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Neighborhood Connected Domination Number of Total Graphs

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Abstract

Let $G = (V, E)$ be a connected graph. A dominating set S of a connected graph G is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ of G is connected. The neighborhood connected domination number $\gamma_{nc}(G)$ is the minimum cardinality of a ncd-set. The total graph $T(G)$ of a graph G is a graph such that the vertex set of $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ if and only if their corresponding elements are either adjacent or incident in G. In this paper we study the concept of neighborhood connected domination in total graphs.

Keywords: *Neighborhood connected domination number, Total graph.*

1 Introduction

The graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2] and Haynes et.al[3,4].

Let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

A set of edges in a graph is independent if no two edges in the set are adjacent. By a matching in a graph G, we mean an independent set of edges in G. The edge independence number $\beta_1(G)$ of a graph G is the maximum cardinality of an independent set of edges. A perfect matching is a matching with every vertex of the graph is incident to exactly one edge of the matching. The graph G^+ is obtained from the graph G by attaching a pendent edge to all the vertices of G. The total graph $T(G)$ of a graph G is a graph such that the vertex set $T(G)$ corresponds to the vertices and edges of G and two vertices are adjacent in $T(G)$ if and only if their corresponding elements are either adjacent or incident in G.

A subset S of V is called a dominating set if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. S. Arumugam and C.Sivagnanam [1] introduced the concept of neighborhood connected domination in graphs. A dominating set S of a connected graph G is called a neighborhood connected dominating set(ncd-set)if the induced subgraph $\langle N(S)\rangle$ is connected.The minimum cardinality of a ncd-set of G is called the neighborhood connected domination number of G and is denoted by $\gamma_{nc}(G)$. A. Thuraiswamy [5] studied the connected domination number and total domination number of total graphs. In this paper we study the concept of neighborhood connected domination in total graphs. we need the following theorem.

Theorem 1.1 (1). *Let G be a graph with $\Delta < n-1$. Then $\lceil \frac{n}{\Delta+1} \rceil \leq \gamma \leq \gamma_{nc} \leq n - \Delta$*

2 Main Results

Theorem 2.1. *For a non-trivial path P_n, $\gamma_{nc}(T(P_n)) = \lceil \frac{2n-1}{5} \rceil$*

Proof. Let $P_n=(v_1, v_2, ..., v_n)$ and $e_i=v_iv_{i+1}, 1 \leq i \leq n-1$. Let $u_i \in V(T(P_n))$ be the vertex corresponding to e_i. Let $S_1 = \{v_i \in V(T(P_n)) : i \equiv 2(mod 5)\}$ and $S_2 = \{u_i \in V(T(P_n)) : i \equiv 4(mod 5)\}$. If $n \equiv 0\, or\, 3(mod 5)$ then $S = S_1 \cup S_2$ is a ncd-set of $T(P_n)$ and $|S| = \lceil \frac{2n-1}{5} \rceil$. If $n \equiv 1$ or 2 or $4(mod 5)$ then $S = S_1 \cup S_2 \cup \{v_n\}$ is a ncd-set of $T(P_n)$ and $|S| = \lceil \frac{2n-1}{5} \rceil$. Hence $\gamma_{nc}(T(P_n)) \leq \lceil \frac{2n-1}{5} \rceil$. Further since $\gamma_{nc}(G) \geq \lceil \frac{n}{\Delta+1} \rceil$,it follows that $\gamma_{nc}(T(P_n)) \geq \lceil \frac{2n-1}{\Delta+1} \rceil \geq \lceil \frac{2n-1}{5} \rceil$. Thus $\gamma_{nc}(T(P_n)) = \lceil \frac{2n-1}{5} \rceil$. □

Theorem 2.2. *For a cycle C_n on n vertices, $\gamma_{nc}(T(C_n)) = \lceil \frac{2n}{5} \rceil$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $e_i = v_iv_{i+1}, 1 \leq i \leq n-1$ and $e_n = v_nv_1$. Let u_i be the vertex corresponding to e_i in $T(C_n)$. Let $S_1 = \{v_i \in V(T(C_n)) : i \equiv 1(mod 5)\}$ and $S_2 = \{u_i \in V(T(C_n)) : i \equiv 3(mod 5)\}$. Then

$S = S_1 \cup S_2$ is a ncd-set of $T(C_n)$ and $|S| = \lceil \frac{2n}{5} \rceil$. Thus $\gamma_{nc}(C_n) \leq \lceil \frac{2n}{5} \rceil$. Further $\gamma_{nc}(G) \geq \lceil \frac{n}{\Delta+1} \rceil$. Hence it follows that $\gamma_{nc}(T(C_n)) \geq \lceil \frac{2n}{5} \rceil$. Thus $\gamma_{nc}(T(C_n)) = \lceil \frac{2n}{5} \rceil$. □

Theorem 2.3. $\gamma_{nc}(T(C_n^+)) = n$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and let u_i be the pendant vertex adjacent to $v_i, 1 \leq i \leq n$. Let x_i be the vertex of $T(C_n^+)$ corresponding to the edge $u_i v_i$ in C_n^+. Then $S = \{v_1, v_2, \ldots, v_n\}$ is a ncd-set of $T(C_n^+)$. Thus $\gamma_{nc}(T(C_n^+)) \leq n$. Further, any dominating set of $T(C_n^+)$ must contains at least one of v_i, u_i, x_i for all i and hence $|S| \geq n$, so that $\gamma_{nc}(T(C_n^+)) \geq n$. Thus $\gamma_{nc}(T(C_n^+)) = n$. □

Theorem 2.4. $\gamma_{nc}(T(K_n)) = \lceil \frac{n}{2} \rceil$

Proof. Suppose n is odd. Let $v \in V(K_n)$ and let $S = S_1 \cup \{v\}$ where S_1 is a set of vertices in $T(K_n)$ corresponding to a perfect matching of $K_n - v$. It is clear that S is a dominating set of $T(K_n)$. Let X be the set of vertices of $T(K_n)$ corresponding to the vertices of K_n and Y be the set of vertices of $T(K_n)$ corresponding to the edges of K_n. Then $N(S) = [V(K_n) \cup Y] - S = V(T(K_n)) - S$. Since every vertex in $Y - S_1$ is adjacent to a vertex in $X - \{v\}$, $\langle N(S) \rangle$is connected.Hence S is a ncd-set of $T(K_n)$ and $|S| = 1 + \frac{n-1}{2} = \lceil \frac{n}{2} \rceil$.

Suppose n is even. Let S_2 be a set of vertices in $T(K_n)$ corresponding to a perfect matching F of K_n. It is clear that S_2 is a dominating set of $T(K_n)$. Since $K_n - F$ is a connected graph and every edge of $K_n - F$ incident with a vertex of $K_n, \langle N(S_2) \rangle$ is connected. Hence S_2 is a ncd-set of $T(K_n)$. Thus $\gamma_{nc}(T(K_n)) \leq \lceil \frac{n}{2} \rceil$.

Also, $|V(T(K_n)| = \frac{n(n+1)}{2}$ and $\Delta(T(K_n)) = 2(n-1)$. Further $\gamma_{nc}(G) \geq \lceil \frac{n}{\Delta+1} \rceil$. Hence it follows that $\gamma_{nc}(T(K_n)) \geq \lceil \frac{n}{2} \rceil$. Thus we have $\gamma_{nc}(T(K_n)) = \lceil \frac{n}{2} \rceil$. □

Theorem 2.5. $\gamma_{nc}(T(K_{r,s})) = min\{r, s\}$

Proof. Let (V_1, V_2) be the bipartition of $K_{r,s}$ with $V_1 = \{v_1, v_2, \ldots, v_r\}$ and $V_2 = \{u_1, u_2, \ldots, u_s\}$. Let us assume $r \leq s$. Let $S = V_1$. It is clear that S is a dominating set of $T(K_{r,s})$ and $\langle N(S) \rangle = \langle T(K_{r,s}) - V_1 \rangle$. Let X be the set of vertices in $T(K_{r,s})$ corresponding to the edges in $K_{r,s}$.

Claim. $\langle N(S) \rangle$ is connected

Let $x, y \in \langle N(S) \rangle$. If $x, y \in V_2$ then x and y are not adjacent vertices. Let $x = u_i$ and $y = u_j$ for some i and j. Then $u_i v_1, u_j v_1 \in E(K_{r,s})$. Let x_i and x_j be the vertices in $T(K_{r,s})$ corresponding to $u_i v_1$ and $u_j v_1$ respectively.Hence

(x, x_i, x_j, y) is a $x-y$ path in $\langle N(S)\rangle$. Suppose $x \in V_2$ and $y \in X$. Let v_iu_k be the edge in $K_{r,s}$ corresponding to y. If $u_k = x$ then nothing to prove. Let $x = u_j$ for some $j \neq k$. Then u_j is adjacent to v_i in $K_{r,s}$. Let $x_i \in X$ corresponding the edge u_jv_i. Then (x, x_i, y) be the $x-y$ path in $\langle N(S)\rangle$. Suppose $x, y \in X$. Let u_iv_j and u_kv_l be the edges in $K_{r,s}$ corresponding to the vertices x, y in $T(K_{r,s})$ respectively and let $x_i$be the vertex in $T(K_{r,s})$ corresponding to the edge u_iv_l in $K_{r,s}$. Hence (x, x_i, y) is a $x-y$ path in $\langle N(S)\rangle$. Thus $\langle N(S)\rangle$ is connected. Hence $\gamma_{nc}(T(K_{r,s})) \leq r = min\{r, s\}$.

Let S be any minimum ncd-set of $T(K_{r,s})$. Since every vertex dominates maximum of s vertices of $T(K_{r,s})$ corresponding to the edges in $K_{r,s}$ and $K_{r,s}$ contains rs number of edges, S should contains at least r vertices. Hence $|S| \geq r$. Then $\gamma_{nc}(T(K_{r,s})) \geq r$. Thus $\gamma_{nc}(T(K_{r,s})) = r = min\{r, s\}$. □

Remark 2.6. *If G is a star graph then $\gamma_{nc}(T(G)) = 1$.*

Theorem 2.7. *For any graph $G, \gamma_{nc}(T(G)) \leq n - \lceil \frac{\Delta}{2} \rceil$.*

Proof. Let $v \in V(G)$ with $degv = \Delta$. Let M be a maximum matching of the induced subgraph $\langle N(v)\rangle$ of G so that $|M| \leq \lfloor \frac{\Delta}{2} \rfloor$. Let $M = \{u_1v_1, u_2v_2, \ldots, u_kv_k\}$. Then $S = (V - N(v)) \cup \{u_1, u_2, \ldots, u_k\}$ is a dominating set of $T(G)$. Also $N(S) = V(T(G)) - X$ where X is the set of isolates in $\langle S\rangle$.

Claim. $\langle N(S)\rangle$ is connected.

Let $x, y \in N(S)$. If $x, y \notin N[v]$ then $x, y \in S - X$ and we can find the paths $x - v_i, v_i - v - v_j$ and $v_j - y$ where $v_i, v_j \in N[v]$ in $\langle N(S)\rangle$. If $x \in N[v]$ and $y \notin N[v]$ then $y \in S - X$ and there is a $y - v_j, v_j \in N(v)$ path in $\langle N(S)\rangle$. Thus there is a $x - v - v_j - y$ path in $\langle N(S)\rangle$. If $x, y \in N[v]$ then $x - v - y$ is a $x - y$ path in $\langle N(S)\rangle$. This gives $\langle N(S)\rangle$ is connected. Then S is a ncd-set of $T(G)$ and $|S| \leq n - \Delta + \lfloor \frac{\Delta}{2} \rfloor = n - \lceil \frac{\Delta}{2} \rfloor$. Hence $\gamma_{nc}(T(G)) \leq n - \lceil \frac{\Delta}{2} \rceil$. □

Remark 2.8. $\gamma_{nc}(T(K_3)) = 2 = n - \lceil \frac{\Delta}{2} \rceil$ *and hence the bound given in theorem 2.7 is sharp.*

Theorem 2.9. *If G has a perfect matching then $\gamma_{nc}(T(G)) \leq \beta_1(G)$.*

Proof. Let M be a perfect matching of the graph G. Let X be the set of vertices in $T(G)$ corresponding to the edges of G and let S be the set of vertices in $T(G)$ corresponding to the edges in M. It is clear that S is a dominating set of $T(G)$. Since $V(G) \subseteq N(S), G$ is connected and every vertex in $X - S$ is adjacent to two vertices in $V(T(G)) - X, \langle N(S)\rangle$ is connected.Hence S is a ncd-set of $T(G)$. Thus $\gamma_{nc}(T(G)) \leq |S| \leq \beta_1(G)$. □

Theorem 2.10. *Let G be a graph with $G - v$ has a perfect matching, for some $v \in V(G)$. Then $\gamma_{nc}(T(G)) \leq \beta_1(G) + 1$.*

Proof. Let M be a perfect matching of the graph $G - v$. Let S be the set of vertices in $T(G)$ corresponding to the edges in M. Then $S_1 = S \cup \{v\}$ is a dominating set of $T(G)$.

Claim. $\langle N(S_1) \rangle$ is connected

Suppose $G - v$ is disconnected. Let $G_1, G_2, \ldots, G_k$ be the components of $G - v$. Let $v_i \in G_i, 1 \leq i \leq k$ be the vertex adjacent to v in G. Let x_i be the vertex in $T(G)$ corresponding to the edge $vv_i \in E(G)$. Then the subgraph induced by the vertex set $\{x_1, x_2, \ldots, x_k\}$ is a complete graph. Then the graph $\langle N(S_1) \rangle$ is isomorphic to a graph obtained from $T(G_1) \cup T(G_2) \cup \cdots \cup T(G_k) \cup K_k$ by joining a vertex x_i to the vertex v_i and the vertices corresponding to the edges incident with v_i and removing the vertex in S. Hence $\langle N(S_1) \rangle$ is connected. If $G - v$ is connected then $\langle N(S_1) \rangle = T(G) - S_1$ is connected.Thus $\gamma_{nc}(T(G)) \leq \beta_1(G) + 1$. □

Theorem 2.11. *Let G be a graph with G and $G - v$ has no perfect matching. Then $\gamma_{nc}(T(G)) \leq \beta_1(G)$.*

Proof. Let M be a maximum matching in G. Let D be the set of vertices not covered by M. Let $A = N(D) \cap (V - D)$. Let X be the set of vertices of $T(G)$ corresponding to the edges of M. Let Y be the set of edges in M incident with a vertex in A and Z be the set of vertices in $T(G)$ corresponding to the edges in Y. Let $S = (X - Z) \cup A$. It is clear that S is a dominating set of $T(G)$. Since G and $G - v$ contain no perfect matching $|A| \leq |Z|$ and hence $|S| \leq |X| = \beta_1(G)$. Also $\langle N(S) \rangle = T(G) - A$. Since every vertices corresponding to the edges incident to a vertex in A are adjacent in $T(G)$ gives $\langle N(S) \rangle$ is connected. Hence $\gamma_{nc}(T(G)) \leq \beta_1(G)$. □

Corollary 2.12. *If G is any connected graph then $\gamma_{nc}(T(G)) \leq \beta_1(G) + 1$.*

Corollary 2.13. *If G is a connected graph of order n, then $\gamma_{nc}(T(G)) \leq \lceil \frac{n}{2} \rceil$.*

Proof. If n is odd, then $\beta_1(G) \leq \frac{n-1}{2}$ then $\gamma_{nc}(T(G)) \leq \beta_1(G) + 1 \leq \frac{n-1}{2} + 1 = \lceil \frac{n}{2} \rceil$.Suppose n is even. If $\beta_1(G) = \frac{n}{2}$, then G has a perfect matching, and so $\gamma_{nc}(T(G)) \leq \beta_1(G) = \frac{n}{2}$. If $\beta_1(G) \leq \frac{n}{2} - 1$, then $\gamma_{nc}(T(G)) \leq \beta_1(G) + 1 \leq \frac{n}{2}$. Hence $\gamma_{nc}(T(G)) \leq \lceil \frac{n}{2} \rceil$. □

Remark 2.14. *Since $\gamma_{nc}(T(K_n)) = \lceil \frac{n}{2} \rceil$ the bound given in corollary 2.13 is sharp.*

Problem 2.15. *Characterize the classes of graphs for which $\gamma_{nc}(T(G)) = \lceil \frac{n}{2} \rceil$.*

3 Conclusion

In this paper we computed the exact value of the neighborhood connected domination number for total graphs of paths, cycles, complete graphs, complete bipartite graphs and some special graphs. Also we found some upper bounds for neighborhood connected domination number for total graph of a graph.

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