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Application of Soft Sets to Determine Hamilton Cycles in a Graph

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Abstract

Traditionally graphs can be represented by pictures, relations or matrices. Now a new technique is available to represent graphs with the help of soft sets. Cycles, Euler cycles and Hamilton cycles are very important concepts in graph theory due to their various applications. In this paper it is shown that soft sets are very handy to determine the presence of Euler's cycle and Hamilton's cycle in a graph.

Keywords: *Graphs, Soft Sets, Circuits.*

1 Introduction

Graph theory is one of the classic branch of mathematics, its applications in many fields make it a common interest of researchers, engineers and academicians. Usually graphs have their natural visual representation, however to study certain properties of graphs sometimes matrix representation is also very convenient. Ali et al. in [4] have shown that graphs can be represented with the help of soft sets. In general, soft set over multi-sets can represent any graph, however simple graphs can also be represented by soft sets over ordinary set of vertices. This representation of graphs by soft sets can be very helpful to deal with many graph theoretic problems.

Moldtsov in [13] introduced the concept of soft sets. It is a new mathematical tool to deal with uncertainty. In soft sets there are enough number of parameters available so that, these are free from problems associated with other techniques dealing with uncertainty for details see [10, 13]. This availability of parameters, makes soft set theory very easy to apply in different fields. Maji et al. in [10] has defined some operations on soft sets which are further strengthened by Ali et al. [2, 3] in soft sets and fuzzy soft sets. The ability of hybridization with other techniques such as fuzzy sets and rough sets is the beauty of soft set theory. This aspect of soft sets can be viewed in [2, 3, 7, 10, 6].

Theory of multi-sets allows membership of an element more than once. Therefore set theory is a particular type of multi-set theory. During recent years interest of researchers has increased in multi-set theory due to its wide range of applications, particularly in computer sciences. Moreover, in many daily life situations multi-set theory can be very useful.

In this paper we study the soft set representation of graphs and discuss the application of soft sets to study cycles, circuits, cut vertices, bridges and Hamilton cycles in a graph. Notions studied in this paper are as the following.

In section 2, some basic definitions regarding multi-set theory and soft set theory are given which will be required in later sections. Moreover notions about soft set representation of a graph are also given here. Section 3, is reserved to represent some fundamental concepts of graph theory with the help of soft sets, especially cycles, circuits, cut vertex and bridges. In section 4, it is seen that soft sets can be very useful to determine the presence or absence of Hamilton cycles in a given graph.

2 Preliminaries

Some basic notions about multi-sets are presented in the following from [12]. A crisp multi-set $M(V)$ of V, where $V = \{v_1, ...v_p\}$ is a finite universal set, characterized by a function $CountM(.)$ whereby a non negative integer corresponds to each $v \in V : CountM : V \to \{0, 1, 2...\}$.

A crisp multi-set can be expressed through different notations, such as

$$M(V) = \{k_1/v_1...k_p/v_p\}$$

and

$$M(V) = \left\{ \overbrace{v_1, ..., v_1}^{k_1}, ..., \overbrace{v_p, ..., v_p}^{k_p} \right\}$$

Here the element v_1 appears k_1 times and so on the element v_p appears k_p times. Throughout this work both V and $M(V)$ are finite. Basic relations and operations for crisp multi-sets are as follow.

(1) $M(V) \subseteq N(V) \Longleftrightarrow CountM(v) \leq CountN(v)$, $\forall v \in V$. (inclusion)
(2) $M(V) = N(V) \Longleftrightarrow CountM(x) = CountN(v)$, $\forall v \in V$. (Equality)
(3) $Count(M \cup N)(v) = max\{CountM(v), CountN(v)\}$. (Union)
(4) $Count(M \cap N)(v) = min\{CountM(v), CountN(v)\}$. (Intersection)
(5) $Count(M \oplus N)(v) = CountM(v) + CountN(v)$. (Addition)

Now some basic notions related to soft sets are given in the following.

Definition 2.1 *[13] A soft set over a universe U is a pair (F, B), where B is set of parameters and F is a mapping given by $F : B \to P(U)$.*

Definition 2.2 *[11] Let U be a universe and (F, B), (G, A) are two soft sets over U. Then (F, B) is called soft subset of (G, A) if*
(1) $B \subseteq A$ and
(2) $F(b) \subseteq G(b)$ for all $b \in B$.

We write $(F, B) \widetilde{\subseteq} (G, A)$. Here (G, A) is called soft super set of (F, B).

Definition 2.3 *[11] Let U be a universe and (F, B), (G, A) are two soft sets over U. Then (F, B) and (G, A) are said to be soft equal if (F, B) is a soft subset of (G, A) and (G, A) is a soft subset of (F, B).*

As graphs have a natural visual representation, however matrices are also used to represent graphs to study their certain properties. Here is a new technique, as mentioned above, to represent a graph based on soft sets, which is introduced by Ali et al in [4] is studied here. As the operations like extended union, restricted union, extended intersection, restricted intersection and restricted difference are available in soft sets, so these operations are successfully applied to graphs in [3]. This approach, to use such operations on graph, is more dynamic and fruitful. In this section representation of graphs, di-graphs, degree of graph and bi-partite graphs through soft sets are briefly studied.

Theorem 2.4 *[4] Every graph $G = (U, E)$ can be represented by a unique soft set (F, U) over $M(U)$.*

Definition 2.5 *[4] Let (F, U) and (G, V) over $M(U)$ represent two graphs. Then graph represented by (G, V) is the sub-graph of (F, U) if*
(1) $V \subseteq U$
(2) $G(u) \subseteq F(u)$ for all $u \in V$.

Theorem 2.6 *[4] Let $M(U)$ be a multi-set over U, where U is a non-empty set of vertices. Then every soft set (F, U) over $M(U)$ represents either graph or a directed graph.*

A very nice criteria to determine whether a soft set (F, U) over $M(U)$ is representing a graph or a directed graph also given in the following proposition from [4]

Proposition 2.7 *[4] A soft set (F, U) over $M(U)$ represents a di-graph if for some $i \neq j$, $u_j \in F(u_i)$ implies that $u_i \notin F(u_j)$, where $u_i, u_j \in U$.*

Definition 2.8 *[4] Let a soft set (F, U) over $M(U)$ represents graph G. Then a soft set (H, Y) over $M(U)$ represents a sub-graph if*

(1) $Y \subseteq U$

(2) $H(y) \subseteq F(y)$ for all $y \in Y$.

(3) $y_j \in H(y_i)$ implies that $y_i \in H(y_j)$ for all $i \neq j$, where $y_i, y_j \in M(U)$.

Here the third condition is used to ensure that the sub-graph is not a directed graph. Without third condition the sub-graph may be a directed graph. Third condition can be removed if the choice of graph or directed graph is not concerned.

Proposition 2.9 *[4] A soft set (F, U) over $M(U)$ represents a graph. Then at $u_i \in U$ graph has a loop if and only if $u_i \in F(u_i)$.*

Definition 2.10 *[4] Let a soft set (F, U) over $M(U)$ represents a graph. Then for a vertex u number of vertices adjacent to u is the degree of u.*

If there is such a vertex with which no other vertex is adjacent then its degree is 0 and is called an isolated vertex.

Theorem 2.11 *[4] If a soft set (F, U) over $M(U)$ represents a graph G. Then $\sum_{v_i \in V} |F(u_i)| + l$ is the total degree of G, where $|F(u_i)|$ represents the number of vertices adjacent to u_i and l represents number of loops in G.*

Proposition 2.12 *[4] A graph represented by soft set (F, U) over U, with $u_i \notin F(u_i)$ for all $u_i \in U$ is a simple graph.*

Proposition 2.13 *[4] A simple graph represented by a soft set (F, U) over U, is complete if and only if for all $u_i \in U$, $F(u_i) = U - \{u_i\}$.*

Proposition 2.14 *Let a soft set (F, U) over U represents a graph, where $U = \{u_1, u_2, ...u_m\}$ be a set of $m-$vertices with two disjoint subsets $U' = \{u'_1, u'_2, ...u'_p\}$ and $U'' = \{u''_1, u''_2, ...u''_q\}$ such that $U' \cup U'' = U$. Then graph is a bi-partite if and only if,*

$F(u'_i) \subseteq U''$ and $F(u''_j) \subseteq U'$ for all $1 \leq i \leq p$ and $1 \leq j \leq q$.

Proposition 2.15 *Let a soft set* (F,U) *over* U *represents a graph, where* $U = \{v_1, u_2, ...u_m\}$ *be a set of* $m-$*vertices with two disjoint subsets* $U' = \{u'_1, u'_2, ...u'_p\}$ *and* $U'' = \{u''_1, u''_2, ...u''_q\}$ *such that* $U' \cup U'' = U$. *Then graph is a* $(p,q)-$*bi-partite if and only if,*

(1) $F(u'_i) = U''$ *and* $F(u''_j) = U'$ *for all* $1 \leq i \leq p$ *and* $1 \leq j \leq q$.

Definition 2.16 $(1,n)$ *complete bipartite graph is called a star graph.*

Proposition 2.17 *Let a soft set* (F,U) *over* U *represents a graph* G. *Then* G *is a star graph if there exists a vertex* $v \in U$ *such that* $F(v) = U - \{v\}$ *and* $F(u_i) = \{v\}$ *for all* $u_i \in U - \{v\}$.

3 Soft Set Representation of Cycles and Circuits

This work is mainly for the study of connected graphs through soft sets. A connected graph consists of single component while disconnected graph consists of more than one component. A graph, is connected if a path is there from any vertex to any other vertex within the graph, otherwise a graph is disconnected. Following proposition describes how soft sets can represent a graph having components.

Proposition 3.1 *Let a graph* G *be represented by a soft set* (F,U) *over* U. *Then* G *has* n *disconnected components if there are* n *subsets* $U_1, U_2, ...U_n$ *of* U, *such that* $U_1 \cup U_2 ... \cup U_n = U$ *and* $U_i \cap U_j = \emptyset$ *for any* $i \neq j$, *where* $1 \leq i \leq n$, $1 \leq j \leq n$ *such that for all* $u_i \in U_i$ *implies* $F(u_i) \subseteq U_i$

Proof: Let a graph G be represented by a soft set (F,U) over U, with $U_1 \cup U_2 ... \cup U_n = U$ and $U_i \cap U_j = \emptyset$ for any $i \neq j$. Let $u_i \in U_i$ be such that $F(u_i) \subseteq U_i$, for all $u_i \in U_i$ and $u_j \in U_j$ such that $F(u_j) \subseteq U_j$ for all $u_j \in U_j$. Since $F(u_i) \subseteq U_i$, so $F(u_i) \cap U_j = \emptyset$, therefore for any $u_j \in U_j$ implies $u_j \notin F(u_i)$. That is, no vertex in U_i is adjacent to any vertex in U_j. Therefore for each $U_i \subseteq U$ with $u_i \in U_i$ such that $F(u_i) \subseteq U_i$ gives a disjoint component of graph G.

Corollary 3.2 *A graph* G, *represented by a soft set* (F,U) *over* U, *is connected if and only if there exists no proper subset* U_i *of* U *with condition* $F(u_i) \subseteq U_i$ *for all* $u_i \in U_i$.

Now soft set representation of cycles is given in the following. If all the vertices of a graph G are connected in a chain, then it is called a cyclic graph. This means, it consists of a single cycle. In a cyclic graph number of vertices is equal to number of edges. Usually a cyclic graph with n vertices is denoted by C_n. In the following soft set representation of a cyclic graph is given.

Proposition 3.3 *Let a connected graph G be represented by a soft set (F,U) over U. Then (F,U) represents a single cycle if and only if $|F(u)| = 2$, for all $u \in U$.*

Proof: Let a connected graph G be represented by the soft set (F,U) over U, with $|F(u)| = 2$, for all $u \in U$. Since $|F(u)| = 2$, so every vertex is adjacent with two other vertices of U. Consider $u_i \in U$ as $|F(u_i)| = 2$, so u_i is adjacent to two vertices say $u_{i-1}, u_{i+1} \in U$, that is, $F(u_i) = \{u_{i-1}, u_{i+1}\}$. Further as $|F(u_{i-1})| = |F(u_{i+1})| = 2$, so u_{i-1} is adjacent to u_i and u_{i-2}, similarly u_{i+1} is adjacent to u_i and one more vertex u_{i+2}, continuing this process gives $|F(u_n)| = 2$, so u_n is adjacent to two vertices u_{n-1}, $u_1 \in U$, Also $|F(u_1)| = 2$, so u_1 is adjacent to u_n and u_2. Hence this completes the cycle

Conversely, let the graph represented by soft set (F,U) be a single cycle. Then every vertex approaches a single vertex and at the same time being approached by a single vertex. Thus every vertex has the degree 2, therefore $|F(u)| = 2$, for all $u \in U$. As graph is a single cycle so it is connected.

Proposition 3.4 *Let G be represented by a soft set (F,U) over U. Then G has a cycle through all the vertices if and only if there exists a soft subset (H,V) of (F,U) representing a connected sub-graph such that*

1. *for $v_i \in H(v_j)$ implies $v_j \in H(v_i)$, where $v_i, v_j \in V$.*

2. *$|H(v)| = 2$, for all $v \in V$.*

Proof: Let a graph G be represented by a soft set (F,U) over U which has a cycle, if this cycle consists of vertices say $V = \{v_i,\ v_{i+1},\ v_{i+2},\ ...v_{i+m}\}$ in G. Then it can be represented by soft subset (H,V) of (F,U). As (H,V) represents cycle so $|H(v)| = 2$, for all $v \in V$. Since a cycle is not a directed graph so (1) is also satisfied.

Conversely, let (H,V) be a soft subset of (F,U) with above mentioned conditions, (1) implies (H,V) represents sub-graph of graph G, (2) implies (H,V) is a cycle. As (H,V) is contained in (F,U) therefore G contains a cycle.

Now we are able to see how many cycles are there in a graph with the help of soft sets. So the following proposition will serve the purpose.

Corollary 3.5 *Let a graph G be represented by a soft set (F,U) over U and V be a subset of U, if there are n soft subsets (H_k,V), $1 \leq k \leq n$, of (F,U) such that each (H_k,V) over V represents a cycle. Then there are n cycles in G through vertices set V.*

Remark 3.6 *Let a graph G with set of vertices U, then for $V \subseteq U$ if there are n soft subsets (H_k, V), $1 \leq k \leq n$, of (F, U) such that each (H_k, V) over V represents a cycle. Then V is said to have $Cl - Counting = n$ and it will be represented as $ClC(V) = n$*

Remark 3.7 *Let G be a graph defined by the soft set (F, U) over U. Then total number of cycles in G is the sum of $Cl - Countings$ of all subsets of U.*

It is well known that, if in a walk from any vertex and trace back the starting vertex through all vertices without using any edge more than once, such graph is called an Euler's circuit. Here is a proposition for an Euler's circuit.

Proposition 3.8 *Let a connected graph G be represented by a soft set (F, U) over U. Then G is an Euler's circuit if and only if $|F(u)| = 2n$, $n \in N$, for all $u \in U$.*

Proof: Let a connected graph G be represented by soft set (F, U) over U. Let G be an Euler's circuit, so in walk from any vertex, the starting vertex can be traced back walking through all vertices without using any edge more than once. It is possible only if a vertex is approached by k vertices then it also approaches k other vertices, this suggests that every vertex has an even degree, hence $|F(u)| = 2n$, where $n \in N$, for all $u \in U$.

Conversely, let a connected graph G be represented by soft set (F, U) over U, with $|F(u)| = 2n$, $n \in N$, for all $u \in U$, implies every vertex has even number of edges, therefore total number of edges in G is even. Therefore it is possible to divide these edges in to two sets, the set of incoming edges and the set of outgoing edges. Hence there is no point in this sub-graph where we stuck, hence in our walk from any vertex, the starting vertex can be traced back, Therefore (F, U) represents an Euler's circuit.

As a circuit is a closed path in a graph G and it is well known that when talking about some closed path in a graph, it means a walk through the vertices of graph that ends on the starting point without repeating any edge, and there is no such point through which we enter and do not leave it. It suggests us that in every closed path number of times a vertex is approached is equal to the number of times the same vertex approach the other vertices. So in every closed path, that is, in a circuit degree of each vertex is even.

Proposition 3.9 *Let a graph G be represented by a soft set (F, U) over U. Then G has a circuit if and only if there exists a soft subset (H, V) of (F, U) such that (H, V) is a connected sub-graph with $|H(v_i)| = 2n$, for all $v_i \in V$, $n \in N$.*

Proof: Let G be a graph represented by soft set (F, U) over U having a circuit. Let $V = \{v_1, v_2, ...v_p\}$ be the set of vertices on the circuit . Now consider any $v_i \in V$, $0 \leq i \leq p$. As v_i is on a circuit therefore if it has been approached from n vertices and then n vertices have been approached from v_i. Therefore $|H(v_i)| = 2n$, for all $v_i \in V$, $n \in N$. Thus circuit can be represented by soft subset (H, V) of (F, U) such that $v_i \in H(v_j) \implies v_j \in H(v_i)$ and $|H(v_i)| = 2n$, for all $v_i \in V$, $n \in N$.
Conversely, consider (H, V) is a soft subset of (F, U) representing a connected sub-graph, say G', with $|H(v_i)| = 2n$, for all $v_i \in V$, $n \in N$, of graph G . Since $|H(v_i)| = 2n$ implies in G' every vertex is adjacent to even number of vertices, so the total edges in G' are even in number. So, for every vertex, the edges incident to it can be divided into two categories, the incoming edges and the outgoing edges. Hence there is no such vertex in this sub-graph where we are stuck. Therefore if we start a walk from any point, it can be traced back, so walk becomes closed path, hence (H, V) represents a circuit. As (H, V) is contained in (F, U) therefore G contains a circuit.

A cycle is a special case of circuit and it is called a simple circuit.

Remark 3.10 *Let a graph G with vertices set U, and $V \subseteq U$ be a set of some of the vertices from G, and if there are n soft subsets (H_k, V), $1 \leq k \leq n$, of (F, U) such that each (H_k, V) over V represents an Euler's circuit. Then there are n Euler's circuits in G through vertices set V.*

Remark 3.11 *Let a graph G with vertices set U, and $V \subseteq U$ be a set of some of the vertices from G, and if there are n soft subsets (H_k, V), $1 \leq k \leq n$, of (F, U) such that each (H_k, V) over V represents an Euler's circuit. Then V is said to have $C - Counting = n$ and it will be represented as $CC(V) = n$*

Remark 3.12 *Let G be a graph defined by soft set (F, U) over U. Then total number of circuits in G is the sum of $C - Countings$ of all subsets of U.*

As every graph is not complete, but there might be some set of vertices within the graph which constitute a complete sub-graph. A complete sub-graph of graph G is called clique of G. Here is a proposition to represent clique using soft sets.

Proposition 3.13 *Let G be a graph defined by soft set (F, U) over U. Then sub-graph defined by soft subset (H, V) of (F, U) is clique of G if and only if $H(v_i) = V - \{v_i\}$ for all $v_i \in V$.*

Proof: Since (H, V) is soft subset of (F, U) and $H(v_i) = V - \{v_i\}$ for all $v_i \in V$ implies every vertex of sub-graph represented by soft subset (H, V) is adjacent to all other vertices of the sub-graph. So this sub-graph is a complete sub-graph of G, hence is clique of G, so (H, V) represents clique of G.

Conversely, let soft subset (H, V) of (F, U) represents clique G' of G then G' is a complete sub-graph of G. Since G' is complete sub-graph so every vertex of G' is adjacent to all other vertices of G' except itself implies $H(u_i) = V - \{v_i\}$ for all $v_i \in V$.

Concept of removal of an edge or a vertex is very important in graph theory. Soft set theory can be very handy to study related concepts. If u_k is a vertex which is required to remove from a graph G along with all its adjacencies and remaining graph unchanged, then the resulting graph is denoted by $G - u_k$. In fact $G - u_k$ is the sub-graph of G which is induced by $U - \{u_k\}$, that is, the graph induced by removing vertex u_k from the given graph G.

Let a graph G represented by soft set (F, U) over U. Then $G - u_k$, where $u_k \in U$, is represented by soft set $(F_{u'_k}, U - \{u_k\})$ over $U - \{u_k\}$ and defined as $F_{u'_k}(u_i) = F(u_i) - \{u_k\}$ if $i \neq k$ for all $u_i \in U$.

A cut vertex of a graph G is a vertex that splits a connected graph into disconnected components. For instance, a vertex u in graph G is a cut vertex if $G - u$ has greater components then G.

Proposition 3.14 *Let G be a connected graph defined by soft set (F, U) over U then $u_k \in U$ is cut vertex in G if and only if the soft subset $(F_{u'_k}, U - \{u_k\})$ of (F, U), gives a partition of $U - \{u_k\}$ such that U_1, U_2, U_3, $...U_n \subseteq U - \{u_k\}$, $U_i \cap U_j = \emptyset$, where $i \neq j$, and $\bigcup_{i=1}^{n} U_i = U - \{u_k\}$,$1 \leq i \leq n$ and, for any $u' \in U_i$, $F(u') \subseteq U_i$.*

Proof: Let G be a connected graph represented by soft set (F, U) over U and $u_k \in U$ is a cut vertex in G this implies $G - u_k$ is represented by soft subset $\left(F_{u'_k}, U - \{u_k\}\right)$ of (F, U) defined by $F_{u'_k}(u_i) = F(u_i) - \{u_k\}$, $i \neq j$ for all $u_i \in U$ is disconnected so a partition of $U - \{u_k\}$ arises such that U_1, U_2, U_3, $...U_n \subseteq U - \{u_k\}$, $U_i \cap U_j = \emptyset$, for $i \neq j$, $\bigcup_{i=1}^{n} U_i = U - \{u_k\}$,$1 \leq i \leq n$. and, for any $u' \in U_i$, $F(u') \subseteq U_i$.

Conversely, consider a soft set (F, U) over U represents a connected graph and for the soft subset $\left(F_{u'_k}, U - \{u_k\}\right)$ a partition of $U - \{u_k\}$ arises such that U_1, U_2, U_3, $...U_n \subseteq U - \{u_k\}$, $U_i \cap U_j = \emptyset$, $\bigcup_{i=1}^{n} U_i = U - \{u_k\}$, $1 \leq i \leq n$ and, for any $u' \in U_i$, $F(u') \subseteq U_i$. As $\left(F_{u'_k}, U - \{u_k\}\right)$ represents $G - u_k$ and is disconnected so u_k is a cut vertex.

In visual representation, adjacency relations between the vertices are denoted by edges. If two vertices $u_i, u_j \in U$ are adjacent in a graph G, that is, (u_i, u_j) edge is there in graph G. For the soft set (F, U) representing graph G, $u_i \in F(u_j)$ if and only if $u_j \in F(u_i)$ implies (u_i, u_j) edge in G [4]. In this subsection the removal of an edge from a graph and the bridge in graph is studied using soft sets. If (u_i, u_j) is an edge which is to be removed from G, remaining graph unchanged, the resulting graph is denoted by $G - (u_i, u_j)$.

Proposition 3.15 *Let a graph G be represented by soft set (F,U) over U then soft subset $(F_{-(u,v)},U)$ represents $(u,v)-edge-free$ sub-graph of G containing all vertices and edges of G except (u,v) edge if and only if*

1. $F_{-(u,v)}(u_i)=F(u_i)$ *if* $u_i\neq u, u_i\neq v$
2. $F_{-(u,v)}(u_i)=F(u_i)-\{u\}$ *if* $u_i=v$
3. $F_{-(u,v)}(u_i)=F(u_i)-\{v\}$ *if* $u_i=u$, *for all* $u_i\in U$.

Proof: Let a graph G be represented by a soft set (F,U) over U. Then soft subset $(F_{-(u,v)},U)$ of (F,U) over U satisfies the following

(1) $F_{-(u,v)}(u_i)=F(u_i)$ if $u_i\neq u, u_i\neq v$

(2) $F_{-(u,v)}(u_i)=F(u_i)-\{u\}$ if $u_i=v$

(3) $F_{-(u,v)}(u_i)=F(u_i)-\{v\}$ if $u_i=u$ for all $u_i\in U$

(1) implies sub-graph represented by soft subset $(F_{-(u,v)},U)$ has all the adjacencies for all the vertices as in graph except u and v. (2) implies that v also has all the adjacencies with all other vertices in G but is not adjacent to u. Whereas (3) implies that u also has all the adjacencies with all other vertices in G but is not adjacent to v. So $(F_{-(u,v)},U)$ over U with above conditions represents $(u,v)-edge-free$ sub-graph of G containing all vertices and edges of G except (u,v) edge.

Conversely, consider $(F_{-(u,v)},U)$ over U represents $(u,v)-edge-free$ sub-graph of G containing all vertices and edges of G except (u,v) edge. Then it is obvious that $(F_{-(u,v)},U)$ over U has all the adjacencies as (F,U) over U except (u,v). Therefore $F_{-(u,v)}(u_i)=F(u_i)$ if $u_i\neq u, u_i\neq v$. As v has all the adjacencies with all other vertices as in G. except with u. Therefore $F_{-(u,v)}(u_i)=F(u_i)-\{u\}$ if $u_i=v$. Lastly as u has all the adjacencies with all other vertices as in G. except with u. Therefore $F_{-(u,v)}(u_i)=F(u_i)-\{u\}$ if $u_i=v$.

An edge is called a bridge whose deletion increases the number of components in the graph, if graph G is connected then removal of the bridge edge splits it into two disjoint components. For instance, an edge (u,v) in graph G is a bridge if $G-(u,v)$ has greater components than G.

Proposition 3.16 *Let G be a connected graph represented by soft set (F,U) over U then (u,v) edge is a bridge in G if and only if for the soft subset $(F_{-(u,v)},U)$, a partition of U arises such that $U',U''\subseteq U$, $U'\cap U''=\emptyset$, $U'\cup U''=U$, for all $u'\in U'$, $F(u')\subseteq U'$, and for all $u''\in U''$, $F(u'')\subseteq U''$.*

Proof: Let a connected graph G be represented by soft set (F,U) over U. Let (u,v) be the edge which is a bridge in G, this implies $G-(u,v)$ represented by $(F_{-(u,v)},U)$ is disconnected, so a partition of U consisting of two subsets arises such that U', $U'' \subseteq U$, $U' \cap U'' = \emptyset$, $U' \cup U'' = U$, for all $u' \in U'$, $F(u') \subseteq U'$, and for all $u'' \in U''$, $F(u'') \subseteq U''$.

Conversely, consider a soft set (F, U) over U represents a connected graph and for the soft subset $(F_{-(u,v)}, U)$ with a partition of U such that U', $U'' \subseteq U$, $U' \cap U'' = \emptyset$, $U' \cup U'' = U$, for all $u' \in U'$, $F(u') \subseteq U'$, and for all $u'' \in U''$, $F(u'') \subseteq U''$. As $(F_{-(u,v)}, U)$, representing $G - (u, v)$, is disconnected therefore (u, v) edge is a bridge in G .

4 Hamilton Cycle

As in a graph, the cycle through all the vertices is called a Hamilton's cycle. To determine a Hamilton cycle in a graph is not straight forward. In the following it is seen that soft set theory may help to find presence or absence of Hamilton's cycle in a graph.

Proposition 4.1 *Let G be a connected graph represented by a soft set (F, U) over U, then G has a Hamilton's cycle if and only if there exists a soft subset (H, U) of (F, U) such that (H, U) represents a connected sub-graph with $|H(u_i)| = 2$, for all $u_i \in U$.*

Proof: Let G be a connected graph represented by a soft set (F, U) over U. Let G has a Hamilton's cycle, that is, if a walk is started from any vertex, then it can be traced back the starting vertex through all vertices without using any vertex more than once. Consequently G has a cycle through all the vertices. This cycle can be represented by soft subset (H, U) of (F, U), such that, $|H(u_i)| = 2$, for all $u_i \in U$.

Conversely, let a graph G be represented by soft set (F, U) over U, and G has cycle through all the vertices which is represented by soft subset (H, U) of (F, U), such that, $|H(u_i)| = 2$, for all $u_i \in U$. This implies there is a cycle in G such that if a walk is started from any vertex, then it can be traced back the starting vertex through all vertices without using any vertex more than once. As such a walk in a graph represents a Hamilton's cycle. So G has a Hamilton's cycle.

In a graph, if there is a walk from any vertex to the starting vertex, walking through all vertices without using any vertex more than once. This implies that Hamilton's cycle is there in the graph. Here is a proposition using soft sets for this case.

Proposition 4.2 *Let G be a connected graph represented by a soft set (F, U) over U. Then the necessary condition that G has a cycle through all the vertices is $|F(u)| \geq 2$, for all $u \in U$.*

Proof: Let G be a connected graph represented by a soft set (F, U) over U. If G has a cycle through all the vertices then it is represented by a soft subset (H, U) of (F, U) such that $|H(u)| = 2$, for all $u \in U$. Since (F, U) is soft super set of (H, U) therefore without any loss of generality $|F(u)| \geq 2$, for all $u \in U$.

Proposition 4.3 *Let G be a connected graph represented by a soft set (F, U) over U. If $u \in U$ is a vertex adjacent to at least three more vertices say u_i, u_{i+1} and u_{i+2} such that $|F(u_i)| = |F(u_{i+1})| = |F(u_{i+2})| = 2$. Then G has no Hamilton's cycle.*

Proof: Let G be a connected graph represented by a soft set (F, U) over U with a vertex u in G such that $|F(u)| \geq 3$. Let u_i, u_{i+1} and u_{i+2} be adjacent to u such that $|F(u_i)| = |F(u_{i+1})| = |F(u_{i+2})| = 2$. If possible let G has a Hamilton's cycle. This implies there exists a soft subset (H, U) of (F, U) such that $|H(u')| = 2$, for all $u' \in U$. In order to obtain this soft subset (H, U) of (F, U) at least one of the vertices from u_i, u_{i+1} and u_{i+2} is to be removed. Therefore u must be removed from any one of $F(u_i)$, $F(u_{i+1})$ or $F(u_{i+2})$. Without any loss of generality let u_i is removed from $F(u)$ as a result u will be deleted from the set $F(u_i)$. This removal of u from $F(u_i)$ will make $|H(u_i)| = 1$. Which is a contradiction to Proposition 4.1. Hence (H, U) can not represent a cycle. Consequently the graph G has no Hamilton's cycle.

Proposition 4.4 *Let G be a connected graph represented by a soft set (F, U) over U. if there are two overlapping subsets U' and U'' of U such that $U' \cap U'' \neq \emptyset$, and (H, U') represents cycle, such that for $u' \in U'$ and $u' \notin U' \cap U''$, $u' \notin F(u'')$, for any $u'' \in U''$. Then G has no Hamilton's cycle.*

Proposition 4.5 *Let a connected graph G be represented by soft set (F, U) over U. If G has a cut vertex then G has no Hamilton's cycle.*

Proof: Let G be a connected graph represented by a soft set (F, U) over U, with cut vertex v in G. Therefore there exist two subsets U' and U'' of U, such that, $U' \cup U'' = U$ and $U' \cap U'' = \{v\}$ and for any u' other than v in U', $u' \notin F(u'')$ for all u'' other than v in U''. As v is cut vertex so $v \in F(u'_i)$ for some $u'_i \in U'$, and $v \in F(u''_j)$ for some $u''_j \in U''$. If possible let G has a Hamilton's cycle. This implies that there exists a soft subset (H, U) of (F, U) such that $|H(u)| = 2$, for all $u \in U$. Therefore $|H(v)| = 2$. Let $u'_{i+1}, u''_{i+2} \in H(v)$, where $u'_{i+1} \in U'$, and $u''_{i+2} \in U''$. The vertex v provides us a way to shift form vertices of U' to U''. Now to complete the cycle, there is no other vertex, say w, such that $|H(w)| = 2$, and $u'_{j+1} \in H(w)$ and $u''_{j+2} \in H(w)$, such that $u'_{j+1} \in U'$, and $u''_{j+2} \in U''$. Hence (H, U) can not represent a cycle. Consequently the graph G has no Hamilton's cycle.

Example 4.6: Let G be a graph represented by a soft set (F, U) over $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$, as shown in figure 1,defined as $F(u_1) = \{u_3, u_7\}$, $F(u_2) = \{u_6, u_7\}$, $F(u_3) = \{u_1, u_4, u_7\}$, $F(u_4) = \{u_3, u_5, u_7\}$, $F(u_5) = \{u_4, u_6, u_7\}$, $F(u_6) = \{u_2, u_5\}$, $F(u_7) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$. To find the Hamilton cycle in G let's start from u_1. Here $F(u_1) = \{u_3, u_7\}$ let's chose u_3 next to u_1 to start walk. As $F(u_3) = \{u_1, u_4, u_7\}$, Here exclude u_1, also u_7, because $u_7 \in F(u_1)$. So the sub set of our interest, of $F(u_3)$, is $\{u_4\}$. Now $F(u_4) = \{u_3, u_5, u_7\}$ ignoring u_3 and u_7, the sub set of our interest, of $F(u_4)$, is $\{u_5\}$. $F(u_5) = \{u_4, u_6, u_7\}$ ignoring u_4 and u_7, the sub set of our interest, of $F(u_5)$, is $\{u_6\}$. Again $F(u_6) = \{u_2, u_5, u_7\}$ ignoring u_5 and u_7, the sub set of our interest, of $F(u_6)$, is $\{u_2\}$. $F(u_2) = \{u_6, u_7\}$ ignoring u_6, the sub set of our interest, of $F(u_2)$, is $\{u_7\}$. And $F(u_7) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ ignoring u_2, u_3, u_4, u_5 and u_6, the sub set of our interest, of $F(u_7)$, is $\{u_1\}$. At the end when all vertices are used u_7 approaches the first vertex. So we have the soft subset, say (H, V), such that, $H(u_1) = \{u_3, u_7\}$, $H(u_2) = \{u_6, u_7\}$, $H(u_3) = \{u_1, u_4\}$, $H(u_4) = \{u_3, u_5\}$, $H(u_5) = \{u_4, u_6\}$, $H(u_6) = \{u_2, u_5\}$, $H(u_7) = \{u_1, u_2\}$, which represents the cycle, as shown in figure 2.

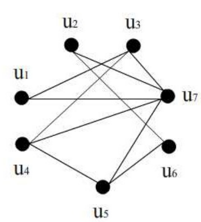

Figure 1: Graph G having Hamilton Cycle

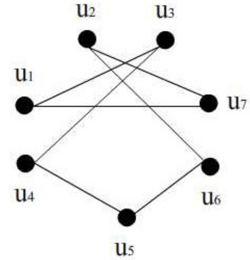

Figure 2: Hamilton Cycle in graph G

5 Conclusion

In the present paper, it is shown that graphs particularly cycles can be represented by soft sets. This representation has many advantages due to operations available in soft sets. On the other hand this representation helps us to determine existence of Euler's cycle and Hamiltons cycle in graphs.

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