Gen. Math. Notes, Vol. 3, No. 2, April 2011, pp.66-72
ISSN 2219-7184; Copyright ©ICSRS Publication, 2011
www.i-csrs.org
Available free online at http://www.geman.in

Some Theorems on Fixed Point

Sampada Navshinde[1] and J. Achari[2]

[1]Asst. Prof. SGGSI&T, Nanded-431605, (Maharashtra) India
E-mail: snavshinde@gmail.com
[2]Retired H.O.D.Mathematics, N.E.S.'Science College,
Nanded-431605,(Maharashtra) India

(Received: 24-12-10/ Accepted: 23-3-11)

Abstract

Fixed point theorems for a class of mappings using rational symmetric expression involving four points of the space under consideration have been studied.

Keywords: *Clouser, Common fixed point, commuting mappings.*

1 Introduction

The chief aim of this paper is to introduce a class of mappings by using rational symmetric expression and which involve four points of the space under consideration. A fixed point theorem with this mapping has been proved. Finally some related results with this type of mappings have been proved.

Let (M,d) be a complete metric space. Let $\psi_i : \tilde{P} \to [0, \infty\ )$ (P is the range of d and $\tilde{P}$ is the closure of P) be an upper semicontinuous function from the right on P and satisfies the condition

$$\psi_i(t) < \frac{t}{3} \quad for \quad t > 0 \quad and \quad \psi_i(0) = 0 \;\; i = 1, 2, 3. \tag{1.1}$$

Also, let f be a mapping of M into itself such that

$$d(fu_1, fu_2) \leq \frac{\psi_1(d(u_2, fu_4))[1+\psi_1(d(u_1, fu_3))]}{1+\psi_1(d(u_1, u_2))} + \frac{\psi_2(d(u_1, fu_4))[1+\psi_2(d(u_2, fu_3))]}{1+\psi_2(d(u_1, u_2))} + \frac{\psi_3(d(u_1, fu_3))[1+\psi_3(d(u_2, fu_4))]}{1+\psi_3(d(u_1, u_2))} \tag{1.2}$$

$for\ \ u_1,\ u_2,\ u_3,\ u_4\ \in\ M.$

2 The Main Results

Theorem 2.1. *If f be mapping of M into itself satisfying (1.2), then f has a unique fixed point.*

Proof. Let $x,\ y \in M$ and we define $u_1 = fy,\ \ u_2 = fx,\ \ u_3 = x,\ \ u_4 = y$ Then (1.2) takes the form

$$\begin{aligned} d(f(fy), f(fx)) \leq & \frac{\psi_1(d(fx, fy))[1+\psi_1(d(fy, fx))]}{1+\psi_1(d(fy, fx))} \\ & + \frac{\psi_2(d(fy, fy))[1+\psi_2(d(fx, fx))]}{1+\psi_2(d(fy, fx))} \\ & + \frac{\psi_3(d(fy, fx))[1+\psi_3(d(fx, fy))]}{1+\psi_3(d(fy, fx))} \\ \leq & \ \ \psi_1(d(fx, fy)) \ + \ \psi_3(d(fx, fy)) \end{aligned} \tag{2.1}$$

Let $x_0 \in M$ be arbitrary and construct a sequence $\{x_n\}$ defined by
$fx_{n-1} = x_n,\ \ fx_n = x_{n+1},\ \ fx_{n+1},\ \ \ n = 1, 2, \cdots$
Let us put $x = x_{n-1},\ \ y = x_n$ in (2.1), then we have,

$$\begin{aligned} d(f(fx_n), f(fx_{n-1})) &\leq \ \psi_1(d(fx_{n-1}, fx_n)) \ + \ \psi_3(d(fx_{n-1}, fx_n)) \\ i.e.\ d(x_{n+1}, x_{n+2}) &\leq \ \psi_1(d(x_n, x_{n+1})) \ + \ \psi_3(d(x_n, x_{n+1})) \end{aligned} \tag{2.2}$$

Now set $C_n = d(x_{n-1}, x_n)$. Then

$$\begin{aligned} C_{n+2} &= d(x_{n+1}, x_{n+2}) \\ &\leq \psi_1(d(x_n, x_{n+1})) \ + \ \psi_3(d(x_n, x_{n+1})) \\ &\leq \psi_1(C_{n+1}) \ + \ \psi_3(C_{n+1}) \end{aligned} \tag{2.3}$$

From (2.3) it follows that C_n decreases with n and hence $C_n \to C$ say as $n \to \infty$. Then since ψ_i is upper semicontinous we obtain in the limit as $n \to \infty$

$$C \leq \psi_1(C) + \psi_3(C) < \frac{2}{3}C$$

which is impossible unless $C = 0$.

Next, we shall show that the sequence $\{x_n\}$ is Cauchy. Suppose that it is not so. Then there exist an $\epsilon > 0$ and sequence of integers $\{m(k)\}$,$\{n(k)\}$ with $m(k) > n(k) \geq k$ such that

$$d_k = d(x_{m(k)}, x_{n(k)}) \geq \epsilon, k = 1, 2, 3, \cdots \tag{2.4}$$

If m(k) is the smallest integer exceding n(k) for which (2.4) holds, then from the well ordering principle we have,

$$d(x_{m(k)-1}, x_{n(k)}) \leq \epsilon \tag{2.5}$$

Then $d_k = d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) \leq C_{m(k)} + \epsilon < C_k + \epsilon$ which implies that $d_k \rightarrow \epsilon \; as \; n \rightarrow \infty$.
Also we have,

$$\begin{aligned} d_k &= d(x_m, x_n) \\ &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n) \\ &\leq C_{m+1} + C_{n+1} + d(fx_n, fx_m) \\ &\leq C_{m+1} + C_{n+1} + \frac{\psi_1(d(x_m, fx_{m-1}))[1 + \psi_1(d(x_n, fx_{n-1}))]}{1 + \psi_1(d(x_n, x_m))} \\ &\quad + \frac{\psi_2(d(x_n, fx_{m-1}))[1 + \psi_2(d(x_m, fx_{n-1}))]}{1 + \psi_2(d(x_n, x_m))} \\ &\quad + \frac{\psi_3(d(x_n, fx_{n-1}))[1 + \psi_3(d(x_m, fx_{m-1}))]}{1 + \psi_3(d(x_n, x_m))} \end{aligned}$$

(By putting $u_1 = x_n, \; u_2 = x_m, \; u_3 = x_{n-1}, \; u_4 = x_{m-1}$)

$$\begin{aligned} d_k = d(x_m, x_n) &\leq C_{m+1} + C_{n+1} + \psi_2(d(x_n x_m)) + \psi_3(d(x_n, x_m)) \\ &\leq C_{m+1} + C_{n+1} + \psi_2(d_k) + \psi_3(d_k) \end{aligned}$$

letting k $\rightarrow \infty$ we have

$$\epsilon \; \leq \; \psi_2(\epsilon) + \psi_3(\epsilon) \; < \; \frac{2}{3}\epsilon$$

which is a contradiction if $\epsilon > 0$.

This leads us to conclude that $\{x_n\}$ is a Cauchy sequence and since M is complete, there exists a point $z \in M$ such that $x_n \rightarrow z \; as \; n \rightarrow \infty$. We shall show that z is a fixed point of f.

Now putting $u_1 = x_{n-1},\ \ u_2 = z,\ \ u_3 = x_{n+1},\ \ u_4 = x_n$ in (1.2) we have,

$$\begin{aligned} &d(fx_{n-1}, fz) \\ &\quad \le \frac{\psi_1(d(z, fx_n))[1 + \psi_1(d(x_{n-1}, fx_{n+1}))]}{1 + \psi_1(d(x_{n-1}, z))} \\ &\qquad + \frac{\psi_2(d(x_{n-1}, fx_n))[1 + \psi_2(d(z, fx_{n+1}))]}{1 + \psi_2(d(x_{n-1}, z))} \\ &\qquad + \frac{\psi_3(d(x_{n-1}, fx_{n+1}))[1 + \psi_3(d(z, fx_n))]}{1 + \psi_3(d(x_{n-1}, z))} \\ &\quad \le \frac{\psi_1(d(z, x_{n+1}))[1 + \psi_1(d(x_{n-1}, x_{n+2}))]}{1 + \psi_1(d(x_{n-1}, z))} \\ &\qquad + \frac{\psi_2(d(x_{n-1}, x_{n+1}))[1 + \psi_2(d(z, x_{n+2}))]}{1 + \psi_2(d(x_{n-1}, z))} \\ &\qquad + \frac{\psi_3(d(x_{n-1}, x_{n+2}))[1 + \psi_3(d(z, x_{n+1}))]}{1 + \psi_3(d(x_{n-1}, z))} \end{aligned} \tag{2.6}$$

Letting $n \to \infty$ we get $d(z, fz) \le 0$ which implies $z = fz$. Thus z is a fixed point of f.
If possible, let there be another fixed point $w (\neq z)$, then putting $u_1 = u_4 = z,\ \ u_2 = u_3 = w$ in (1.2) we get,

$$\begin{aligned} d(z, w) &= d(fz, fw) \\ &\le \frac{\psi_1(d(w, fz))[1 + \psi_1(d(z, fw))]}{1 + \psi_1(d(z, w))} + \frac{\psi_2(d(z, fz))[1 + \psi_2(d(w, fw))]}{1 + \psi_2(d(z, w))} \\ &\quad + \frac{\psi_3(d(z, fw))[1 + \psi_3(d(w, fz))]}{1 + \psi_3(d(z, w))} \\ &\le \psi_1(d(z, w)) + \psi_2(d(z, w)) \ < \ \frac{2}{3} d(z, w) \end{aligned}$$

which is impossible. Hence $z = w$.

Theorem 2.2. *Let (M,d) be a complete metric space and* f_k $(k = 1, 2, \cdots, n)$ *be a family of mappings of M into itself. If* f_k *(k =1,2,$\cdots$,n) satisfies*
(i) $f_k f_m = f_m f_k \quad (m, k = 1, 2, \cdots n)$
(ii) there is a system of positive integers $m_1, m_2, \cdot m_n$ *such that*

$$\begin{aligned} &d(f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_2) \\ &\quad \le \frac{\psi_1(d(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))[1 + \psi_1(d(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))]}{1 + \psi_1(d(u_1, u_2))} \\ &\qquad + \frac{\psi_2(d(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))[1 + \psi_2(d(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))]}{1 + \psi_2(d(u_1, u_2))} \\ &\qquad + \frac{\psi_3(d(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))[1 + \psi_3(d(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))]}{1 + \psi_3(d(u_1, u_2))} \end{aligned}$$

for $u_1,\ u_2,\ u_3,\ u_4\ \in\ M$ *and* $\psi_i(t)$ *satisfies (1.1) , then* f_k $(k = 1, 2, \cdots n)$ *have a unique common fixed point.*

Proof. Let $f = f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n}$. Then (ii) takes the form (iii)

$$\begin{aligned} d(fu_1, fu_2) \leq & \frac{\psi_1(d(u_2, fu_4))[1 + \psi_1(d(u_1, fu_3))]}{1 + \psi_1(d(u_1, u_2))} \\ & + \frac{\psi_2(d(u_1, fu_4))[1 + \psi_2(d(u_2, fu_3))]}{1 + \psi_2(d(u_1, u_2))} \\ & + \frac{\psi_3(d(u_1, fu_3))[1 + \psi_3(d(u_2, fu_4))]}{1 + \psi_3(d(u_1, u_2))} \end{aligned}$$

Then by Theorem [2.1], f has a unique fixed point z in M. Therefore $fz = z$, then we have,

$$f_k(fz) = f_k z, \quad k = 1, 2, \cdots n$$

By commutativity of f_k we have,

$$f(f_k z) = f_k z, \quad k = 1, 2, \cdots n$$

Since f has a unique common fixed point z, we obtain $f_k z, \quad k = 1, 2, \cdots n$. Hence z is a common fixed point of the family f_k. Let z,w be common fixed point of f_k, then by (ii) we have by putting $u_1 = u_4 = z, \ \ u_2 = u_3 = w$

$$\begin{aligned} d(z, w) &= d(fz, fw) \\ &\leq \frac{\psi_1(d(w, fz))[1 + \psi_1(d(z, fw))]}{1 + \psi_1(d(z, w))} + \frac{\psi_2(d(z, fz))[1 + \psi_2(d(w, fw))]}{1 + \psi_2(d(z, w))} \\ &\quad + \frac{\psi_3(d(z, fw))[1 + \psi_3(d(w, fz))]}{1 + \psi_3(d(z, w))} \\ &\leq \psi_1(d(z, w)) + \psi_2(d(z, w)) \ < \ \frac{2}{3} d(z, w) \end{aligned}$$

which implies $z = w$. Hence the proof.

Theorem 2.3. *Let M be a metric space with d and p and* $f_k \quad (k = 1, 2, \cdots, n)$ *be a family of mappings of M into itself.*
Suppose that
(i) $d(x, y)\ \leq\ p(x, y)$ *to all x,y* $\in M$
(ii) M is f-orbitally complete w.r.t. d.
(iii) $f_k f_m = f_m f_k \quad (m, k = 1, 2, \cdots n)$
(iv) there is a system of positive integers $m_1, m_2, \cdot m_n$ *such that*

$$p(f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_2)$$
$$\leq \frac{\psi_1(p(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))[1 + \psi_1(p(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))]}{1 + \psi_1(p(u_1, u_2))}$$
$$+ \frac{\psi_2(p(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))[1 + \psi_2(p(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))]}{1 + \psi_2(p(u_1, u_2))}$$
$$+ \frac{\psi_3(p(u_1, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_3))[1 + \psi_3(p(u_2, f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n} u_4))]}{1 + \psi_3(p(u_1, u_2))}$$

for $u_1,\ u_2,\ u_3,\ u_4\ \in\ M$ *and* $\psi_i(t) < \frac{t}{3}$ *for* $t > 0$ *and* $\psi_i(0) = 0$ $, i = 1, 2, 3$ *Then* f_k $(k = 1, 2, \cdots, n)$ *have a unique common fixed point.*

Proof. As in Theorem [2.1] put $f = f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n}$ then(iv) takes the form

$$p(fu_1, fu_2) \leq \frac{\psi_1(p(u_2, fu_4))[1 + \psi_1(p(u_1, fu_3))]}{1 + \psi_1(p(u_1, u_2))}$$
$$+ \frac{\psi_2(p(u_1, fu_4))[1 + \psi_2(p(u_2, fu_3))]}{1 + \psi_2(p(u_1, u_2))}$$
$$+ \frac{\psi_3(p(u_1, fu_3))[1 + \psi_3(p(u_2, fu_4))]}{1 + \psi_3(p(u_1, u_2))}$$

for $u_1,\ u_2,\ u_3,\ u_4\ \in\ M$

Following the lines of arguments of the proof of Theorem [2.1], it can be shown that the sequence of iterates $\{x_n\}$ is Cauchy with respect to p. Since $d(x, y) \leq p(x, y)$ for all x,y ∈ M , so $\{x_n\}$ is Cauchy with respect to d also. Again M being f-orbitally complete with respect to d, so we have $\{x_n\}$ has a limit u in M. From the proof of Theorem [2.1] it can be easily shown that u is the unique common fixed point of the family f_k .

Acknowledgements

The authors thank the referee for his/her suggestions and comments.

References

[1] J. Achari, *Resultate der Mathematik*, (1) (1979), 1-6.

[2] J. Achari, *Comp. Rend. L'Acad. Bulgar Sci*, (32) 1979, 703-706.

[3] Lj.B. Ciric, *Proc. Amer. Math. Soc.*,(45) (1974), 267-273.

[4] M. Edelstein, *Proc. Amer. Math. Soc*, (12) (1961), 7-10.

[5] R. Kannan, *Amer. Math. Monthly*, (76) (1969), 405-408.

[6] F. Pittnauer, *Archive der Math*, (26) (1975), 421-426.

[7] F. Pittnauer, *Periodica Math. Hungarica*, (12) (1979), 1-6.

[8] S. Reich, *Canad. Math. Bull.*, (14) (1971), 121-124.

[9] B.E. Rhoades, A fixed point theorem in metric space, .

[10] C.S. Wong, *Pacific J. Math.*, (48) (1973), 299-312.