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Semiderivations and Commutativity In Semiprime Rings[1]

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Abstract

Let R be a semiprime ring. An additive mapping $f : R \to R$ is called a semiderivation if there exists a function $g : R \to R$ such that $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$ for all $x, y \in R$. In the present paper we investigate commutativity of R satisfying any one of the properties (i) $[f(x), f(y)] = 0$, (ii) $[f(x), f(y)] = [x, y]$, (iii) $[f(x), d(y)] = [x, y]$, d is a derivation on R, or (iv) $f([x, y]) = \pm[x, y]$, for all x, y in some appropriate subset of R. Also we extend two results of Bell and Martindale from prime rings to semiprime rings.

Keywords: *prime ring, semiprime ring, essential ideal, derivation, semiderivation, commuting mapping, strong commutativity-preserving mapping.*

1 Introduction

Throughout, R will be an associative ring. R is said to be 2-torsion-free, if $2x = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ for $x, y \in R$ will be denoted by $[x, y]$. We shall use basic commutator identities $[x, yz] = [x, y]z + y[x, z]$ and $[xy, z] = [x, z]y + x[y, z]$, for $x, y, z \in R$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$ for every $a, b \in R$, and

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is semiprime if $aRa = (0)$ implies $a = 0$, for every $a \in R$. An ideal U of R is essential if for every nonzero ideal K of R we have $U \cap K \neq (0)$. If R is a ring with center Z, a mapping f from R to R is called centralizing on $S \subseteq R$ if $[x, f(x)] \in Z$ for all $x \in S$; in the special case where $[x, f(x)] = 0$ for all $x \in S$, the mapping f is said to be commuting on S. A mapping $f : R \to R$ is called strong commutativity-preserving (scp) on $S \subseteq R$ if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. A derivation $d : R \to R$ is an additive map which satisfies $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$.

The present paper has been motivated by the works of Chang [7], Daif [9], Bell and Daif [3], Daif and Bell [8], and Bell and Martindale [5]. Bergen [6] has introduced the following notion. An additive mapping f of a ring R into itself is called a semiderivation if there exists a function $g : R \to R$ such that $f(xy) = f(x)g(y) + xf(y) = f(x)y + g(x)f(y)$ and $f(g(x)) = g(f(x))$ for all $x, y \in R$. For $g = 1$ a semiderivation is of course a derivation. The other main motivating examples are of the form $f(x) = x - g(x)$ where g is any ring endomorphism of R. Then f is a semiderivation of R with associated map g which is not a derivation. In [11], Herstein has shown that if R is a prime ring admitting a nonzero derivation d such that $[d(x), d(y)] = 0$ for all $x, y \in R$, then R is commutative whenever $charR \neq 2$, and if $charR = 2$, then either R is commutative or is an order in a simple algebra which is 4-dimensional over its center. In [7], Chang has given an extension of the above mentioned result of Herstein in the following way. Let $f \neq 0$ be a semiderivation of a prime ring R associated with an epimorphism g of R such that $[f(R), f(R)] = \{0\}$. Then, if char $(R) \neq 2$, R is a commutative, and if char $(R) = 2$, R is commutative or is an order in a simple algebra which is $4-$dimensional over its center. In [9], Daif has generalized the previously mentioned result of Herstein in the following way. Let R be a two-torsion-free semiprime ring and U a nonzero ideal of R. If R admits a derivation d which is nonzero on U and $[d(x), d(y)] = 0$ for all $x, y \in U$, then R contains a nonzero central ideal. In [8], Daif and Bell have proved that a semiprime ring R is commutative if it admits a derivation d for which either $d([x, y]) = [y, x]$ for all $x, y \in R$ or $d([x, y]) = [x, y]$ for all $x, y \in R$. In [3], Bell and Daif have shown that if a semiprime ring R admits a strong-commutativity preserving derivation on a nonzero right ideal U of R, then $U \subseteq Z$, the center of R. In [5], Bell and Martindale have proved the following three results.

(i) Let $f \neq 0$ be a semiderivation of a prime ring R of characteristic not 2 with associated endomorphism g of R and $U \neq 0$ be an ideal of R. Suppose that $a \in R$ such that $af(U) = 0$. Then $a = 0$.

(ii) Let f be a semiderivation of a prime ring R of characteristic not 2 with associated endomorphism g of R. If there exists a nonzero ideal U of R for which $U \cap g(R) = 0$, then there exists $\lambda \in C$ (the extended centroid of R) such that $f(x) = \lambda(x - g(x))$ for all $x \in R$.

(iii) Let f be a semiderivation of a prime ring R of characteristic not 2 with associated endomorphism g of R. If g is not one-one and $V \neq 0$ is an ideal of R contained in ker g, then $f(V)$ is a nonzero ideal of R, and there exists $\lambda \in C$ such that $f(x) = \lambda(x - g(x))$ for all $x \in R$.

In [1], Ali and Huang have proved the following theorem. Let R be a 2−torsion free semiprime ring and I a nonzero ideal of R. Let d be a derivation of R. If one of the following conditions holds:

(i) $[d(x), d(y)] = [x, y]$ for all $x, y \in I$,

(ii) $[d(x), d(y)] = -[x, y]$ for all $x, y \in I$,

(iii) for all $x, y \in I$, either $[d(x), d(y)] = [x, y]$ or $[d(x), d(y)] = -[x, y]$,

then d is commuting on I. Further, if $d(I) \neq 0$, then R has a nonzero central ideal.

In [10], De Filippis, Mamouni and Oukhtite have showed the following result. Let R be a prime ring of characteristic not 2 and I a nonzero ideal of R . If R admits a nonzero semiderivation f with associated function g such that $f([x, y]) = [x, y]$ for all $x, y \in I$, then one of the following holds:

(1) R is commutative;

(2) $f(x) = x - g(x)$ for all $x \in R$, with $g([R, R]) = 0$;

(3) $f(x) = x$, for all $x \in I$ and $g(I) = 0$.

Our aim in this work is to investigate the commutativity of semiprime rings admitting semiderivations. In the first section we extend the above mentioned result of Chang [7, Theorem 2] for prime rings to semiprime rings, extend two results of Bell and Martindale ([5, Lemma 4], [5, Lemma 5]) for prime rings to semiprime rings, and give a counter example to [5, Lemma 2] in the semiprime ring case. In the second section we study commutativity for a semiprime ring R admitting a semiderivation f associated with an epimorphism g of R which satisfies $[f(x), f(y)] = [x, y]$ for all x, y belonging to an ideal of R, or satisfies $f([x, y]) = \pm[x, y]$ for all $x, y \in R$, or admits an additive map f and a derivation d which satisfy $[f(x), d(y)] = [x, y]$ for all x, y belonging to an ideal of R.

In order to prove our aims we need the following results:

Theorem 1.1. [2, *Theorem* 2.3.2]. *Let R be a semiprime ring, $Q = Q_{mr}(R)$, the maximal right ring of quotients of R, ${}_RU_R \subseteq_R Q_R$ a subbimodule of Q and $f :_R U_R \rightarrow_R Q_R$ a homomorphism of bimodules. Then there exists an element $\lambda \in C$ (the extended centroid of R) such that $f(u) = \lambda u$ for all $u \in U$.*

Lemma 1.2. [8, *Lemma*1]. *Let R be a semiprime ring and I a nonzero ideal of R. If x in R centralizes the set $[I, I]$, then x centralizes I.*

Lemma 1.3. [3, *Lemma* 1]. *If R is a semiprime ring, the center of a nonzero one-sided ideal is contained in the center of R; in particular, any commutative one-sided ideal is contained in the center of R.*

Remark 1.4. [2, *Remark* 2.1.4]. *If U is an essential two-sided ideal of a semiprime ring R, then $l(U) = r(U) = (0)$.*

2 Semiderivations on Semiprime Rings

In this section we begin with a theorem that extends Chang's theorem ([7, Theorem 2]) from prime rings to semiprime rings, and also generalizes Daif's theorem ([9, Theorem 2.1]) for derivations to semiderivations. To achieve this goal we modify Theorem 3 of [4] from the case of derivations to the case of semiderivations. Also we extend two results of Bell and Martindale ([5, Lemma 4], [5, Lemma 5]) on derivations to semiderivations, and give a counter example to [5, Lemma 2] in the semiprime ring case.

Lemma 2.1. *Let R be a semiprime ring. If R admits a nonzero semiderivation f with associated surjective map g of R which is commuting on R, then R contains a nonzero central ideal.*

Proof. We have for all $x \in R$ that $[x, f(x)] = 0$. Replacing x by $u+v$, we get

$$[u, f(v)] + [v, f(u)] = 0 \text{ for all } u, v \in R. \tag{2.1}$$

Replacing u by x and v by yx, and using our hypothesis and (2.1), we get

$$[x, g(y)]f(x) = 0 \text{ for all } x, y \in R. \tag{2.2}$$

Since g is onto we have

$$[x, y]f(x) = 0 \text{ for all } x, y \in R. \tag{2.3}$$

Replacing y by wy and using (2.3), we get $[x, w]yf(x) = 0$, which implies that

$$[x, w]Rf(x) = \{0\} \text{ for all } x, w \in R. \tag{2.4}$$

Since R is semiprime, consider the set $\{P_\alpha\}$ of prime ideals of R such that $\cap P_\alpha = \{0\}$. Then for each P_α either

(a)

$$[x, w] \in P_\alpha \text{ for all } x, w \in R, \tag{2.5}$$

or

(b)

$$f(x) \in P_\alpha \text{ for all } x \in R. \tag{2.6}$$

Call P_α a type-one prime if it satisfies (a), and call P_α a type-two prime if it satisfies (b). Let P_1 and P_2 be, respectively, the intersections of all type-one and type-two primes. Note that $P_1 \cap P_2 = \{0\}$.

We now investigate a typical type-two prime $P = P_\alpha$. From (b), we have

$$Rf(R) \subseteq P \tag{2.7}$$

Now consider the left ideal $V = Rf(R)$; we shall show that V is commutative, hence a two-sided central ideal. A typical element of V is a sum of elements of the form $rf(s)$, where $r, s \in R$. Thus we need only show that commutators of the form $[r_1 f(s_1), r_2 f(s_2)]$ are all trivial, clearly this commutator is in P_1 by (a) and in P_2 by (2.7), hence belongs to $P_1 \cap P_2 = \{0\}$.

Assume that $V = \{0\}$ in which case $Rf(R) = \{0\}$, hence $f(R)Rf(R) = \{0\}$, since R is semiprime we have $f(R) = \{0\}$ which is a contradiction. Hence $V \neq \{0\}$. By Lemma 1.3, R contains a nonzero central ideal. □

Now, we are ready to prove the first theorem of this section.

Theorem 2.2. *If R is a two torsion free semiprime ring and f is a nonzero semiderivation of R associated with an epimorphism g of R such that $[f(R), f(R)] = \{0\}$, then R contains a nonzero central ideal.*

Proof. We have $[f(x), f(y)] = 0$ for all $x, y \in R$, replacing y by $yf(z)$, then yields

$$[f(x), f(y)]f(z) + f(y)[f(x), f(z)] + g(y)[f(x), f^2(z)] + [f(x), g(y)]f^2(z) = 0 \text{ for all } x, y, z \in R. \tag{2.8}$$

Using our hypothesis, then $[f(x), g(y)]f^2(z) = 0$ for all $x, y, z \in R$. Since g is onto, we have

$$[f(x), y]f^2(z) = 0 \text{ for all } x, y, z \in R. \tag{2.9}$$

Replacing y by yw and using (2.9), we get

$$[f(x), y]Rf^2(z) = \{0\} \text{ for all } x, y, z \in R. \tag{2.10}$$

Consider the set of prime ideals P_α of R such that $\cap P_\alpha = \{0\}$. For each P_α, from (2.10) we either have

(a) $[f(x), y] \in P_\alpha$ for all $x, y \in R$,

or

(b) $f^2(R) \subseteq P_\alpha$.

Call P_α an (a)-prime ideal or a (b)-prime according to which of these conditions is satisfied.

Now consider a (b)-prime ideal P_α. Since $f^2(xy) = f^2(x)g^2(y) + f(x)f(g(y)) + f(x)f(g(y)) + xf^2(y)$, then $2f(x)f(g(y)) \in P_\alpha$, and since g is onto we get

$$2f(x)f(y) \in P_\alpha, \text{ for all } x, y \in R. \tag{2.11}$$

Now replacing y by zy, we get $2f(x)f(z)g(y) + 2f(x)zf(y) \in P_\alpha$, which implies

$$2f(x)zf(y) \in P_\alpha, \text{ for all } x, y, z \in R. \tag{2.12}$$

Since P_α is prime, we either have $2f(x) \in P_\alpha$ for all $x \in R$ or $f(y) \in P_\alpha$ for all $y \in R$. In either case, we have $2[f(x), y] \in P_\alpha$ for all (b)-prime P_α. Also from (a), $2[f(x), y] \in P_\alpha$ for all (a)-prime P_α. So $2[f(x), y] \in \cap P_\alpha = \{0\}$. Since R is two torsion free, then $[f(x), y] = 0$ for all $x, y \in R$, in particular $[f(x), x] = 0$ for all $x \in R$. By Lemma 2.1, R contains a nonzero central ideal. □

Lemma 2.3. [*see* 5, *Lemma* 1] *Let R be a semiprime ring. If $f \neq 0$ is a semiderivation on R associated with a function g of R, and U is an essential ideal of R, then $f \neq 0$ on U.*

Proof. Suppose $f(U) = 0$. Then for $u \in U, x \in R$ we have $0 = f(ux) = f(u)g(x) + uf(x) = uf(x)$, which implies $0 = Uf(x)$. From Remark 1.4, we have $f(x) = 0$, which is a contradiction. □

Theorem 2.4. [*see* 5, *Lemma* 4] *Let R be a semiprime ring, and f be a semiderivation on R associated with an endomorphism g of R. If there exists a nonzero essential ideal U of R for which $U \cap g(R) = 0$, then there exists $\lambda \in C$ (the extended centroid of R) such that $f(x) = \lambda(x - g(x))$ for all $x \in R$.*

Proof. We let W be the ideal $\sum U(x - g(x))U$ and note that $W \neq 0$ (otherwise g would be the identity mapping, contradicting that $U \cap g(R) = 0$). We define a mapping $\phi : W \to R$ according to the rule $\sum u_i(x_i - g(x_i))v_i \to u_i f(x_i)v_i$ where $u_i, v_i \in U$ and $x_i \in R$. Of course our main problem is to prove that ϕ is well-defined, consequently ϕ is an $(R, R)-$ bimodule map of W into R. Suppose that

$$\sum u_i(x_i - g(x_i))v_i = 0. \tag{2.13}$$

We attempt to show that $\phi(\sum u_i(x_i - g(x_i))v_i) = 0$, i.e., $u_i f(x_i)v_i = 0$. Applying f to 2.13, we see that $0 = f(\sum u_i(x_i - g(x_i))v_i)$
$= \sum[u_i f(x_i v_i) + f(u_i)g(x_i v_i) - f(u_i g(x_i))g(v_i) - u_i g(x_i)f(v_i)]$
$= \sum[u_i f(x_i)v_i + u_i g(x_i)f(v_i) + f(u_i)g(x_i)g(v_i)$
$- f(u_i)g(x_i)g(v_i) - g(u_i)f(g(x_i))g(v_i) - u_i g(x_i)f(v_i)]$
$= \sum[u_i f(x_i)v_i - g(u_i)f(g(x_i))g(v_i)]$
$= \sum u_i f(x_i)v_i - g(\sum u_i f(x_i)v_i)$. Therefore $\sum u_i f(x_i)v_i = g(\sum u_i f(x_i)v_i) \in U \cap g(R) = 0$, which implies $\sum u_i f(x_i)v_i = 0$, then ϕ is well-defined. Since ϕ is an $(R, R)-$bimodule map of W into R, from Theorem 1.1, there exists

$\lambda \in C$ (the extended centroid of R) such that $\lambda w = \phi(w)$ for all $w \in W$. Now, regarding R as a subring of the central closure RC, we have for all $u, v \in U$ and $x \in R$ that $u\lambda(x - g(x))v = \lambda(u(x - g(x))v) = \phi(u(x - g(x))v) = uf(x)v$, which implies $u[\lambda(x - g(x)) - f(x)]v = 0$ for all $u, v \in U, x \in R$, i.e., $U[\lambda(x - g(x)) - f(x)]v = 0$ for all $v \in U, x \in R$. From Remark 1.4, we have $[\lambda(x - g(x)) - f(x)]v = 0$ for all $v \in U, x \in R$, i.e., $[\lambda(x - g(x)) - f(x)]U = 0$ for all $x \in R$. From Remark 1.4 we have $\lambda(x - g(x)) - f(x) = 0$, which implies $f(x) = \lambda(x - g(x)), \lambda \in C$. □

Theorem 2.5. [*see* 5, *Lemma* 5]*Let R be a semiprime ring, and $f \neq 0$ be a semiderivation of R associated with an endomorphism g of R. If g is not one-one and V is an essential ideal of R contained in $ker g$, then*

(*a*) *$f(V)$ is a nonzero ideal of R, and*

(*b*) *there exists $\lambda \in C$ such that $f(x) = \lambda(x - g(x))$ for all $x \in R$.*

Proof. (a) For $v \in V$ and $r \in R$, we see immediately from $f(vr) = f(v)r + g(v)f(r) = f(v)r$ and $f(rv) = rf(v) + f(r)g(v) = rf(v)$ that $f(V)$ is an ideal of R. Furthermore $f(V) \neq 0$ in view of Lemma 2.3, and so (a) is proved.

(b) The argument establishing (a) also shows that f is an $(R, R)-$bimodule map of V into R. From Theorem 1.1, there exists $\lambda \in C$ such that $\lambda v = f(v)$ for all $v \in V$. For $v \in V$ and $r \in R$ we then see that $\lambda vr = f(vr) = vf(r) + f(v)g(r) = vf(r) + \lambda vg(r)$. In other words, $v(f(r) + \lambda g(r) - \lambda r) = 0$, which implies $V(f(r) + \lambda g(r) - \lambda r) = 0$, and from Remark 1.4, we get $f(r) + \lambda g(r) - \lambda r = 0$, which yields $f(r) = \lambda(r - g(r))$ for all $r \in R$. □

In the next remark we give a counter example to [5, Lemma 2] when R is semiprime.

Remark 2.6. *We notice that* [5, *Lemma* 2] *is not true in the case when R is semiprime. Let $R = R_1 \bigoplus R_2$ where R_1 and R_2 are prime rings, R is a semiprime ring. Let $\alpha : R_1 \to R_2$ be an additive map and $\beta : R_2 \to R_2$ be a nonzero left and right R_2-module map which is not a derivation. Define $f : R \to R$ such that $f((r_1, r_2)) = (0, \beta(r_2))$ and $g : R \to R$ such that $g((r_1, r_2)) = (\alpha(r_1), 0)$, $r_1 \in R_1, r_2 \in R_2$. Then f is a semiderivation on R. Consider the subset $U = \{(0, r_2), r_2 \in R_2\}$, then U is an ideal of R. Let $a = (a_1, 0) \neq 0$ be an element of R, we see that $af(U) = 0$ but neither a nor $f(U)$ is zero.*

3 Commutativity Results for Semiprime Rings with Derivations and Semiderivations

In this section, we study commutativity for a semiprime ring R admitting a semiderivation f associated with an epimorphism g of R which satisfies

$[f(x), f(y)] = [x, y]$ for all x, y belonging to an ideal of R, or satisfies $f([x, y]) = \pm[x, y]$ for all $x, y \in R$, or admits an additive map f and a derivation d which satisfy $[f(x), d(y)] = [x, y]$ for all x, y belonging to an ideal of R. We generalize [3, Theorem 1] of Bell and Daif and [8, Theorem 2] of Daif and Bell from the case of derivations to the case of semiderivations.

Theorem 3.1. *Let R be a semiprime ring admitting a semiderivation f associated with an epimorphism g of R. Suppose that U is a nonzero ideal of R such that f is scp on U and $g(U) = U$. Then $U \subseteq Z$.*

Note that: The condition $g(U) = U$ may be sead as U is a $g-$ideal.
Proof. For $x, y \in U$, we have $[x, xy] = [f(x), f(xy)]$, which yields

$$f(x)[f(x), g(y)] + [f(x), x]f(y) = 0 \text{ for all } x, y \in U. \tag{3.1}$$

Replacing y by $yr, r \in R$, gives

$$f(x)[f(x), g(y)]g(r)+f(x)g(y)[f(x), g(r)] + [f(x), x]f(y)g(r) + [f(x), x]yf(r) = 0 \text{ for all } x, y \in U, r \in R. \tag{3.2}$$

Comparing with (3.1) yields

$$f(x)g(y)[f(x), g(r)] + [f(x), x]yf(r) = 0 \text{ for all } x, y \in U, r \in R. \tag{3.3}$$

Since $g(U) = U$, letting $x = g(x)$, we see that $f(g(x))g(y)[f(g(x)), g(r)] + [f(g(x)), g(x)]yf(r) = 0$ for all $x, y \in U, r \in R$. Letting $r = f(x)$, we see that

$$[f(g(x)), g(x)]yf^2(x) = 0 \text{ for all } x, y \in U. \tag{3.4}$$

Therefore (3.4) implies that

$$[f(g(x)), g(x)]URf^2(x) = \{0\} \text{ for all } x \in U. \tag{3.5}$$

Since R is semiprime, it must contain a family $\{P_\alpha | \alpha \in \wedge\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of these and $x \in U$, (3.5) shows that $f^2(x) \in P$ or $[f(g(x)), g(x)]U \subseteq P$. For a fixed P, the sets of $x \in U$ for which these two conditions hold are additive subgroups of U whose union is U; therefore

$$f^2(U) \subseteq P \textit{ or } [f(g(x)), g(x)]U \subseteq P \text{ for all } x \in U. \tag{3.6}$$

Suppose that $f^2(U) \subseteq P$, then for each $y \in U$ we get $[x, yf(x)] = [f(x), f(yf(x))]$, expanding this equation to $y[x, f(x)] = [f(x), g(y)]f^2(x)+g(y)[f(x),$

$f^2(x)]$ implies $y[x, f(x)] \in P$, then so $UR[x, f(x)] \subseteq P$. By the primeness of P we reach to $U \subseteq P$ or $[x, f(x)] \in P$ for all $x \in U$. Either of these cases implies

$$[x, f(x)]U \subseteq P \text{ for all } x \in U. \tag{3.7}$$

From (3.6) now suppose that $[f(g(x)), g(x)]U \subseteq P$ for all $x \in U$, since $g(U) = U$ we get

$$[f(x), x]U \subseteq P \text{ for all } x \in U. \tag{3.8}$$

From (3.7) and (3.8) we have $[x, f(x)]U = \{0\}$ and from (3.3) we have

$f(x)g(y)[f(x), g(r)] = 0$ for all $x, y \in U, r \in R$. Since g is onto, $f(x)g(y)[f(x), r] = 0$. Moreover, since $g(U) = U$ we have $f(x)y[f(x), r] = 0$, which implies

$$f(x)UR[f(x), r] = \{0\} \text{ for all } x \in U, r \in R. \tag{3.9}$$

Since R is semiprime, it must contain a family $\{P_\alpha | \alpha \in \wedge\}$ of prime ideals such that $\cap P_\alpha = \{0\}$. If P is a typical member of these and $x \in U$, (3.9) shows that $f(x)U \subseteq P$ for all $x \in U$ or $[f(x), r] \in P$ for all $x \in U, r \in R$. For a fixed P, the sets of $x \in U$ for which these two conditions hold are additive subgroups of U whose union is U; therefore

$$f(U)U \subseteq P \textit{ or } [f(U), R] \subseteq P. \tag{3.10}$$

Suppose that $f(U)U \subseteq P$, then $f(U)RU \subseteq P$, that is, $f(U) \subseteq P$ or $U \subseteq P$. In either event $[f(U), f(U)] \subseteq P$. Now (3.10) yields $[f(U), f(U)] = \{0\}$, then [U,U]={0}, U is commutative, by Lemma 1.3, $U \subseteq Z$. □

The following two corollaries are immediate from the previous theorem.

Corollary 3.2. *Let R be a semiprime ring. If R admits a semiderivation f which is scp on R associated with an epimorphism g of R, then R is commutative.*

Corollary 3.3. *Let R be a prime ring, U a nonzero ideal, and R admit a semiderivation f which is scp on U associated with an epimorphism g of R. If $g(U) = U$, then R is commutative.*

Theorem 3.4. *Let R be a semiprime ring and U a nonzero ideal of R. If R admits an additive map f and a derivation d such that $[f(x), d(y)] = [x, y]$ for all $x, y \in U$, then $U \subseteq Z$.*

Proof. For $x, y \in U$, we have $[x, xy] = [f(x), d(xy)]$, which yields

$$d(x)[f(x), y] + [f(x), x]d(y) = 0 \text{ for all } x, y \in U. \tag{3.11}$$

Replacing y by yr gives

$$d(x)[f(x), yr] + [f(x), x]d(yr) = 0 \text{ for all } x, y \in U, r \in R. \tag{3.12}$$

Comparing with (3.11) yields

$$d(x)y[f(x),r]+[f(x),x]yd(r)=0 \text{ for all } x,y\in U, r\in R. \tag{3.13}$$

Letting $r=f(x)$, we see that $[f(x),x]yd(f(x))=0$ for all $x,y\in U$, which implies

$$[f(x),x]Ud(f(x))=0=[f(x),x]URd(f(x)) \text{ for all } x\in U. \tag{3.14}$$

Since R is semiprime, it must contain a family $\{P_\alpha|\alpha\in\wedge\}$ of prime ideals such that $\cap P_\alpha=\{0\}$. If P is a typical member of these and $x\in U$, (3.14) shows that $d(f(x))\in P$ or $[f(x),x]U\subseteq P$. For a fixed P, the sets of $x\in U$ for which these two conditions hold are additive subgroups of U whose union is U. Therefore,

$$d(f(U))\subseteq P \textit{ or } [f(x),x]U\subseteq P \text{ for all } x\in U. \tag{3.15}$$

Suppose that $d(f(U))\subseteq P$, for $x,y\in U$, we get $[x,yf(x)]=[f(x),d(yf(x))]$, which implies $U[x,f(x)]\subseteq P$ and $UR[x,f(x)]\subseteq P$, by the primness of P we reach to $U\subseteq P$ or $[x,f(x)]\in P$ for all $x\in U$. In either case

$$[x,f(x)]U\subseteq P \text{ for all } x\in U. \tag{3.16}$$

From (3.15) we have $[x,f(x)]U=\{0\}$ and from (3.13) we have $d(x)y[f(x),r]=0$ and

$$d(x)UR[f(x),r]=\{0\} \text{ for all } x\in U,\ r\in R. \tag{3.17}$$

Since R is semiprime, it must contain a family $\{P_\alpha|\alpha\in\wedge\}$ of prime ideals such that $\cap P_\alpha=\{0\}$. If P is a typical member of these and $x\in U$, (3.17) shows that $d(x)U\subseteq P$ or $[f(x),R]\subseteq P$. For a fixed P, the sets of $x\in U$ for which these two conditions hold are additive subgroups of U whose union is U. Therefore,

$$d(U)U\subseteq P \textit{ or } [f(U),R]\subseteq P. \tag{3.18}$$

Suppose that $d(U)U\subseteq P$, then $d(U)RU\subseteq P$. By the primeness of P we reach to $d(U)\subseteq P$ or $U\subseteq P$, in either case $Ud(U)\subseteq P$, then $y[f(x),d(z)]\in P$ for all $x,y,z\in U$. By our hypothesis, then $y[x,z]\in P$ which implies that $UR[U,U]\subseteq P$, by the primness of P we reach to $U\subseteq P$ or $[U,U]\subseteq P$. In either case $[U,U]\subseteq P$. By our hypothesis $[f(U),d(U)]\subseteq P$. From (3.18) we have $[f(U),d(U)]=\{0\}$, then $[U,U]=\{0\}$, U is commutative, by Lemma 1.3, $U\subseteq Z$. □

The following three corollaries are immediate from the previous theorem.

Corollary 3.5. *Let R be a semiprime ring and U a nonzero ideal of R. If R admits a semiderivation f and a derivation d such that $[f(x),d(y)]=[x,y]$ for all $x,y\in U$, then $U\subseteq Z$.*

Corollary 3.6. *Let R be a semiprime ring .If R admits a semiderivation f and a derivation d such that $[f(x), d(y)] = [x, y]$ for all $x, y \in R$, then R is commutative.*

Corollary 3.7. *Let R be a prime ring and U a nonzero ideal of R. If R admits a semiderivation f and a derivation d such that $[f(x), d(y)] = [x, y]$ for all $x, y \in U$, then R is commutative.*

In the next theorem, we prove Daif and Bell result ([8, Theorem 2]) in the setting of semiderivations.

Theorem 3.8. *Let R be a semiprime ring admitting a semiderivation f associated with an epimorphism g of R for which either $xy + f(xy) = yx + f(yx)$ for all $x, y \in R$, or $xy - f(xy) = yx - f(yx)$ for all $x, y \in R$. Then R is commutative.*

Proof. Suppose first

$$xy + f(xy) = yx + f(yx) \text{ for all } x, y \in R. \tag{3.19}$$

This can be written as

$$[x, y] = -f([x, y]) \text{ for all } x, y \in R. \tag{3.20}$$

From (3.19) replace x by $[x, y]$ and y by z and using (3.20) and our hypothesis we get, $[g(x), g(y)]f(z) = f(z)[g(x), g(y)]$. Since g is onto we have $[x, y]f(z) = f(z)[x, y]$, which shows that $f(z)$ centralizes $[R, R]$. From Lemma 1.2, $f(z)$ centralizes R. By using (3.19), we get

$$[x, y] \in Z(R) \text{ for all } x, y \in R. \tag{3.21}$$

From Lemma 1.2, R centralizes R, which implies that R is commutative. □

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