*Gen. Math. Notes, Vol. 18, No. 1, September, 2013, pp. 64-87*
*ISSN 2219-7184;* 
*www.i-csrs.org*
*Available free online at http://www.geman.in*

# Lorentzian Beltrami-Meusnier Formula

**Soley Ersoy[1] and Murat Tosun[2]**

[1, 2] Department of Mathematics
Faculty of Arts Sciences, Sakarya University
54187, Sakarya, Turkey
[1] E-mail: sersoy@sakarya.edu.tr
[2] E-mail: tosun@sau.edu.tr

(Received: 4-6-13 / Accepted: 9-7-13)

**Abstract**

*In this paper, the relationship between the arbitrary sectional curvatures and normal sectional curvatures of generalized timelike ruled surface with timelike generating space in n-dimensional Minkowski space* $\mathbb{R}_1^n$ *are investigated. Three different types of relation are obtained and called I., II., and III. type of Lorentzian Beltrami-Meusnier formula.*

**Keywords:** *Sectional Curvature, Ruled surface, Beltrami-Meusnier Formula.*

## 1 Introduction

Meusnier formula and Beltrami formula are well known theorems from classical surface theory, (see for example, [3], [7], etc.) An analogue of Meusnier formula was obtained in [6] by the way of applying this formula to tangential sections of generalized ruled surface in $n-$dimensional Euclid space $E^n$, and was called as Beltrami-Meusnier formula. In [5], authors calculated the first fundamental form and the metric coefficients of generalized timelike ruled surfaces with timelike generating space given in [1] and [2]. Moreover, according to Christoffel Symbols, Riemann-Christoffel curvatures of these surfaces were obtained. Furthermore, the principal sectional curvatures of generalized timelike ruled surface with timelike generating space and central ruled surface were found to be with respect to the determinant of the first fundamental form of the surface for spacelike and timelike tangential sections, separately. Lastly, four different types of Lorentzian Beltrami-Euler formulas were constituted for generalized timelike ruled surface with timelike generating space in [5]. In this paper, we will consult to these theorems, in case of necessity. To avoid from repetition of the basic concepts in Minkowski-Lorentz space, which are known from [8] and [9], will not be repeated.

## 2 Generalized Timelike Ruled Surfaces with Timelike Generating Space in *n*-Dimensional Minkowski Space

A $(k+1)-$ dimensional timelike ruled surface in $n-$ dimensional Minkowski space, $\mathbb{R}_1^n$ is given parametrically as

$$\begin{aligned} &\phi(t,u_1,...u_k)=\alpha(t)+\sum_{i=1}^{k}u_i e_i(t) \\ &\operatorname{rank}\left(\phi_t,e_1(t),...,e_k(t)\right)=k+1 \end{aligned} \tag{2.1}$$

where the base curve $\alpha$ is a spacelike curve and generating space $E_k(t)=Sp\{e_1(t),...,e_k(t)\}$ is a timelike subspace.

$$A(t)=Sp\left\{e_1(t),...,e_k(t),\dot{e}_1(t),...,\dot{e}_k(t)\right\}$$

is called asymptotic bundle of $M$ with respect to $E_k(t)$. It is clear that $A(t)$ is a timelike subspace. If $\dim A(t)=k+m$ , $0\le m\le k$ , then one can find an orthonormal base for $A(t)$ containing $E_k(t)$ such as $\{e_1(t),...e_k(t),a_{k+1}(t),...a_{k+m}(t)\}$ . Furthermore, for the orthonormal base $\{e_1(t),\ldots,e_k(t)\}$, the following equations hold [2]

$$\begin{aligned} &\dot{e}_\sigma=\sum_{\mu=1}^{k}\alpha_{\sigma\mu}e_\mu+\kappa_\sigma a_{k+\sigma} \quad , \quad 1\le\sigma\le m \\ &\dot{e}_{m+\rho}=\sum_{\mu=1}^{k}\alpha_{(m+\rho)\mu}e_\mu \quad , \quad 1\le\rho\le k-m \end{aligned} \tag{2.2}$$

where

$$\varepsilon_\mu\alpha_{\nu\mu}=-\varepsilon_\nu\alpha_{\mu\nu} \tag{2.3}$$

and

$$\kappa_1>\kappa_2>\ldots>\kappa_m>0. \tag{2.4}$$

The subspace

$$Sp\left\{e_1(t),...,e_k(t),\dot{e}_1(t),...,\dot{e}_k(t),\dot{\alpha}(t)\right\}$$

is called tangential bundle of $M$ with respect to $E_k(t)$ and denoted as $T(t)$. It can be easily seen that

$$k+m \leq boyT(t) \leq k+m+1,\ 0 \leq m \leq k\ .$$

If $\dim T(t)=k+m$, then $\{e_1(t),...e_k(t),a_{k+1}(t),...a_{k+m}(t)\}$ is the base for both asymptotic and tangential bundles. If $\dim T(t)=k+m+1$, then one can find an orthonormal base for $T(t)$ as $\{e_1(t),...e_k(t),a_{k+1}(t),...a_{k+m}(t),a_{k+m+1}(t)\}$. For both cases tangential bundle $T(t)$ is a timelike subspace, [2].

Let $\dim T(t)=k+m+1$. In this case $(k+1)-$dimensional timelike ruled surface has a $(k-m)-$dimensional subspace called central space of $M$ and denoted as $Z_{k-m}(t)\subset E_k(t)$. The subspace $Z_{k-m}(t)$ is either spacelike or timelike subspace. If the base curve $\alpha$ of $M$ is chosen to be the base curve and $Z_{k-m}(t)$ to be generating space, we get a $(k-m+1)-$dimensional ruled surface contained by $M$ in $\mathbb{R}_1^n$. This is denoted by $\Omega$ and called the central ruled surface. If $Z_{k-m}(t)$ is spacelike (timelike) then central ruled surface $\Omega$ becomes spacelike (timelike) ruled surface, [1].

A base curve $\alpha$ of $(k+1)$-dimensional ruled surface $M$ is a base curve of central surface $\Omega\subset M$ as well, iff its tangent vector has the form

$$\dot{\alpha}(t)=\sum_{\nu=1}^{k}\zeta_\nu e_\nu+\eta_{m+1}a_{k+m+1} \quad , \qquad \eta_{m+1}\neq 0 \tag{2.5}$$

where $\eta_{m+1}\neq 0$, $a_{k+m+1}$ is a unit vector well defined up to the sign with the property that $\{e_1,...,e_k,a_{k+1},...,a_{k+m},a_{k+m+1}\}$ is an orthonormal base of the tangential bundle of $M$, [1]. The tangential space of $M$ at the central points is perpendicular to the asymptotic bundle $A(t)$. Considering the equation (2.1) at the central point of central ruled surface $\Omega\subset M$ we see that

$$u_\sigma=0 \quad , \qquad 1\leq\sigma\leq m. \tag{2.6}$$

For the spacelike base curve $\alpha$ of $(k+1)-$dimensional timelike ruled surface $M$, if $\eta_{m+1}\neq 0$ we call $m-$magnitudes

$$P_\sigma = \frac{\eta_{m+1}}{\kappa_\sigma} \quad , \quad 1 \le \sigma \le m \tag{2.7}$$

is called the $\sigma^{\text{th}}$ principal distribution parameter of $M$ , [1].

The canonical base of the tangential of $M$ is

$$\left\{ \sum_{\nu=1}^{k} \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) e_\nu + \sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma a_{k+\sigma} + \eta_{m+1} a_{k+m+1}, e_1, e_2, \ldots, e_k \right\}. \tag{2.8}$$

We can evaluate the first fundamental form of $M$ and the metric coefficients with respect to this canonical base. In conventional notation we choose $u_0 = t$ and calculate the metric coefficients of $M$ as follows;

$$\begin{aligned}
g_{00} &= -g + \sum_{\nu=1}^{k} \varepsilon_\nu \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right)^2 \\
g_{\nu 0} &= \varepsilon_\nu \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) && , \quad 1 \le \nu \le k \\
g_{\nu\mu} &= \varepsilon_\nu \delta_{\nu\mu} && , \quad 0 \le \nu, \mu \le k \\
g &= \det\left[ g_{ij} \right] = -\sum_{\sigma=1}^{m} \left( u_\sigma \kappa_\sigma \right)^2 - \eta_{m+1}^2 && , \quad 0 \le i, j \le k.
\end{aligned} \tag{2.9}$$

In addition to these, the coefficients of the inverse matrix $\left[ g^{ij} \right]$ of the matrix $\left[ g_{ij} \right]$, $0 \le i, j \le k$, are as follows

$$\begin{aligned}
g^{00} &= -g^{-1} \\
g^{\nu 0} &= \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) g^{-1} && , \quad 1 \le \nu \le k \\
g^{\nu\lambda} &= \left( \varepsilon_\nu \delta_{\nu\lambda} g - \left( \zeta_\nu + \sum_{\mu=1}^{k} \alpha_{\nu\mu} u_\mu \right) \left( \zeta_\lambda + \sum_{\mu=1}^{k} \alpha_{\lambda\mu} u_\mu \right) \right) g^{-1} && , \quad 1 \le \nu, \lambda \le k.
\end{aligned} \tag{2.10}$$

Substituting the equations (2.9) and (2.10) into the Koszul equation (given in [4]) the Christoffel symbols are obtained for $1 \le \nu, \mu, \lambda \le k$ ,

$$\Gamma_{00}^{0}=\frac{1}{2g}\left[\frac{\partial g}{\partial u_0}+\sum_{\nu=1}^{k}\left(\zeta_\nu+\sum_{\mu=1}^{k}\alpha_{\nu\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu}\right],$$

$$\Gamma_{00}^{\lambda}=\frac{1}{2g}\left[-\left(\zeta_\lambda+\sum_{\mu=1}^{k}\alpha_{\lambda\mu}u_\mu\right)\left(\frac{\partial g}{\partial u_0}+\sum_{\nu=1}^{k}\left(\zeta_\nu+\sum_{\mu=1}^{k}\alpha_{\nu\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu}\right)\right.$$
$$\left.+2g\left(\left(\dot{\zeta}_\lambda+\sum_{\mu=1}^{k}\dot{\alpha}_{\lambda\mu}u_\mu\right)+\sum_{\nu=1}^{k}\left(\zeta_\nu+\sum_{\mu=1}^{k}\alpha_{\nu\mu}u_\mu\right)\alpha_{\lambda\nu}+\frac{1}{2}\varepsilon_\lambda\frac{\partial g}{\partial u_\lambda}\right)\right],$$

$$\Gamma_{\nu\mu}^{0}=\Gamma_{\mu\nu}^{0}=0, \tag{2.11}$$

$$\Gamma_{\nu\mu}^{\lambda}=\Gamma_{\mu\nu}^{\lambda}=0,$$

$$\Gamma_{\lambda 0}^{0}=\Gamma_{0\lambda}^{0}=\frac{1}{2g}\frac{\partial g}{\partial u_\lambda},$$

$$\Gamma_{\nu 0}^{\lambda}=\Gamma_{0\nu}^{\lambda}=\frac{1}{2g}\left[-\left(\zeta_\lambda+\sum_{\mu=1}^{k}\alpha_{\lambda\mu}u_\mu\right)\frac{\partial g}{\partial u_\nu}+2g\left(\alpha_{\lambda\nu}\right)\right].$$

Adopting that the base of tangential space in the neighborhood of coordinate system $\{u_0,u_1,\ldots,u_k\}$ is $\{\partial_0,\partial_1,\ldots,\partial_k\}$ ( $\frac{\partial}{\partial u_i}=\partial_i$ , $0\le i\le k$ ), the Riemann curvature tensor of the generalized timelike surface $M$ is given by

$$R_{hlij}=\sum_{r=0}^{k}g_{rh}\left(\frac{\partial}{\partial u_i}\Gamma_{jl}^{r}-\frac{\partial}{\partial u_j}\Gamma_{il}^{r}-\sum_{s=0}^{k}\Gamma_{il}^{s}\Gamma_{js}^{r}+\sum_{s=0}^{k}\Gamma_{jl}^{s}\Gamma_{is}^{r}\right). \tag{2.12}$$

Considering the equations (2.9) and (2.11), $R_{ij00}$, $R_{ij\nu\mu}$, $R_{\nu 0\mu 0}$ are found to be (in terms of the determinant of the first fundamental form of $M$ , the first and second order partial differentials of $g$ )

$$\begin{aligned} R_{ij00}&=0 &&, \ 0\le i,j\le k\\ R_{ij\nu\mu}&=0 &&, \ 0\le i,j\le k\ ,1\le \nu,\mu\le k \\ R_{\nu 0\mu 0}&=\frac{1}{2}\frac{\partial^2 g}{\partial u_\nu\partial u_\mu}-\frac{1}{4g}\frac{\partial g}{\partial u_\nu}\frac{\partial g}{\partial u_\mu} &&, \ 1\le \nu,\mu\le k,\end{aligned} \tag{2.17}$$

# 3 Lorentzian Beltrami-Meusnier Formula for Generalized Timelike Ruled Surfaces with Timelike Generating Space in *n*-Dimensional Minkowski Space

Two-dimensional subspace $\Pi$ of $(k+1)-$dimensional time-like ruled surface at the point $\xi \in T_M(\xi)$ is called tangent section of $M$ at point $\xi$. If $\vec{v}$ and $\vec{w}$ form a basis of the tangent section $\Pi$, then $Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2$ is a non-zero quantity iff $\Pi$ is non-degenerate. This quantity represents the square of the Lorentzian area of the parallelogram determined by $\vec{v}$ and $\vec{w}$. Using the square of the Lorentzian area of the parallelogram determined by the basis vectors $\{\vec{v},\vec{w}\}$, one has the following classification for the tangent sections of the time-like ruled surfaces:

$$Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2 < 0 \quad , \quad \text{(time-like plane)},$$

$$Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2 = 0 \quad , \quad \text{(degenerate plane)},$$

$$Q(\vec{v},\vec{w}) = \langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2 > 0 \quad , \quad \text{(space-like plane)}.$$

For the non-degenerate tangent section $\Pi$ given by the basis $\{\vec{v},\vec{w}\}$ of $M$ at the point $\xi$

$$K_\xi(\vec{v},\vec{w}) = \frac{\langle R_{\vec{v}\vec{w}}\vec{v},\vec{w}\rangle}{Q(\vec{v},\vec{w})} = \frac{\sum R_{ijkm}\beta_i\gamma_j\beta_k\gamma_m}{\langle \vec{v},\vec{v}\rangle\langle \vec{w},\vec{w}\rangle - \langle \vec{v},\vec{w}\rangle^2} \tag{3.1}$$

is called sectional curvature of $M$ at the point $\xi$, where $\vec{v} = \sum \beta_i \frac{\partial}{\partial x_i}$ and $\vec{w} = \sum \gamma_j \frac{\partial}{\partial x_j}$. Here the coordinates of the basis vectors $\vec{v}$ and $\vec{w}$ are $(\beta_0,\beta_1,\ldots,\beta_k)$ and $(\gamma_0,\gamma_1,\ldots,\gamma_k)$, respectively, [4].

Let the base curve $\alpha$ of timelike ruled surface $M$ with timelike generating space be also a base curve of central ruled surface $\Omega$ of $M$ in $\mathbb{R}_1^n$. In this case, a normal tangent vector $n$ of $M$ orthogonal to $E_k(t)$ is defined to be

$$n = \sum_{\sigma=1}^{m} u_\sigma \kappa_\sigma(t) a_{k+\sigma}(t) + \eta_{m+1} a_{k+m+1}(t) \quad , \quad (\eta_{m+1} \neq 0) \tag{3.2}$$

at the point $\forall \xi(t,u_v)$, where this normal tangent vector field is always spacelike since it is orthogonal to generating space $E_k(t)$.

The central ruled surface $\Omega$ of the generalized timelike ruled surface $M$ with timelike generating space in $\mathbb{R}_1^n$ is either spacelike or timelike. Therefore, at the central point $\forall\zeta\in\Omega$, any unit vector $a$ is defined to be

$$a=\lambda_0\frac{n}{\|n\|}+\lambda_1 e_1+\lambda_2 e_2+\ldots+\lambda_{s-1}e_{s-1}+\lambda_s e_s+\lambda_{s+1}e_{s+1}+\ldots+\lambda_k e_k \quad , \quad \|a\|=1$$

where $a$ and $e_\sigma$, $1\le\sigma\le m$, are linearly independent. Here $e_s$ is a timelike unit vector which is either in subspace $F_m(t)$ or in central subspace $Z_{k-m}(t)$ and we write

$$\langle a,a\rangle=\lambda_0^2+\lambda_1^2+\lambda_2^2+\ldots+\lambda_{s-1}^2-\lambda_s^2+\lambda_{s+1}^2+\ldots+\lambda_k^2=\pm1.$$

This means that $a$ is either unit spacelike or timelike vector. Thus, there exist the following cases depending on whether central ruled surface $\Omega$ of $M$ is spacelike (i.e., $Z_{k-m}(t)$, spacelike) or timelike (i.e., $Z_{k-m}(t)$, timelike):

**(a)** While the central space $Z_{k-m}(t)$ is spacelike,

**(a1)** Unit vector $a$ is either spacelike or
**(a2)** Unit vector $a$ is timelike.

**(b)** While central $Z_{k-m}(t)$ is timelike,

**(b1)** Unit vector $a$ is either spacelike or
**(b2)** Unit vector $a$ is timelike.

Now let us consider these cases separately.

**(a1) Let the central space be spacelike and unit vector $a$ be spacelike vector:**

Suppose that $M$ is generalized timelike ruled surface with timelike generating space and spacelike central ruled surface in $\mathbb{R}_1{}^n$. The principle frame of $M$ is $\{e_1,\ldots,e_s,\ldots,e_m,\ldots,e_k\}$ and the normal tangent vector $n$ is orthogonal to the generating space $E_k(t)$ of $M$, so at the central point $\forall\zeta\in\Omega$ any spacelike vector $a$ can be written as follows

$$a=\cosh\psi_0\frac{n}{\|n\|}+\sum_{\substack{\nu=1\\ \nu\neq s}}^{k}\cosh\psi_\nu e_\nu+\sinh\psi_s e_s \text{ and } \sum_{\substack{\nu=0\\ \nu\neq s}}^{k}\cosh^2\theta_\nu-\sinh^2\theta_s=1 \qquad (3.3)$$

where $e_s$, $1 \le s \le m$, is a timelike vector in the subspace $F_m(t)$ and the angles $\psi_0, \psi_1, \ldots, \psi_s, \ldots, \psi_k$ are the hyperbolic angles between the spacelike unit vector $a$ and the vectors $n, e_1, \ldots, e_s, \ldots, e_k$.

**(a2) Let the central space be spacelike and unit vector $a$ be timelike vector:**

Suppose that $M$ is generalized timelike ruled surface with timelike generating space and spacelike central ruled surface. As in case (a1), at the point $\forall \zeta \in \Omega$ any timelike vector $a$ is written as follows

$$a = \sinh \psi_0 \frac{n}{\|n\|} + \sum_{\substack{\nu=1 \\ \nu \neq s}}^{k} \sinh \psi_\nu e_\nu + \cosh \psi_s e_s \text{ and } \sum_{\substack{\nu=0 \\ \nu \neq s}}^{k} \sinh^2 \theta_\nu - \cosh^2 \theta_s = -1 \quad (3.4)$$

such that $e_s$, $1 \le s \le m$, is a timelike vector in subspace $F_m(t)$ and the hyperbolic angles between the timelike unit vector $a$ and vectors $n, e_1, \ldots, e_s, \ldots, e_k$ are $\psi_0, \psi_1, \ldots, \psi_s, \ldots, \psi_k$, respectively.

**(b1) Let the central space be timelike and unit vector $a$ be spacelike vector:**

Suppose that $M$ is generalized timelike ruled surface with timelike generating space and timelike central ruled surface. At the $\forall \zeta \in \Omega$ point any spacelike unit vector $a$ can be written as follows

$$a = \cosh \psi_0 \frac{n}{\|n\|} + \sum_{\substack{\nu=1 \\ \nu \neq m+s}}^{k} \cosh \psi_\nu e_\nu + \sinh \psi_{m+s} e_{m+s} \text{ and } \sum_{\substack{\nu=0 \\ \nu \neq m+s}}^{k} \cosh^2 \theta_\nu - \sinh^2 \theta_{m+s} = 1 \quad (3.5)$$

where $e_{m+s}$, $1 \le s \le k-m$, is timelike vector in the central space $Z_{k-m}(t)$ and the angles $\psi_0, \psi_1, \ldots, \psi_{m+s}, \ldots, \psi_k$ are the hyperbolic angles between spacelike unit vector $a$ and base vectors $n, e_1, \ldots, e_{m+s}, \ldots, e_k$, respectively.

**(b2) Let the central space be timelike and unit vector $a$ be timelike vector:**

Suppose that $M$ is generalized timelike ruled surface with timelike generating space and timelike central ruled surface in $\mathbb{R}_1^n$. As in the case (b1) we take $n$ is orthogonal to generating space $E_k(t)$ so that any timelike vector $n$ can be written as follows

$$a = \sinh \psi_0 \frac{n}{\|n\|} + \sum_{\substack{\nu=1 \\ \nu \neq m+s}}^{k} \sinh \psi_\nu e_\nu + \cosh \psi_{m+s} e_{m+s} \text{ and } \sum_{\substack{\nu=0 \\ \nu \neq m+s}}^{k} \sinh^2 \theta_\nu - \cosh^2 \theta_{m+s} = -1 \quad (3.6)$$

where $e_{m+s}$, $1 \le s \le k-m$, is timelike vector in the central space $Z_{k-m}(t)$ and the hyperbolic angles $\psi_0, \psi_1, \ldots, \psi_{m+s}, \ldots, \psi_k$ are the angles between spacelike unit vector $a$ and the base vectors $n, e_1, \ldots, e_{m+s}, \ldots, e_k$, respectively.

Therefore, taking the cases (a1) to (b2) into consideration, we give the following theorems below, about the relationship between the curvatures of the section $(e_\sigma, a)$, $1 \le \sigma \le m$, and the $\sigma^{\text{th}}$ principal section $(e_\sigma, n)$, $1 \le \sigma \le m$.

**Theorem 3.1:** *Let $a$ be any spacelike unit vector of $M$ generalized timelike ruled surface with timelike generating space and with spacelike central ruled surface and $e_s$, $1 \le s \le m$, be timelike base vector within subspace $F_m(t)$, respectively.*

**i.** *Given the unit vector $a$ which is linearly independent with the $\sigma^{th}$ spacelike vector $e_\sigma$, $1 \le \sigma \le m$, ($\sigma \ne s$), there exists the following relation between the curvatures of timelike section $(e_\sigma, a)$ and the $\sigma^{th}$ spacelike principal section $(e_\sigma, n)$ at the point $(\zeta + ue_s) \in M$*

$$\left(1-\cosh^2 \psi_\sigma\right) K_{\zeta+ue_\sigma}(e_\sigma, a) = \cosh^2 \psi_0 K_{\zeta+ue_\sigma}(e_\sigma, n)$$

*where $\psi_0$ is the hyperbolic angle between $a$ and $n$ and $\psi_\sigma$, $1 \le \sigma \le m$, ($\sigma \ne s$) are the hyperbolic angles between $a$ and $e_\sigma$, respectively.*

**ii.** *Given any unit vector $a$ which is linearly independent with the $s^{th}$ timelike vector $e_s$, $1 \le s \le m$, at the point $(\zeta + ue_s) \in M$, there exists the following relation between the curvatures of timelike section $(e_s, a)$ and the $s^{th}$ timelike principal section $(e_s, n)$*

$$\left(1+\sinh^2 \psi_s\right) K_{\zeta+ue_s}(e_s, a) = \cosh^2 \psi_0 K_{\zeta+ue_s}(e_s, n)$$

*where the hyperbolic angles between $a$ and $n$ and between $a$ and $e_s$ are $\psi_0$ and $\psi_s$ respectively.*

**Proof:** Let $a$ be any spacelike unit vector of generalized timelike ruled surface $M$ with timelike generating space and with spacelike central ruled surface in $n-$dimensional Minkowski space $\mathbb{R}_1^n$.

**i.** If we consider the equation (3.1) then the curvature of timelike section is

$$K_{\zeta+ue_\sigma}(e_\sigma,a)=\frac{\beta_\sigma\beta_\sigma\lambda_0\lambda_0R_{\sigma0\sigma0}}{\langle e_\sigma,e_\sigma\rangle\langle a,a\rangle-\langle e_\sigma,a\rangle^2}\quad,\quad 1\le\sigma\le m\quad,\quad \sigma\ne s. \tag{3.7}$$

$(e_\sigma,a)$ at the point $(\zeta+ue_\sigma)\in M$ .

The coordinates of the $\nu^{\text{th}}$ base vector $e_\nu$, $1\le\nu\le k$, and spacelike unit vector $a$ given by the equation (3.3) are $(\beta_0,\beta_1,\ldots,\beta_\nu,\ldots,\beta_k)$ and $(\gamma_0,\gamma_1,\ldots,\gamma_s,\ldots,\gamma_k)$, respectively. Thus, we may write

$$\beta_0=\langle e_\nu,e_0\rangle=0\quad,\quad \beta_\nu=\langle e_\nu,e_\nu\rangle=\varepsilon_\nu\quad,\quad 1\le\nu\le k$$

and

$$\gamma_0=\langle a,e_0\rangle=\frac{\cosh\psi_0}{\|n\|}\quad,\quad \gamma_\nu=\langle a,e_\nu\rangle=\cosh\psi_\nu\quad,\quad 1\le\nu\le k\quad,\quad \nu\ne s$$

$$\gamma_s=\langle a,e_s\rangle=\sinh\psi_s\quad,\quad 1\le s\le m.$$

When we substitute these last equations together with the equation (2.17) into the equation (3.7) we find

$$K_{\zeta+ue_\sigma}(e_\sigma,a)=\frac{\dfrac{\cosh^2\psi_0}{\|n\|^2}\left(\dfrac{1}{2}\dfrac{\partial^2 g}{\partial u_\sigma^2}-\dfrac{1}{4g}\left(\dfrac{\partial g}{\partial u_\sigma}\right)^2\right)}{1-\cosh^2\psi_\sigma}.$$

Since $\|n\|^2=-g$, we obtain

$$K_{\zeta+ue_\sigma}(e_\sigma,a)=\frac{\cosh^2\psi_0\left(-\dfrac{1}{2g}\dfrac{\partial^2 g}{\partial u_\sigma^2}+\dfrac{1}{4g^2}\left(\dfrac{\partial g}{\partial u_\sigma}\right)^2\right)}{1-\cosh^2\psi_\sigma}$$

Considering equation (25) in ref. [5] we find the relation between curvature of the $\sigma^{th}$ spacelike principal section $(e_\sigma,n)$ and curvature of timelike section $(e_\sigma,a)$.

**ii.** Similarly, at the point $(\zeta+ue_s)\in M$, the timelike sectional curvature is given by

$$K_{\zeta+ue_s}(e_s,a)=\frac{\beta_s\beta_s\lambda_0\lambda_0R_{s0s0}}{\langle e_s,e_s\rangle\langle a,a\rangle-\langle e_s,a\rangle^2}\quad, 1\le s\le m,\ s\ne\sigma. \tag{3.8}$$

Substituting the equation (2.17) into the equation (3.8) and considering that $\|n\|^2=-g$ we obtain

$$\left(1+\sinh^2\psi_s\right)K_{\zeta+ue_s}\left(e_s,a\right)=\cosh^2\psi_0\left(\frac{1}{2g}\frac{\partial^2 g}{\partial u_s^2}-\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_s}\right)^2\right).$$

Therefore, taking the equation (26) in ref. [5] into consideration we get the relation between curvature of the $s^{th}$ timelike principal section $\left(e_s,n\right)$ and curvature of timelike section $\left(e_s,a\right)$.

**Theorem 3.2:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and with spacelike central ruled surface in $n-$ dimensional Minkowski space $\mathbb{R}_1^n$. $a$ and $e_s$, $1\le s\le m$, be any timelike unit vector of $M$ and timelike base vector within subspace $F_m\left(t\right)$, respectively. Given any timelike unit vector $a$ which is linearly independent with $e_\sigma$, $1\le\sigma\le m$, $\sigma\ne s$, at the point $\left(\zeta+ue_\sigma\right)\in M$, there exists the following relation between curvature of timelike section $\left(e_\sigma,a\right)$ and curvature of the $\sigma^{th}$ principal spacelike section $\left(e_\sigma,n\right)$*

$$\left(1+\cosh^2\psi_\sigma\right)K_{\zeta+ue_\sigma}\left(e_\sigma,a\right)=-\sinh^2\psi_0 K_{\zeta+ue_\sigma}\left(e_\sigma,n\right).$$

*So that the angles $\psi_0$ and $\psi_\sigma$ are the hyperbolic angles between $a$ and $n$ and between $a$ and $e_\sigma$, respectively.*

**Proof:** At the point $\left(\zeta+ue_\sigma\right)\in M$ the curvature of timelike section $\left(e_\sigma,a\right)$ becomes

$$K_{\zeta+ue_\sigma}\left(e_\sigma,a\right)=\frac{\beta_\sigma\beta_\sigma\lambda_0\lambda_0R_{\sigma0\sigma0}}{\left\langle e_\sigma,e_\sigma\right\rangle\left\langle a,a\right\rangle-\left\langle e_\sigma,a\right\rangle^2}\quad,\quad 1\le\sigma\le m\quad,\ \sigma\ne s., \tag{3.9}$$

The coordinates of the $\nu^{\text{th}}$ base vector $e_\nu$, $1\le\nu\le k$, and timelike unit vector $a$ given by equation (3.4) are $\left(\beta_0,\beta_1,\ldots,\beta_\nu,\ldots,\beta_k\right)$ and $\left(\gamma_0,\gamma_1,\ldots,\gamma_s,\ldots,\gamma_k\right)$, and we write the following equations

$$\beta_0=\left\langle e_\nu,e_0\right\rangle=0\qquad,\quad \beta_\nu=\left\langle e_\nu,e_\nu\right\rangle=\varepsilon_\nu\qquad,\quad 1\le\nu\le k,$$

and

$$\gamma_0=\left\langle a,e_0\right\rangle=\frac{\sinh\psi_0}{\|n\|}\quad,\quad \gamma_\nu=\left\langle a,e_\nu\right\rangle=\sinh\psi_\nu\quad,\quad 1\le\nu\le k\quad,\quad \nu\ne s,$$

$$\gamma_s=\left\langle a,e_s\right\rangle=\cosh\psi_s\quad,\quad 1\le s\le m.$$

Substituting these equations together with the equation (2.17) into the equation (3.9) and considering that $\|n\|^2 = -g$ , we reach

$$\left(-1-\cosh^2\psi_\sigma\right)K_{\zeta+ue_\sigma}\left(e_\sigma,a\right)=\sinh^2\psi_0\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2\right).$$

We obtain, therefore, a relation between curvature of spacelike $\sigma^{\text{th}}$ principal section $\left(e_\sigma,n\right)$, $1\le\sigma\le m$, $\sigma\ne s$, given by the equation (25) in ref. [5] and the curvature of timelike section $\left(e_\sigma,a\right)$ and this completes the proof.

**Theorem 3.3:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and with spacelike central ruled surface in $n-$ dimensional Minkowski space $\mathbb{R}_1^n$. $a$ and $e_{m+s}$, $1\le s\le k-m$, be any spacelike unit vector of $M$ and timelike base vector within the central space $Z_{k-m}\left(t\right)$, respectively. Given the spacelike unit vector $a$ which is linearly independent of spacelike base vector $e_\sigma$, $1\le\sigma\le m$, at the point $\left(\zeta+ue_\sigma\right)\in M$, there exist the following relation between the curvatures of timelike section $\left(e_\sigma,a\right)$ and the $\sigma^{th}$ spacelike principal section $\left(e_\sigma,n\right)$*

$$\left(1-\cosh^2\psi_\sigma\right)K_{\zeta+ue_\sigma}\left(e_\sigma,a\right)=\cosh^2\psi_0 K_{\zeta+ue_\sigma}\left(e_\sigma,n\right)$$

*where the hyperbolic angles between $a$ and $n$, between $a$ and $e_\sigma$ are $\psi_0$ and $\psi_\sigma$, respectively.*

**Proof:** At the point $\left(\zeta+ue_\sigma\right)\in M$ the curvature of timelike section $\left(e_\sigma,a\right)$, $1\le\sigma\le m$, is given by

$$K_{\zeta+ue_\sigma}\left(e_\sigma,a\right)=\frac{\beta_\sigma\beta_\sigma\lambda_0\lambda_0R_{\sigma0\sigma0}}{\left\langle e_\sigma,e_\sigma\right\rangle\left\langle a,a\right\rangle-\left\langle e_\sigma,a\right\rangle^2}\quad,\quad 1\le\sigma\le m. \tag{3.10}$$

The coordinates of the $\nu^{\text{th}}$ base vector $e_\nu$, $1\le\nu\le k$, and spacelike unit vector are $\left(\beta_0,\beta_1,\ldots,\beta_\nu,\ldots,\beta_k\right)$ and $\left(\gamma_0,\gamma_1,\ldots,\gamma_{m+s},\ldots,\gamma_k\right)$, respectively so that

$$\beta_0=\left\langle e_\nu,e_0\right\rangle=0\quad,\quad\beta_\nu=\left\langle e_\nu,e_\nu\right\rangle=\varepsilon_\nu\quad,\quad 1\le\nu\le k,$$

and

$$\gamma_0=\left\langle a,e_0\right\rangle=\frac{\cosh\psi_0}{\|n\|}\quad,\quad\gamma_\nu=\left\langle a,e_\nu\right\rangle=\cosh\psi_\nu\quad,\quad 1\le\nu\le k\ ,\ \nu\ne m+s,$$

$$\gamma_{m+s}=\left\langle a,e_{m+s}\right\rangle=\sinh\psi_{m+s}\quad,\quad 1\le s\le k-m.$$

When these equations together with the equation (2.17) are substituted into the equation (3.10) one finds

$$K_{\zeta+ue_\sigma}(e_\sigma,a)=\frac{\dfrac{\cosh^2\psi_0}{\|n\|^2}\left(\dfrac{1}{2}\dfrac{\partial^2 g}{\partial u_\sigma^2}-\dfrac{1}{4g}\left(\dfrac{\partial g}{\partial u_\sigma}\right)^2\right)}{1-\cosh^2\psi_\sigma}.$$

Considering the last equation together with that $\|n\|^2=-g$ yields us

$$\left(1-\cosh^2\psi_\sigma\right)K_{\zeta+ue_\sigma}(e_\sigma,a)=\cosh^2\psi_0\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2\right).$$

Therefore, we obtain the relation between curvature of spacelike $\sigma^{\text{th}}$ principal section $(e_\sigma,n)$, $1\le\sigma\le m$, given by the equation (25) in ref. [5] and the curvature of timelike section $(e_\sigma,a)$ and this completes the proof.

**Theorem 3.4:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and with spacelike central ruled surface in $n-$ dimensional Minkowski space $\mathbb{R}_1^n$, $a$ be arbitrary timelike unit vector of $M$ and $e_{m+s}$, $1\le s\le k-m$, be a timelike base vector in central space $Z_{k-m}(t)$, respectively. Giving a spacelike vector $e_\sigma$, $1\le\sigma\le m$, and linearly independent with timelike unit vector $a$ there exists a relation at the point $(\zeta+ue_\sigma)\in M$,*

$$\left(1+\cosh^2\psi_\sigma\right)K_{\zeta+ue_\sigma}(e_\sigma,a)=-\sinh^2\psi_0 K_{\zeta+ue_\sigma}(e_\sigma,n)$$

*between the curvatures of timelike section $(e_\sigma,a)$ and the $\sigma^{th}$ spacelike principal section $(e_\sigma,n)$. Here $\psi_0$ and $\psi_\sigma$ are the hyperbolic angles between $a$, $n$, and $a$, $e_\sigma$, $1\le\sigma\le m$, respectively.*

**Proof:** Taking $1\le\sigma\le m$, the sectional curvature at the point $(\zeta+ue_\sigma)\in M$ is given by

$$K_{\zeta+ue_\sigma}(e_\sigma,a)=\frac{\beta_\sigma\beta_\sigma\lambda_0\lambda_0R_{\sigma0\sigma0}}{\langle e_\sigma,e_\sigma\rangle\langle a,a\rangle-\langle e_\sigma,a\rangle^2}\quad,\quad 1\le\sigma\le m. \tag{3.11}$$

If the coordinates of the $\nu^{\text{th}}$ base vector $e_\nu$, $1\le\nu\le k$, and the timelike unit vector given by the equation (3.6) are $(\beta_0,\beta_1,\dots,\beta_\nu,\dots,\beta_k)$ and $(\gamma_0,\gamma_1,\dots,\gamma_{m+s},\dots,\gamma_k)$, respectively, then we may write

$$\beta_0=\langle e_\nu,e_0\rangle=0\qquad,\quad \beta_\nu=\langle e_\nu,e_\nu\rangle=\varepsilon_\nu\qquad,\quad 1\le\nu\le k$$

and

$$\gamma_0 = \langle a, e_0 \rangle = \frac{\sinh \psi_0}{\|n\|} \quad , \quad \gamma_\nu = \langle a, e_\nu \rangle = \sinh \psi_\nu \quad , \quad 1 \le \nu \le k \ , \nu \ne m+s,$$

$$\gamma_{m+s} = \langle a, e_{m+s} \rangle = \cosh \psi_{m+s} , \quad 1 \le s \le k-m.$$

Substituting the last equations and the equation (2.17) into the equation (3.11) and considering that $\|n\|^2 = -g$ we reach that

$$\left(-1-\cosh^2 \psi_\sigma\right) K_{\zeta+ue_\sigma}\left(e_\sigma, a\right) = \sinh^2 \psi_0 \left( -\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2 \right).$$

Taking the equation (25) in ref. [5] into consideration, the relation between curvature of the $\sigma^{\text{th}}$ principal spacelike section $\left(e_\sigma, n\right)$, $1 \le \sigma \le m$, and curvature of timelike section $\left(e_\sigma, a\right)$ is found and this completes the proof.

Taking vector $e$ given by equations (30), (31), (32) and (33) in ref. [5] to be the unit vector in $E_k\left(t\right)$ and unit vector $a$ given by equations (3.3), (3.4), (3.5) and (3.6) to be a unit vector at the central point $\forall \zeta \in \Omega$, the following theorems related to the sectional curvatures of $M$ could be given.

**Theorem 3.5:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and with spacelike central ruled surface in $n-$ dimensional Minkowski space $\mathbb{R}_1^n$, taking vector $a$, which is linearly independent with spacelike unit vector $e$ in generating space $E_k\left(t\right)$ of $M$, to be spacelike unit vector at the central point $\forall \zeta \in \Omega$, the following relation holds between curvature of nondegenerate section $\left(e, a\right)$ of $M$ and curvature of spacelike section $\left(e, n\right)$*

$$K_\zeta\left(e, a\right) = \frac{\cosh^2 \psi_0}{1-\langle e, a\rangle^2} K_\zeta\left(e, n\right)$$

*where $\psi_0$ is the angle between spacelike unit vector $a$ and spacelike normal tangent vector $n$.*

**Proof:** Considering the equation (3.1) the sectional curvature at the central point $\zeta \in \Omega$ is given by

$$K_\zeta\left(e, a\right) = \frac{\sum\limits_{\substack{\sigma=1 \\ \sigma \ne s}}^{m} \beta_\sigma \beta_\sigma \lambda_0 \lambda_0 R_{\sigma 0 \sigma 0} + \beta_s \beta_s \lambda_0 \lambda_0 R_{s0s0}}{\langle e, e\rangle \langle a, a\rangle - \langle e, a\rangle^2}. \tag{3.12}$$

If the coordinates of $e$ given by equation (30) in ref. [5] and tangent vector $a$ given by equation (3.3) $(\beta_0, \beta_1, \ldots, \beta_s, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_s, \ldots, \gamma_k)$, respectively, then we may write

$$\begin{aligned} \beta_0 &= \langle e, e_0 \rangle = 0, \\ \beta_s &= \langle e, e_s \rangle = \sinh \theta_s \quad , \quad 1 \le s \le m \\ \beta_\nu &= \langle e, e_\nu \rangle = \cosh \theta_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne s \end{aligned}$$

and

$$\begin{aligned} \gamma_0 &= \langle a, e_0 \rangle = \frac{\cosh \psi_0}{\|n\|}, \\ \gamma_s &= \langle a, e_s \rangle = \sinh \psi_s \quad , \quad 1 \le s \le m \\ \gamma_\nu &= \langle a, e_\nu \rangle = \cosh \psi_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne s. \end{aligned}$$

Substituting the last equations together with the equation (2.17) into the equation (3.12) we get

$$K_\zeta(e,a) = \frac{\dfrac{\cosh^2 \psi_0}{\|n\|^2}\left(\sum_{\sigma=1}^{m} \cosh^2 \theta_\sigma \left(\dfrac{1}{2}\dfrac{\partial^2 g}{\partial u_\sigma^2} - \dfrac{1}{4g}\left(\dfrac{\partial g}{\partial u_\sigma}\right)^2\right) + \sinh^2 \theta_s \left(\dfrac{1}{2}\dfrac{\partial^2 g}{\partial u_s^2} - \dfrac{1}{4g}\left(\dfrac{\partial g}{\partial u_s}\right)^2\right)\right)}{1 - \langle e, a \rangle^2}.$$

Considering the last equation and $\|n\|^2 = -g$ we reach

$$K_\zeta(e,a) = \frac{\cosh^2 \psi_0 \left(\sum_{\sigma=1}^{m} \cosh^2 \theta_\sigma \left(-\dfrac{1}{2g}\dfrac{\partial^2 g}{\partial u_\sigma^2} + \dfrac{1}{4g^2}\left(\dfrac{\partial g}{\partial u_\sigma}\right)^2\right) + \sinh^2 \theta_s \left(-\dfrac{1}{2g}\dfrac{\partial^2 g}{\partial u_s^2} + \dfrac{1}{4g^2}\left(\dfrac{\partial g}{\partial u_s}\right)^2\right)\right)}{1 - \langle e, a \rangle^2}.$$

Considering curvature of the $\sigma^{th}$ spacelike principal section $(e_\sigma, n)$, $1 \le \sigma \le m$, $\sigma \ne s$, given by the equation (25) in ref. [5] and the curvature of the $s^{th}$ timelike principal section $(e_s, n)$, $1 \le s \le m$, given by the equation (26) in ref. [5] at the central point $\zeta \in \Omega$ we find

$$K_\zeta(e,a) = \frac{\cosh^2 \psi_0 \left(\sum_{\sigma=1}^{m} \cosh^2 \theta_\sigma K_\zeta(e_\sigma, a) - \sinh^2 \theta_s K_\zeta(e_s, a)\right)}{1 - \langle e, a \rangle^2}.$$

If we take I. type Lorentzian Beltrami-Euler formula given by theorem in ref. [5] we complete the proof.

**Theorem 3.6:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and spacelike central ruled surface in $n-$ dimensional Minkowski space $\mathbb{R}_1^n$ and $a$ be a timelike unit vector which is linearly independent with spacelike unit vector $e$ in generating space $E_k(t)$ of $M$. The relationship between curvature of timelike section $(e,a)$ and curvature of spacelike section $(e,n)$ of $M$ at $\forall \zeta \in \Omega$ is*

$$K_\zeta(e,a) = -\frac{\sinh^2 \psi_0}{1+\langle e,a\rangle^2} K_\zeta(e,n)$$

*where $\psi_0$ is an angle between timelike vector $a$ and spacelike normal tangent vector $n$.*

**Proof:** At the point $\zeta \in \Omega$ the sectional curvature is given by

$$K_\zeta(e,a) = \frac{\sum\limits_{\substack{\sigma=1 \\ \sigma\neq s}}^{m} \beta_\sigma \beta_\sigma \lambda_0 \lambda_0 R_{\sigma 0 \sigma 0} + \beta_s \beta_s \lambda_0 \lambda_0 R_{s0s0}}{\langle e,e\rangle\langle a,a\rangle - \langle e,a\rangle^2}. \tag{3.13}$$

If the coordinates of $e$ given by equation (30) in ref. [5] and the unit vector $a$ given by equation (3.4) are $(\beta_0, \beta_1, \ldots, \beta_s, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_s, \ldots, \gamma_k)$, respectively, then we obtain

$$\begin{aligned}
\beta_0 &= \langle e, e_0\rangle = 0, \\
\beta_s &= \langle e, e_s\rangle = \sinh\theta_s \quad , \quad 1\le s\le m \\
\beta_\nu &= \langle e, e_\nu\rangle = \cosh\theta_\nu \quad , \quad 1\le \nu\le k \quad , \quad \nu\neq s,
\end{aligned}$$

and

$$\begin{aligned}
\gamma_0 &= \langle a, e_0\rangle = \frac{\sinh\psi_0}{\|n\|}, \\
\gamma_s &= \langle a, e_s\rangle = \cosh\psi_s \quad , \quad 1\le s\le m \\
\gamma_\nu &= \langle a, e_\nu\rangle = \sinh\psi_\nu \quad , \quad 1\le \nu\le k \quad , \quad \nu\neq s.
\end{aligned}$$

Substituting the last equations together with equation (2.17) into equation (3.13) and considering that $\|n\|^2 = -g$ we reach

$$K_\zeta(e,a) = -\frac{\sinh^2\psi_0\left(\sum_{\sigma=1}^{m}\cosh^2\theta_\sigma\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2\right)+\sinh^2\theta_s\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_s^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_s}\right)^2\right)\right)}{1+\langle e,a\rangle^2}.$$

Considering the curvature of the $\sigma^{th}$ spacelike principal section $(e_\sigma, n)$, $1 \le \sigma \le m$, $\sigma \ne s$, which is given by equation (25) in ref. [5], and curvature of the $s^{th}$ timelike principal section $(e_s, n)$, $1 \le s \le m$, is given by (26) in ref. [5], into the last equation at the point $\zeta \in \Omega$, we get

$$K_\zeta(e,a) = -\frac{\sinh^2\psi_0\left(\sum_{\sigma=1}^{m}\cosh^2\theta_\sigma K_\zeta(e_\sigma,a)-\sinh^2\theta_s K_\zeta(e_s,a)\right)}{1+\langle e,a\rangle^2}.$$

The last equation together with the I. type Lorentzian Beltrami-Euler formula completes the proof.

**Theorem 3.7:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and spacelike central ruled surface in $n-$ dimensional Minkowski space $\mathbb{R}_1^n$. If we suppose the vector $a$, which is linearly independent with timelike unit vector $e$ in the generating space $E_k(t)$ to be a spacelike unit vector at the point $\forall\zeta \in \Omega$ there exists the following relation between curvature of timelike section $(e,a)$ and curvature of timelike section $(e,n)$ of $M$*

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0}{1+\langle e,a\rangle^2}K_\zeta(e,n)$$

*where $\psi_0$ is an angle between spacelike vector $a$ and spacelike normal tangent vector $n$.*

**Proof:** At the central point $\zeta \in \Omega$, the curvature of the timelike section $(e,a)$ is given by

$$K_\zeta(e,a) = \frac{\sum_{\substack{\sigma=1\\ \sigma\ne s}}^{m}\beta_\sigma\beta_\sigma\lambda_0\lambda_0 R_{\sigma0\sigma0}+\beta_s\beta_s\lambda_0\lambda_0 R_{s0s0}}{\langle e,e\rangle\langle a,a\rangle-\langle e,a\rangle^2}. \tag{3.14}$$

If the coordinates of $e$ given by equation (30) in ref. [5] and the tangent vector $a$ are $(\beta_0, \beta_1, \ldots, \beta_s, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_s, \ldots, \gamma_k)$, respectively, then we get

$$\beta_0 = \langle e, e_0 \rangle = 0,$$
$$\beta_s = \langle e, e_s \rangle = \cosh\theta_s \quad , \quad 1 \le s \le m,$$
$$\beta_\nu = \langle e, e_\nu \rangle = \sinh\theta_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne s,$$

and

$$\gamma_0 = \langle a, e_0 \rangle = \frac{\cosh\psi_0}{\|n\|},$$
$$\gamma_s = \langle a, e_s \rangle = \sinh\psi_s \quad , \quad 1 \le s \le m,$$
$$\gamma_\nu = \langle a, e_\nu \rangle = \cosh\psi_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne s.$$

Substituting the last equations together with the equation (2.17) into the equation (3.14) and considering that $\|n\|^2 = -g$ we find

$$K_\zeta(e,a) = -\frac{\cosh^2\psi_0\left(\sum_{\sigma=1}^{m}\sinh^2\theta_\sigma\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2\right)+\cosh^2\theta_s\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_s^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_s}\right)^2\right)\right)}{1+\langle e,a\rangle^2}.$$

If we substitute the curvature of the $\sigma^{th}$ spacelike principal section $(e_\sigma, n)$, $1 \le \sigma \le m$, $\sigma \ne s$, given by equation (25) in ref. [5], and the curvature of the $s^{th}$ timelike principal section $(e_s, n)$, $1 \le s \le m$, given by equation (26) in ref. [5] into the last equation at the point $\zeta \in \Omega$, we get

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0\left(-\sum_{\sigma=1}^{m}\sinh^2\theta_\sigma K_\zeta(e_\sigma,a)+\cosh^2\theta_s K_\zeta(e_s,a)\right)}{1+\langle e,a\rangle^2}.$$

Considering the II. type Lorentzian Beltrami-Euler formula given by Theorem 4.7 in ref. [5] completes the proof.

**Theorem 3.8:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and timelike central ruled surface in $n-$dimensional Minkowski space $\mathbb{R}_1^n$. Suggesting the vector $a$, which is independent with spacelike unit vector $e$ in the generating space $E_k(t)$, to be a spacelike unit vector at the point $\forall\zeta \in \Omega$, there exists the following relation between curvature of nondegenerate section $(e,a)$ and curvature of spacelike section $(e,n)$ of $M$*

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0}{1-\langle e,a\rangle^2}K_\zeta(e,n)$$

*so that* $\psi_0$ *denotes the angle between spacelike unit vector* $a$ *and spacelike normal tangential vector* $n$.

**Proof:** The curvature of the spacelike section $(e,a)$ at the point $\zeta \in \Omega$ is

$$K_\zeta(e,a) = \frac{\sum_{\sigma=1}^{m} \beta_\sigma \beta_\sigma \lambda_0 \lambda_0 R_{\sigma 0 \sigma 0}}{\langle e,e\rangle\langle a,a\rangle - \langle e,a\rangle^2}. \qquad (3.15)$$

If the coordinates of $e$ from the equation (32) in [5] and the tangent vector $a$ given by equation (3.5) are $(\beta_0, \beta_1, \ldots, \beta_{m+s}, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_{m+s}, \ldots, \gamma_k)$, respectively, we find

$$\begin{aligned}
\beta_0 &= \langle e, e_0\rangle = 0,\\
\beta_{m+s} &= \langle e, e_{m+s}\rangle = \sinh\theta_{m+s}, \quad 1 \le s \le k-m,\\
\beta_\nu &= \langle e, e_\nu\rangle = \cosh\theta_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne m+s,
\end{aligned}$$

and

$$\begin{aligned}
\gamma_0 &= \langle a, e_0\rangle = \frac{\cosh\psi_0}{\|n\|},\\
\gamma_{m+s} &= \langle a, e_{m+s}\rangle = \sinh\psi_{m+s}, \quad 1 \le s \le k-m,\\
\gamma_\nu &= \langle a, e_\nu\rangle = \cosh\psi_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne m+s.
\end{aligned}$$

Considering that $\|n\|^2 = -g$ and substituting the last equations together with equation (2.17) into the equation (3.15) yields

$$K_\zeta(e,a) = \frac{\frac{\cosh^2\psi_0}{\|n\|^2} \sum_{\sigma=1}^{m} \cosh^2\theta_\sigma \left( \frac{1}{2}\frac{\partial^2 g}{\partial u_\sigma^2} - \frac{1}{4g}\left(\frac{\partial g}{\partial u_\sigma}\right)^2 \right)}{1 - \langle e,a\rangle^2}.$$

Considering that the curvature of spacelike $\sigma^{\text{th}}$ principal section $(e_\sigma, n)$, $1 \le \sigma \le m$, given by equation (25) in ref. [5] at the point $\zeta \in \Omega$, we obtain

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0 \sum_{\sigma=1}^{m} \cosh^2\theta_\sigma K_\zeta(e_\sigma, a)}{1 - \langle e,a\rangle^2}$$

From III. type Lorentzian Beltrami-Euler formula, we see that there exists the relation between the curvature of spacelike section $(e,a)$ and curvature of spacelike section $(e,n)$ and this completes the proof.

**Theorem 3.9:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and timelike central ruled surface in $n-$dimensional Minkowski space $\mathbb{R}_1^n$. Considering that the vector $a$, which is independent with spacelike unit vector $e$ in the generating space $E_k(t)$, to be a timelike unit vector at the point $\forall\zeta\in\Omega$, there exists the following relation between the curvature of timelike section $(e,a)$ and curvature of spacelike section $(e,n)$ of $M$, such that*

$$K_\zeta(e,a)=-\frac{\sinh^2\psi_0}{1+\langle e,a\rangle^2}K_\zeta(e,n)$$

*where $\psi_0$ is the angle between timelike unit vector $a$ and spacelike normal tangential vector $n$.*

**Proof:** The curvature of timelike section $(e,a)$ at the point $\zeta\in\Omega$ is given by

$$K_\zeta(e,a)=\frac{\sum_{\sigma=1}^{m}\beta_\sigma\beta_\sigma\lambda_0\lambda_0R_{\sigma0\sigma0}}{\langle e,e\rangle\langle a,a\rangle-\langle e,a\rangle^2}. \tag{3.16}$$

Taking the coordinates of $e$ from equation (32) in ref. [5] and the tangent vector $a$ to be $(\beta_0,\beta_1,\ldots,\beta_{m+s},\ldots,\beta_k)$ and $(\gamma_0,\gamma_1,\ldots,\gamma_{m+s},\ldots,\gamma_k)$, respectively, we reach

$$\begin{aligned}\beta_0&=\langle e,e_0\rangle=0,\\ \beta_{m+s}&=\langle e,e_{m+s}\rangle=\sinh\theta_{m+s},\quad 1\le s\le k-m,\\ \beta_\nu&=\langle e,e_\nu\rangle=\cosh\theta_\nu\quad,\quad 1\le\nu\le k\quad,\quad \nu\ne m+s,\end{aligned}$$

and

$$\begin{aligned}\gamma_0&=\langle a,e_0\rangle=\frac{\sinh\psi_0}{\|n\|},\\ \gamma_{m+s}&=\langle a,e_{m+s}\rangle=\cosh\psi_{m+s},\quad 1\le s\le k-m,\\ \gamma_\nu&=\langle a,e_\nu\rangle=\sinh\psi_\nu\quad,\quad 1\le\nu\le k\quad,\quad \nu\ne m+s.\end{aligned}$$

Considering $\|n\|^2=-g$ and substituting the last equations together with the equation (2.17) into the equation (3.16), we find

$$K_\zeta(e,a) = -\frac{\sinh^2\psi_0 \sum_{\sigma=1}^{m} \cosh^2\theta_\sigma \left( -\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2 \right)}{1+\langle e,a\rangle^2}.$$

If we consider the curvature of the $\sigma^{th}$ spacelike principal section $(e_\sigma, n)$, $1 \le \sigma \le m$, given by equation (25) in ref. [5], we find

$$K_\zeta(e,a) = -\frac{\sinh^2\psi_0 \sum_{\sigma=1}^{m} \cosh^2\theta_\sigma K_\zeta(e_\sigma, a)}{1+\langle e,a\rangle^2}.$$

Considering the curvature of spacelike section $(e,n)$, which is given by III. type Lorentzian Beltrami-Euler formula in ref. [5] completes the proof.

**Theorem 3.10** *Let $M$ be a generalized timelike ruled surface with timelike generating space and timelike central ruled surface in $n-$dimensional Minkowski space $\mathbb{R}_1^n$. Considering that the vector $a$, which is independent with timelike unit vector $e$ in the generating space $E_k(t)$ of $M$, to be a timelike unit vector at the central point $\forall\zeta \in \Omega$, there exists the following relation between curvature of timelike section $(e,a)$ and curvature of timelike section $(e,n)$*

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0}{1+\langle e,a\rangle^2} K_\zeta(e,n)$$

*Where $\psi_0$ denotes the angle between spacelike unit vector $a$ and spacelike normal tangential vector $n$.*

**Proof:** The curvature of timelike section $(e,a)$ at the central point $\zeta \in \Omega$ is given by

$$K_\zeta(e,a) = \frac{\sum_{\sigma=1}^{m} \beta_\sigma\beta_\sigma\lambda_0\lambda_0 R_{\sigma 0\sigma 0}}{\langle e,e\rangle\langle a,a\rangle - \langle e,a\rangle^2}. \tag{3.17}$$

If the coordinates of tangent vector $a$ given by equation (3.5) and $e$ given by (33) in ref. [5] are $(\beta_0, \beta_1, \ldots, \beta_{m+s}, \ldots, \beta_k)$ and $(\gamma_0, \gamma_1, \ldots, \gamma_{m+s}, \ldots, \gamma_k)$, respectively, then we write

$$\beta_0 = \langle e, e_0 \rangle = 0,$$
$$\beta_{m+s} = \langle e, e_{m+s} \rangle = \cosh \theta_{m+s}, \quad 1 \le s \le k-m,$$
$$\beta_\nu = \langle e, e_\nu \rangle = \sinh \theta_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne m+s,$$

and

$$\gamma_0 = \langle a, e_0 \rangle = \frac{\cosh \psi_0}{\|n\|},$$
$$\gamma_{m+s} = \langle a, e_{m+s} \rangle = \sinh \psi_{m+s}, \quad 1 \le s \le k-m,$$
$$\gamma_\nu = \langle a, e_\nu \rangle = \cosh \psi_\nu \quad , \quad 1 \le \nu \le k \quad , \quad \nu \ne m+s.$$

Considering that $\|n\|^2 = -g$ and substituting the equation (2.17) together with the last equations into the (3.17) we reach

$$K_\zeta(e,a) = -\frac{\cosh^2 \psi_0 \sum_{\sigma=1}^{m} \sinh^2 \theta_\sigma \left( -\frac{1}{2g}\frac{\partial^2 g}{\partial u_\sigma^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_\sigma}\right)^2 \right)}{1 + \langle e, a \rangle^2}.$$

If we use the curvature of the $\sigma^{\text{th}}$ spacelike principal section $(e_\sigma, n)$, $1 \le \sigma \le m$, given by the equation (25) in ref. [5] at the central point $\zeta \in \Omega$ in the last equation we see that

$$K_\zeta(e,a) = -\frac{\cosh^2 \psi_0 \sum_{\sigma=1}^{m} \sinh^2 \theta_\sigma K_\zeta(e_\sigma, a)}{1 + \langle e, a \rangle^2}.$$

Considering the IV. type Lorentzian Beltrami-Euler formula in Theorem 4.9, [5] completes the proof.

From the last six theorems it can be easily seen that when the central ruled surface of $M$ is either spacelike or timelike, the relations obtained are seemed to be same. Therefore we can give the following corollaries, respectively.

From Theorems 3.6 and 3.8 we give the following corollary.

**Corollary 3.11:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and (spacelike or timelike) central ruled surface in $n-$dimensional Minkowski space $\mathbb{R}_1^n$. The vector $a$ is the spacelike unit vector, which is linearly independent with spacelike unit vector $e$ in $E_k(t)$ of $M$ at the central point $\forall \zeta \in \Omega$. The curvature of spacelike section $(e,a)$ of $M$ depends on the vectors $e$ and $a$, Lorentzian Beltrami-Euler formula for the spacelike section*

*$(e,n)$ and the angle $\psi_0 = \sphericalangle(a,n)$. In this case, there exists the following relation between the curvature of spacelike section $(e,a)$ and curvature of spacelike section $(e,n)$ of $M$*

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0}{1-\langle e,a\rangle^2} K_\zeta(e,n).$$

*This equation is called* ***I. type Lorentzian Beltrami-Meusnier formula*** *for sectional curvature of generalized timelike ruled surface with timelike generating space at the central point.*

From Theorems 3.7 and 3.10 we give the following corollary.

**Corollary 3.12:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and (spacelike or timelike) central ruled surface in $n-$dimensional Minkowski space $\mathbb{R}_1^n$. The vector $a$ be a timelike unit vector, which is linearly independent with spacelike unit vector $e$ in $E_k(t)$ of $M$ at the central point $\forall\zeta\in\Omega$. In this case, the following relation holds between the curvature of timelike section $(e,a)$ and curvature of spacelike section $(e,n)$*

$$K_\zeta(e,a) = -\frac{\sinh^2\psi_0}{1+\langle e,a\rangle^2} K_\zeta(e,n)$$

*and this equation is called* ***II. type Lorentzian Beltrami-Meusnier formula*** *for sectional curvature of generalized timelike ruled surface with timelike generating space at the central point. Here, any curvature of timelike section $(e,a)$ of $M$ depends on the vectors $e$ and $a$, Lorentzian Beltrami-Euler formula for the spacelike section $(e,n)$ and the angle $\psi_0 = \sphericalangle(a,n)$.*

We can give the following corollary from Theorem 3.7 and 3.10.

**Corollary 3.13:** *Let $M$ be a generalized timelike ruled surface with timelike generating space and (spacelike or timelike) central ruled surface in $n-$dimensional Minkowski space $\mathbb{R}_1^n$. We also suppose that the vector $a$ is a spacelike unit vector, which is linearly independent with timelike unit vector in $E_k(t)$ of $M$. In this special case any curvature of timelike section $(e,a)$ of $M$ depends on the vectors $e$ and $a$, Lorentzian Beltrami-Euler formula for the timelike section $(e,n)$ and the angle $\psi_0 = \sphericalangle(a,n)$. The relation*

$$K_\zeta(e,a) = \frac{\cosh^2 \psi_0}{1+\langle e,a\rangle^2} K_\zeta(e,n)$$

*holds between the curvature of timelike section* $(e,a)$ *and curvature of timelike section* $(e,n)$ *and this equation is named* ***III. type Lorentzian Beltrami-Meusnier formula*** *for sectional curvature of generalized timelike ruled surface with timelike generating space at the central point.*

## References


[1] I. Aydemir and N. Kuruoğlu, Edge, center and principal ruled surfaces of ($k$+1)-dimensional generalized time-like ruled surface in the Minkowski space $\mathbb{R}_1^n$, *Pure Appl. Math. Sci.*, 51(1-2) (2000), 19-24.

[2] I. Aydemir and N. Kuruoğlu, Time-like ruled surfaces in the Minkowski space $\mathbb{R}_1^n$, *Int. J. Appl. Math.*, 10(2) (2002), 149-158.

[3] W. Blaschke, *Vorlesungen über Differential Geometrie*, 14-Ayfl. Berlin, (1945).

[4] J.K. Beem, P.E. Ehrlich and K.L. Easley, *Global Lorentzian Geometry*, Marcel Dekker, New York, (1981).

[5] S. Ersoy and M. Tosun, Sectional curvature of timelike ruled surface part I: Lorentzian Beltrami-Euler formula, *Iran. J. Sci. Technol. Trans. A Sci.*, 34(A3) (2010), 197-214.

[6] H. Frank and O. Giering, Zur schnittkrümmung verallgemeinerter regelflachen, *Archiv Der Mathematik*, Fasc., 1(32) (1979), 86-90.

[7] E. Kruppa, *Analytische und Konstruktive Differential Geometrie*, Wien Springer-Verlag, (1957).

[8] B. O'Neill, *Semi-Riemannian Geometry*, Academic Press, New York, (1983).

[9] J.G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Department of Mathematics, Vanderbilt University, (1994).