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# Randomly Weighted Sums of Conditionnally Dependent Random Variables

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**Abstract**

*In this paper we study the asymptotic behavior for the tail probability of the randomly weighted sums and their maximum where the usual assumption of independence of random variable $X$ and indepedence between the variable $X$ and the weighted $\theta$ are relaxed. We suppose that, the variable $X$ follows a conditional dependence intoduced by Geluk and Tang [6] and the pair $(X, \theta)$ follows a certain dependence structure proposed by Asimit and Badescu [1]. This result appears as a direct extension of the results of [7],[11] and [10].*

**Keywords:** *Randomly weighted sums, Long-tailed distribution, Tail probability, Conditional dependence.*

## 1 Introduction

Let $X_1, \ldots, X_n$ be n real-valued random variables (r.v.s) with distribution functions (d.f.s) $F_1, \ldots, F_n$, and the random weights $\theta_1, \ldots, \theta_n$ are nonnegative and nondegenerate at zero r.v.s with d.f.s $G_1, \ldots, G_n$ respectively. In this paper, we discuss the tail probabilities of the randomly weighted sums

$$S_n^\theta = \sum_{k=1}^{n} \theta_k X_k, \tag{1}$$

and their maxima

$$M_n^\theta = \max_{1 \le m \le n} S_m^\theta. \tag{2}$$

The randomly weighted sums and their maximums are frequently encountered in various areas, especially in actuarial and economic situations. For example, in the actuarial context, the r.v.s $X_i$, $i = 1, \ldots, n$ can be interpreted as the liability risks, and the weights $\theta_i$, $i = 1, \ldots, n$ stand for the financial risks, such as the discount factors. Specifically, if we regard $X_i$ as the net loss, i.e. the total amount of premium incomes minus the total amount of claims for an insurance company during period $i$, then the sum $S_n^\theta$ is the discounted losses accumulated from time 0 to time $n$. $M_n^\theta$ is the maximal discounted net loss during the first $n$ periods.

Before discussing the asymptotic properties of probabilities $\mathbb{P}\left(S_n^\theta > x\right)$ and $\mathbb{P}\left(M_n^\theta > x\right)$ we first recall some concepts of heavy-tailed distributions.
We say that a distribution $F$ belongs to the long-tailed class, denoted by $F \in \mathcal{L}$, if $\overline{F}(x) > 0$ for all $c \in (-\infty, +\infty)$ and $\overline{F}(x+t) \sim \overline{F}(x)$ for any $t \in (-\infty, +\infty)$; belongs to the subexponential class, denoted by $F \in \mathcal{S}$, if $F \in \mathcal{L}$ and $\overline{F^{*2}}(x) \sim 2\bar{F}(x)$, where $F^{*2}$ is 2-fold convolution of $F$, belongs to the dominatedly-varying-tailed class, denoted $F \in \mathcal{D}$ if $\bar{F}(x) > 0$ for all $x \in (-\infty, +\infty)$ and $\overline{F}(x) = 0(\bar{F}(xt))$ for all (or some ) $t > 1$. It is known that $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$. For more details about heavy-tailed distributions and their applications, the reader is referred to [4].

In this paper, all limit relationships are for $x \to \infty$ unless mentioned otherwise. For two positive functions $a(.)$ and $b(.)$ satisfying $\limsup a(x)/b(x) = c$, we write $a(x) = 0(b(x))$ if $c < \infty$, write $a(x) \asymp b(x)$ if $a(x) = 0(b(x))$ and $b(x) = 0(a(x))$, write $a(x) \lesssim b(x)$ if $c \leq 1$, write $a(x) \sim b(x)$ if $a(x) \lesssim b(x)$ and $b(x) \lesssim a(x)$ and write $a(x) = o(b(x))$ if $c = 0$. We shall use the symbole $x^+ = \max\{x, O\}$ for a real number $x$ and $\overline{F} = 1 - F$.

The standing assumptions of this paper are that :
Firstly, for each $i = 1, \ldots, n$ the pair sequence $(X_i, \theta_i)$ follows the dependence structure defined in assumption 1. This dependence was introduced by Asimit and Badescu [1].
**Assumption 1**
For each fixed $i = 1, \ldots$ there exists a measurable function $h_i : [0, \infty) \longrightarrow (0, \infty)$ such that

$$P(X_i > x/\theta_i = t) \sim \overline{F}_i(x) h_i(t), \tag{3}$$

uniformly for $t \geq 0$, where the uniformity is understood as

$$\lim_{x\to\infty} \sup_{t\geq 0} \left| \frac{P(X_i > x/\theta_i = t)}{\bar{F}_i(x) h_i(t)} - 1 \right| = 0, \tag{4}$$

when $t$ is not a possible value of some $\theta_i$, the conditional probability in (3) is understood as unconditional and therefore $h_i(t) = 1$ for such $t$.

Some examples of the r.v.s satisfying dependence condition (3) can be found in [1] and [8].

Secondly, as for the conditional dependence structure among nonnegative r.v.s $X_1, \ldots, X_n$ we suppose this assumption introduced by Ko and Tang [7].
**Assumption 2**
for $n \geq 2$ and $D = [0, \infty)$ there exist some large constants $x_0 = x_0(n) > 0$ and $C = C(n)$ such that for every $2 \leq j \leq n$,

$$\limsup_{x_0 \leq t \leq x - x_0} \frac{\mathbb{P}(S_{j-1} > x - t / X_j = t)}{\mathbb{P}(S_{j-1} > x - t)} \leq C. \tag{5}$$

Our objective in this work is to establish the following result under Assumption1, Assumption 2.

$$\mathbb{P}\left(S_n^\theta > x\right) \sim \mathbb{P}\left(M_n^\theta > x\right) \sim \mathbb{P}\left(\max_{1 \leq i \leq n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\theta_i X_i\right). \tag{6}$$

The motivation behind our assumption is given by the fact that in the last years, several works we can cite [2],[9], establish the tail behavior of the randomly weighted sums by assuming the sequence $\{X_i, 1 \leq i \leq n\}$ to be independent, and some of them example [5],[6] even require the mutually independent between $\{X_i, 1 \leq i \leq n\}$ and $\{\theta_i, 1 \leq i \leq n\}$. We remark that the assumption of independence among the underlying random variables appears far too unrealistic in most practical situations especially in insurance and economic situations, it considerably limits the usefulness of obtained resuls.
Note that Yang and al [12] obtained relation (6) in the case of dependence (3), when the sequence r.v.s $\{X_i, 1 \leq i \leq n\}$ are i.i.d with common distribution $F_i \in \mathcal{L}$ and $\{\theta_i, 1 \leq i \leq n\}$ are bounded i.e $\mathbb{P}\left(0 \leq \theta_k \leq b\right) = 1$ for all $1 \leq k \leq n$ and some positive constant $b$.
Yang and al [11] in the case of dependence (3) consider more general case where $\{X_i, 1 \leq i \leq n\}$ are note identically distributed, $\{\theta_i, 1 \leq i \leq n\}$ can be unbounded but $(X_1, \theta_1), \ldots, (X_n, \theta_n)$ are mutually independent random vectors.

In this paper, inspired by those of [7], [10] and [11], we suppose that, for each $i = 1, \ldots, n$, the pair $(X_i, \theta_i)$ satisfies (3), and the r.v.s $\{X_i, 1 \leq i \leq n\}$ follows the dependence structure defined by relation (5). The mutually independence between $(X_1, \theta_1), \ldots, (X_n, \theta_n)$ are relaxed. Then this paper appears as a direct extension of the results of [7], [10] and [11].
The main contribution of this paper is the following:

# 2 Main result

Recall the randomly weighted sums (1), and their maximum defined by (2), where $X_1, \ldots, X_n$ are n real-valued r.v.s with d.f.s $F_1, \ldots, F_n$, and the random weights $\theta_1, \ldots, \theta_n$ are nonnegative and nondegenerate at zero r.v.s with d.f.s $G_1, \ldots, G_n$ respectively. The main result of this work is the following:

**Theorem 2.1** *Assume that $X_1, \ldots, X_n$ are real-valued r.v.s with d.f.s $F_1, \ldots, F_n$, satisfy Assumption 2. Suppose that $\theta_1, \ldots, \theta_n$ are positive r.v.s with d.f.s $G_1, \ldots, G_n$ respectively and for each fixed $i = 1, \ldots, n$, the pair $(X_i, \theta_i)$ satisfies Assumption 1.*
*Then the relation*

$$\mathbb{P}\left(S_n^\theta > x\right) \sim \mathbb{P}\left(M_n^\theta > x\right) \sim \mathbb{P}\left(\max_{1 \leq i \leq n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\theta_i X_i\right), \qquad (7)$$

*holds if one of the following conditions is valid*

1. *$F_i \in \mathcal{L} \cap \mathcal{D}$, for each $i = 1, \ldots, n$,*
2. *$F_i \in \mathcal{S}$ for all $i = 1, \ldots, n$ and either $\overline{G}_i(x) = 0(\overline{F}_i(x))$.*

During the proof of Theorem 2.1, we shall need the following series of lemmas. The first is due to [12].

**Lemma 2.2** *Let $\xi$ be a real-valued r.v. with distribution $F_\xi$ and let $\eta$ be a nonnegative and nondegenerate at zero r.v. with distribution $F_\eta$. Assume that there exists a measurable function $h : [0, \infty) \longrightarrow (0, \infty)$ such that*

$$P(\xi > x/\eta = t) \sim \overline{F}_\xi(x)h(t) \qquad (8)$$

*uniformly for $t \geq 0$.If $F_\xi \in \mathcal{L}$ and $\overline{F}_\eta(x) = 0(\overline{F}_\xi(cx))$ for some $c > 0$ then the d.f $F_{\xi\eta}$ of the product $\xi\eta$ belongs to $\mathcal{L}$.*

We have this similar Lemma for the class $\mathcal{L} \cap \mathcal{D}$ due to [11].

**Lemma 2.3** *Let $\xi$ be a real-valued r.v. with distribution $F_\xi$ and let $\eta$ be a nonnegative and nondegenerate at zero r.v. with distribution $F_\eta$. Assume that the relation (3) holds. If $F_\xi \in \mathcal{D}$ and $\bar{F}_\eta(x) = 0(\overline{F}_\xi(x))$, then $F_{\xi\eta} \in \mathcal{D}$.*

The following Lemma is due to [3] and shows that the class $\mathcal{S}$ is closed under convolution of different d.f.s.

**Lemma 2.4** *Let $F_1$ and $F_2$ be two d.f.s on $[0, \infty)$. If $F_1 \in \mathcal{S}$ and $\bar{F}_2(x) = 0(\bar{F}_1(x))$, then $F_1 * F_2 \in \mathcal{S}$ and $\overline{F_1 * F_2}(x) = \overline{F_1}(x) + \overline{F_2}(x)$*

The next lemma follows from Theorem 3.1 in [7].

**Lemma 2.5** *Let $X_1,\dots,X_n$ be a $n$ real-valued random with distributions $F_1,\dots,F_n$, concentrated on $[0,\infty)$ satisfy Assumption 2, let $X_{(n)}=\max\{X_1,\dots,X_n\}$. Then the ralations*

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > x\right) \sim \mathbb{P}\left(X_{(n)} > x\right) \sim \sum_{i=1}^{n} \overline{F}_i(x) \tag{9}$$

*holds if one of the following conditions is valid*

1. *If $F_i \in \mathcal{L}\cap\mathcal{D}$ for all $i=1,\dots,n$,*
2. *$F_i \in \mathcal{S}$ for all $i=1,\dots,n$ and either $\overline{F}_i(x)=0(\overline{F}_j(x))$ or $\bar{F}_j(x)=0(\overline{F}_i(x))$ for all $1\le i<j\le n$.*

**Lemma 2.6** *Assume that $X_1,\dots,X_n$ are real-valued r.v.s with d.f.s $F_1,\dots,F_n$, satisfy Assumption 2. Suppose that $\theta_1,\dots,\theta_n$ are positive r.v.s with d.f.s $G_1,\dots,G_n$ respectively and for each fixed $i=1,\dots,n$, the pair $(X_i,\theta_i)$ satisfies Assumption 1. Then for each $i$, $X_i\theta_i$ satisfies Assumption 2.*

**Proof:** With $x_0>O$ sufficiently large, it holds, for every $j=2,\dots,n$, that

$$\begin{aligned}
\sup_{x_0\le t\le x}\frac{\mathbb{P}(S^\theta_{j-1}>x-t/X_j\theta_j=t)}{\mathbb{P}(S^\theta_{j-1}>x-t)} &\le \sup_{x_0\le t\le x}\sum_{i=1}^{j-1}\frac{\mathbb{P}(\theta_iX_i>\frac{x-t}{j-1}/X_j\theta_j=t)}{\mathbb{P}(\theta_iX_i>x-t)}\\
&\le \sum_{i=1}^{j-1}\sup_{x_0\le t\le x}\mathbb{E}\left[\frac{\mathbb{P}(\theta_iX_i>\frac{x-t}{j-1}/X_j\theta_j=t/\theta_i=t_i/\theta_j=t_j)}{\mathbb{P}(\theta_iX_i>x-t/\theta_i=t_i)}\right]\\
&\le \sum_{i=1}^{j-1}\sup_{x_0\le t\le x}\mathbb{E}\left[\frac{\mathbb{P}(X_i>\frac{t_i^{-1}(x-t)}{j-1}/X_j=t_j^{-1}t)}{\mathbb{P}(X_i>t_i^{-1}(x-t))\mathbb{P}(\theta_j=t_j)}\right]\\
&\le \sum_{i=1}^{j-1}\sup_{x_0\le t\le x}\mathbb{E}\left[\frac{\mathbb{P}(X_i>\frac{t_i^{-1}(x-t)}{j-1}/X_j=t_j^{-1}t)}{\mathbb{P}(X_i>\frac{t_i^{-1}(x-t)}{j-1})\mathbb{P}(\theta_j=t_j)}\frac{\bar{F}_i(\frac{t_i^{-1}(x-t)}{j-1})}{\bar{F}_i(t_i^{-1}(x-t))}\right]\\
&= C
\end{aligned}$$

This prove Lemma 2.6.

**Proof of Theorem2.1**

1. First in the case $F_i\in\mathcal{L}\cap\mathcal{D}$, combining Lemma 2.2, Lemma 2.3, Lemma 2.5 and Lemma 2.6, it follows that

$$\mathbb{P}\left(S^\theta_n>x\right)\sim\mathbb{P}\left(\max_{1\le i\le n}\theta_iX_i>x\right)\sim\sum_{i=1}^{n}\mathbb{P}\left(\theta_iX_i>x\right) \tag{10}$$

For proving (7) it suffices to show that

$$\mathbb{P}\left(M_n^\theta > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\theta_i X_i > x\right). \tag{11}$$

Since

$$S_n^\theta \leq M_n^\theta \leq \sum_{i=1}^{n} \theta_i X_i^+ \tag{12}$$

it suffices to prove

$$\mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i^+ > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\theta_i X_i > x\right). \tag{13}$$

By Lemma 2.2 and Lemma 2.3, for each $i$, $\theta_i X_i^+$ belongs to $\mathcal{L} \cap \mathcal{D}$. By Lemma 2.6, $\theta_i X_i^+$ satisfies Assumption 2, then from Lemma 2.5 we get

$$\mathbb{P}\left(\sum_{i=1}^{n} \theta_i X_i^+ > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\theta_i X_i^+ > x\right). \tag{14}$$

combining (14) and the fact that for $x \geq 0$ $\mathbb{P}\left(\theta_i X_i^+ > x\right) = \mathbb{P}\left(\theta_i X_i > x\right)$ we obtain (13).
From (12), (13), and (10) we get (11).

2. In the case $F_i \in \mathcal{S}$ from Lemma 2.2 and Lemma 2.4, we get for each $i = 1, \ldots, n$ the random variable $\theta_i X_i$ belongs to $\mathcal{S}$.
From Lemma 2.5, it follows that (10) holds. It suffices now to prove (11). $\theta_i X_i^+$ belongs to $\mathcal{S}$, from Lemma 2.4, we have (14), combining (14) and the fact that for $x \geq 0$, $\mathbb{P}\left(\theta_i X_i^+ > x\right) = \mathbb{P}\left(\theta_i X_i > x\right)$ it follows that (11) holds.

This ends the proof of Theorem 2.1.

# References

[1] A.A. Asimit and A.L. Badescu, Extremes on the discounted aggregate claims in a time dependent risk model, *Scand. Actuar. J.*, 2(2010), 93-104.

[2] Y. Chen, W.N. Kai and W.N. Xie, On the maximum of randomly weighted sums with regularly varying tails, *Statist. Probab. Lett.*, 76(2006), 971-975.

[3] P. Embrechts, C.M. Goldie and N. Veraverbeke, Subexponentiality and infinite divisibility, *Z. Wahrsch. Verw. Geb.*, 49(1979), 335-347.

[4] P. Embrechts, C. Klüppelberg and T. Mikosch, *Modelling Extremal Events for Insurance and Finance*, Springer-Verlag, Berlin, (1997).

[5] S. Diouf and A. Diop, On the maximum of randomly weighted sums with subexponential tails, *Theoretical Mathematics and Application*, 3(2013), 74-86.

[6] J. Geluk and Q. Tang, Asymptotic tail probabilities of sums of dependent subexponential random variables, *J. Theoret. Probab.*, 22($n^o$ 4) (2009), 871-882.

[7] B. Ko and Q. Tang, Sums of dependent nonnegative random variables with subexponential tails, *J. Appl. Probab.*, 45($n^o$ 1) (2008), 85-95.

[8] J. Li, Q. Tang and R. Wu, Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model, *Adv. Appl. Probab.*, 42(2010), 1126-1146.

[9] Q. Tang and G. Tsitsiashvili, Randomly weighted sums of subexponential random variables with application to ruin theory, *Extremes*, 6(2003), 171-188.

[10] J. Tao, G. Qingwu and W. Yuebao, Max-sum equivalence of conditionally dependent random variables, *Statist. Probab. Lett.*, 84(2014), 60-66.

[11] Y. Yang, W. Kaiyong, L. Remigijus and S. Jonas, A note on the max-sum equivalence of randomly weighted sums of heavy-tailed random variables, *Nonlinear Analysis: Modelling and Control*, 84(2013), 519-525.

[12] Y. Yang, R. Leipus and J. Siaulys, Tail probability of randomly weighted sums of subexponential random variables under a dependence structure, *Statist. Probab. Lett.*, 82(2012), 1727-1736.

[13] Y. Zhang, X. Shen and C. Weng, Approximation of the tail probability of randomly weighted sums and applications, *Stochastic Processes and their Applications*, 119($n^o$ 2) (2009), 655-675.