\end{cases} \tag{5}$$

and

$$\mathrm{v}_2(x,y,s,t,\underline{u}(s,t,r),\overline{u}(s,t,r)) = \begin{cases} k(x,y,s,t)\overline{u}(s,t,r) & k(x,y,s,t) \geq 0, \\ k(x,y,s,t)\underline{u}(s,t,r) & k(x,y,s,t) < 0. \end{cases} \quad (6)$$

for each $a \leq x \leq b$ and $c \leq y \leq d$ and $0 \leq r \leq 1$. We can see that (4) is a system of linear Fredholm integral equations of the second kind with three variables in crisp case.

4 Using of Homotopy Analysis Method

consider

$$N(y(x)) = 0$$

where N is a nonlinear operator, $y(x)$ is unknown function and x is an independent variable. let $y_0(x)$ denote an initial guess of the exact solution $y(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and L an auxiliary linear operator with the property $L[r(x)] = 0$ when $r(x) = 0$. Then using $q \in [0, 1]$ as an embedding parameter, we construct a homotopy as follows:

$$(1-q)L[\phi(x;q)-y_0(x)]-qhH(x)N[\phi(x;q)] = H[\phi(x;q); y_0(x), H(x), h, q]. \quad (7)$$

It should be emphasized that we have great freedom to choose the initial guess $y_0(x)$, the auxiliary linear operator L, the non-zero auxiliary parameter h, and the auxiliary function $H(x)$. Enforcing the homotopy (7) to be zero, i.e.,

$$H[\phi(x;q); y_0(x), H(x), h, q] = 0, \quad (8)$$

we have the so-called zero-order deformation equation

$$(1-q)L[\phi(x;q) - y_0(x)] = qhH(x)N[\phi(x;q)]. \quad (9)$$

When q = 0, the zero-order deformation (9) becomes

$$\phi(x;0) = y_0(x), \quad (10)$$

and when $q = 1$, since $h \neq 0$ and $H(x) \neq 0$, the zero-order deformation (9) is equivalent to

$$\phi(x;1) = y(x). \quad (11)$$

Thus, according to (10) and (11), as the embedding parameter q increases from 0 to $1, \phi(x;q)$ varies continuously from the initial approximation $y_0(x)$ to the exact solution $y(x)$. Such a kind of continuous variation is called deformation in homotopy [7]. Due to Taylors theorem,$\phi(x;q)$ can be expanded in a power series of q as follows

$$\phi(x;q) = y_0(x) + \Sigma_{m=1}^{\infty} y_m(x)q^m, \tag{12}$$

where

$$y_m(x) = (1/m!)(\partial^m \phi(x;q)/\partial q^m) \mid_{q=0} . \tag{13}$$

Let the initial guess $y_0(x)$, the auxiliary linear operator L, the nonzero auxiliary parameter h and the auxiliary function $H(x)$ be properly chosen so that the power series (12) of $\phi(x;q)$ converges at $q = 1$, then we have under these assumptions the solution series

$$y(x) = \phi(x;1) = y_0(x) + \Sigma_{m=1}^{\infty} y_m(x). \tag{14}$$

From (12), we can write (9) as follows:

$$(1-q)L[\phi(x;q) - y_0(x)] = (1-q)L[\Sigma_{m=1}^{\infty} y_m(x)q^m] = qhH(x)N(\phi(x;q))$$

so:

$$L[\Sigma_{m=1}^{\infty} y_m(x)q^m] - qL[\Sigma_{m=1}^{\infty} y_m(x)q^m] = qhH(x)N(\phi(x;q)). \tag{15}$$

By differentiating (15) m times with respect to q, we obtain

$$m!L[y_m(x) - y_{m-1}(x)] = hH(x)m[\partial^{m-1} N(\phi(x;q)]/\partial q^{m-1} \mid_{q=0} .$$

Therefore,

$$L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)\Re_m(y_{m-1}(x)), \tag{16}$$

where,

$$\Re_m(y_{m-1}(x)) = (1/(m-1)!).[\partial^{m-1} N(\phi(x;q))]/\partial q^{m-1} \mid_{q=0}, \tag{17}$$

And

$$\chi_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \quad (18)$$

5 Description of HAM for Sloving 2D-FFIE-2

In this section, the homotopy analysis method is used to solve the two-dimensional fuzzy Fredholm integral equation of second kind (2D-FFIE-2).
First, consider $k(x,y,s,t) \geq 0$ on S, so the parametric form of (3) is:

$$\underline{u}(x,y,r) = \underline{f}(x,y,r) + \int_a^b \int_c^d k(x,y,s,t)\underline{u}(s,t,r)dsdt, \tag{19}$$

$$\overline{u}(x,y,r) = \overline{f}(x,y,r) + \int_a^b \int_c^d k(x,y,s,t)\overline{u}(s,t,r))dsdt, \tag{20}$$

HAM is applied for equations (19) and (20) respectively.
To explain this method for (19), we reconstitute this equation as:

$$N(\underline{u}(x,y,r)) = \underline{u}(x,y,r) - \underline{f}(x,y,r) - \int_a^b \int_c^d k(x,y,s,t)\underline{u}(s,t,r)dsdt \tag{21}$$

according to (14) and (16) we have

$$\begin{aligned} \underline{u}(x,y,r) &= \underline{u_0}(x,y,r) + \Sigma_{m=1}^{\infty}\underline{u_m}(x,y,r), \\ L[\underline{u_m}(x,y,r) - \chi_m\underline{u_{m-1}}(x,y,r)] &= hH(x)\Re_m(\underline{u_{m-1}}(x,y,r)), \end{aligned} \tag{22}$$

and define $\Re_m(\underline{u_{m-1}}(x,y,r))$ in the following formula:

$$\begin{aligned} \Re_m(\underline{u_{m-1}}(x,y,r)) &= (1/(m-1)!)[\partial^{m-1}N(\Sigma_{m=0}^{\infty}\underline{u_m}(x,y,r)q^m)]/\partial q^{m-1} \mid_{q=0} \\ &= (1/(\text{m-1})!)[\partial^{m-1}(\Sigma_{m=0}^{\infty}\underline{u_m}(x,y,r)q^m - \underline{f}(x,y,r) \\ &\quad - \textstyle\int_a^b \int_c^d k(x,y,s,t)\Sigma_{m=0}^{\infty}\underline{u_m}(s,t,r)q^m dsdt)]/\partial q^{m-1} \mid_{q=0}, \end{aligned}$$

so:

$$\begin{aligned} \Re_m(\underline{u_{m-1}}(x,y,r)) &= \underline{u_{m-1}}(x,y,r) - (1-\chi_m)\underline{f}(x,y,r) \\ &\quad - \textstyle\int_a^b \int_c^d k(x,y,s,t)\underline{u_{m-1}}(s,t,r)dsdt. \end{aligned} \tag{23}$$

We take an initial guess $\underline{u_0}(x,y,r) = \underline{f}(x,y,r)$, an auxiliary linear operator $L[r(x,y)] = r(x,y)$ and auxiliary function $H(x) = 1$. These are substituted into (22) and with substituting (18) and (23) in (22), we have following recurrence relation :

$$\begin{aligned} &\underline{u_1}(x,y,r) = -h \textstyle\int_a^b \int_c^d k(x,y,s,t)\underline{u_0}(s,t,r)dsdt \\ &\underline{u_m}(x,y,r) = (1+h)\underline{u_{m-1}}(x,y,r) - h \textstyle\int_a^b \int_c^d k(x,y,s,t)\underline{u_{m-1}}(s,t,r)dsdt) \qquad m > 1 \end{aligned} \tag{24}$$

With the same procedure, we can obtain the approximate solution of (20) as follows:

$$\begin{aligned} &\overline{u_1}(x,y,r) = -h \textstyle\int_a^b \int_c^d k(x,y,s,t)\overline{u_0}(s,t,r)dsdt \\ &\overline{u_m}(x,y,r) = (1+h)\overline{u_{m-1}}(x,y,r) - h \textstyle\int_a^b \int_c^d k(x,y,s,t)\overline{u_{m-1}}(s,t,r)dsdt) \qquad m > 1 \end{aligned} \tag{25}$$

So we obtain the approximate solution $u(x,y,r) = (\underline{u}(x,y,r), \overline{u}(x,y,r))$ for (3).

6 Numerical Result

To show the efficiency of the HAM described in the previous section, we present some examples. We use $n+1$ terms in evaluating the approximate solution $u_{approx[n]}(s,t,r,h) = \Sigma_{m=0}^{n} \underline{u_m}(s,t,r,h)$.

Example 6.1 *[12] Consider the following two-dimensional fuzzy Fredholm integral equation*

$$u(x,y) = f(x,y) + \int_0^1 \int_0^1 x^2 ysu(s,t)dsdt, \quad 0 \le x, y \le 1$$

where

$$(f(x,y))(r) = \left(x \sin(y/2)(r^2+r), x\sin(y/2)(4-r^3-r)\right), \quad 0 \le r \le 1$$

The exact solution, by using direct metod, is

$$\underline{u}(x,y,r) = x\left(\sin(y/2) - 16/21\left(\cos(1/2)-1\right)xy\right)(r^2+r),$$
$$\overline{u}(x,y,r) = x\left(\sin(y/2) - 16/21\left(\cos(1/2)-1\right)xy\right)(4-r^3-r).$$

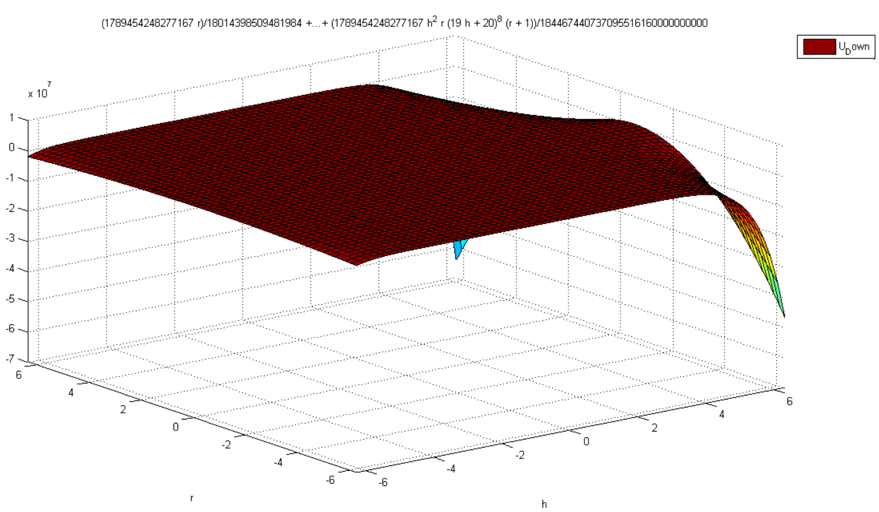

Fig.1. The h-curves of 10th-order of approximation solution given by HAM $(\underline{u}_{approx[n]}(s,t,r,h))$ for the Example 1.

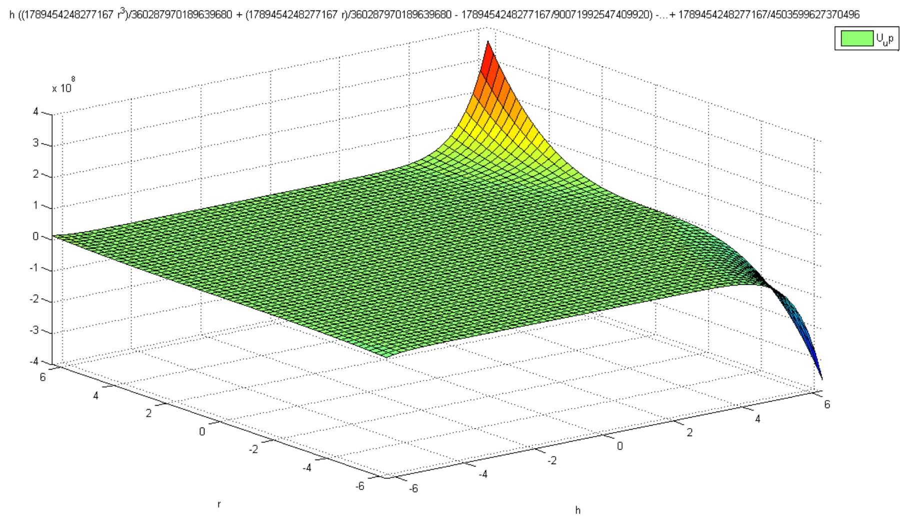

Fig.2. The h-curves of 10th-order of approximation solution given by HAM ($\overline{u}_{approx[n]}(s,t,r,h)$) for the Example 1.

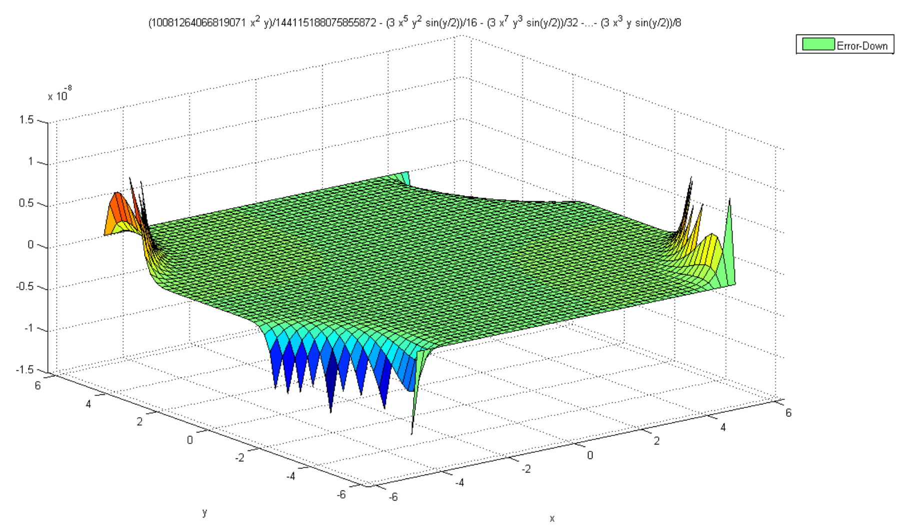

Fig. 3. Comparison between the exact solution and the approximate solution given by HPM ($\underline{u}_{approx[10]}(s,t,r,-1)$) by metric D for Example 1.

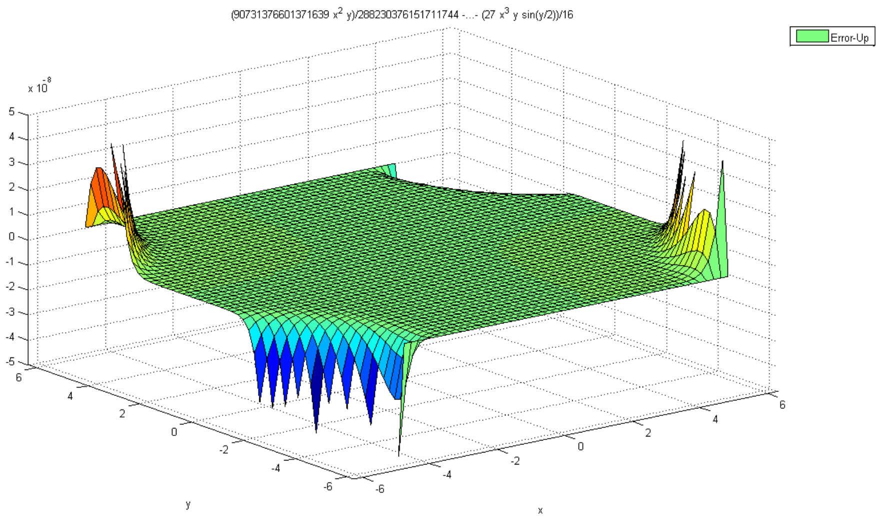

Fig. 4. Comparison between the exact solution and the approximate solution given by HPM ($\overline{u}_{approx[10]}(s,t,r,-1)$) by metric D for Example 1.

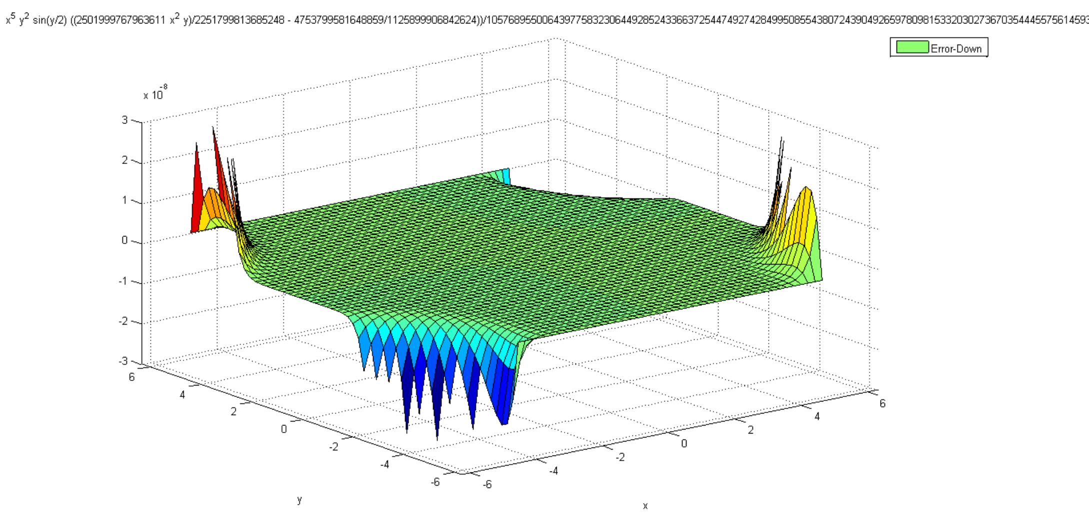

Fig. 5. Comparison between the exact solution and the approximate solution given by HAM ($\underline{u}_{approx[10]}(s,t,r,1.1)$) by metric D for Example 1.

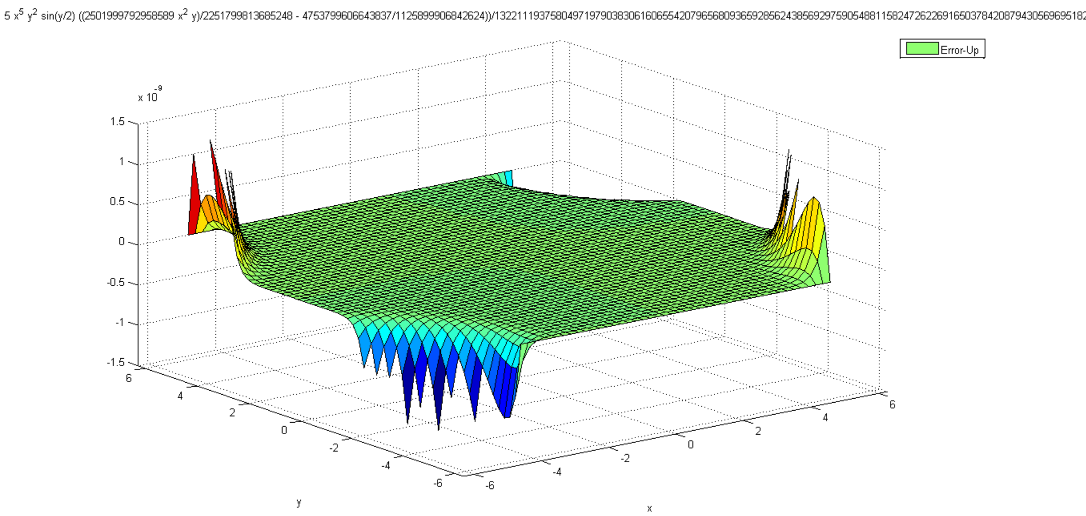

Fig. 6. Comparison between the exact solution and the approximate solution given by HAM ($\overline{u}_{approx[10]}(s,t,r,1.1)$) by metric D for Example 1.

7 Comparison and Discussion

Figs. 1 and 2 show the convergent regions of the solution series given by HAM ($\underline{u}_{approx[10]}(s,t,r,h), \overline{u}_{approx[10]}(s,t,r,h)$) for Examples 1.
Comparison between the exact solution and the approximate solution given by HAM ($\underline{u}_{approx[n]}(s,t,r,1.1), \overline{u}_{approx[10]}(s,t,r,1.1)$) with metric D for Example 1 is given by Fig5 and Fig6 respectively, and the same comparison for HPM ($\underline{u}_{approx[10]}(s,t,r,-1), \overline{u}_{approx[10]}(s,t,r,-1)$) solutions is given by Fig3 and Fig4. It is clear that we yield the exact solution by using the homotopy analysis method in interval $[0,1]$, and approximate solution given by HAM is more accurate than the approximate solutions given by HPM in interval $[0,5]$. In fact, For Example 1, For HAM solution, we have:
$\max D(\underline{u}_{approx[10]}(s,t,r,1.1), \underline{u}(s,t,r,h)) = 2.5 \times 10^{-8}$
and
$\max D(\overline{u}_{approx[10]}(s,t,r,1.1), \overline{u}(s,t,r,h)) = 1.1 \times 10^{-9}$
and for HPM solutions we obtain:
$\max D(\underline{u}_{approx[10]}(s,t,r,-1), \underline{u}(s,t,r,h)) = 1.2 \times 10^{-8}$
and
$\max D(\overline{u}_{approx[10]}(s,t,r,-1), \overline{u}(s,t,r,h)) = 3.5 \times 10^{-8}$

8 Conclusion

In this paper the homotopy analysis method is applied for solving two-dimensional fuzzy Fredholm integral equation of the second kind. In Section 5, the HAM was employed for solving two-dimensional fuzzy Fredholm integral equation of the second kind which can be extended for the other types of system of integral equations such as nonlinear system of two-dimensional fuzzy Fredholm integral equation of second kind and linear/nonlinear system of two-dimensional fuzzy Volterra-type integral equations.
To check the method, it is applied to a problem with known exact solution. The numerical results confirm the validity and the low cost of the method, and suggest it is a viable alternative to existing numercal method for solving the problem under consideration.

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