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An Optimal Distributed Control for Age-dependent Population Diffusion System

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Abstract

The optimal distributed control problem for age-dependent population diffusion system governed by integral partial differential equations is investigated in this paper. As new results, the existence and uniqueness of the optimal distributed control are proposed and proved, a necessary and sufficient conditions for the control to be optimal are obtained, and the optimality system consisting of integro-partial differential equations and variational inequalities are constructed in which the optimal controls can be determined. The applications of penalty shifting method for infinite dimensional systems to approximate solutions of control problems for the population system are researched. An approximation program is structured, and the convergence of the approximating sequences in appropriate Hilbert spaces is derived. The results in this paper may significantly provide theoretical reference for the practical research of the control problem in population systems.

Keywords:*population diffusion system, optimaldistributed control, necessary and sufficient condition, optimality system, penalty shifting method.*

1 Introduction

We consider the foollowing age-dependent population diffusion system($\mathcal{P}$)Ref.1-2; 10-11):

$$Lp \equiv p_r + p_t - k\triangle_x p + \mu(r,t,x)p = v(r,t,x), \quad \text{in } Q = \Theta \times \Omega, \tag{1}$$
$$p(0,t,x) = \int_0^A \beta(r,t,x)p(r,t,x)\mathrm{d}r, \quad \text{in } \Omega_T = (0,T) \times \Omega, \tag{2}$$
$$p(r,0,x) = p_0(r,x), \quad \text{in } \Omega_A = (0,A) \times \Omega, \tag{3}$$
$$p(r,t,x) = 0, \quad \text{on } \Sigma = \Theta \times \partial\Omega, \tag{4}$$

where $p = p(r,t,x)$ is the population density of age $r > 0$ at time $t > 0$ and at spatial position $x \in \Omega$, Ω being a bounded domain in $R^N (1 \le N \le 3)$, μ is the death rate and β is the fertility rate, $0 < r < A$, A is the highest age ever attained by individual of the population and then

$$p(r,t,x) = 0 \ \text{ if } \ r \ge A. \tag{5}$$

$k > 0$ is the dispersal modulus. $\theta = (0,A) \times (0,T)$. p_0 is an initial density. (4) denotes that boundary $\partial\Omega$ of domain Ω is supposed to be extremely inhospitable. In (1) v is the distributed control. The reference [3] proved the controllability for the system (1)- (4) under the hypothesis $p(r,t,x) = 0$ and $\beta(r,t,x)$=$\beta(r)$ without researches of the optimal control problem for the system (1)-(4). In the present paper, the optimal control problem for the system (1)-(4) with μ=$\mu(r,t,x)$ and β=$\beta(r,t,x)$ is investigated. The existence and uniqueness of the optimal control are proved. The necessary and sufficient conditions for a control to be optimal are obtained. By means of the penalty shifting principle, the approximate solution of the optimal control is researched, an approximation program is structured, and the convergence of the approximating sequence on appropriate Hilbert Space is derived.

The following assumptions are made throughout the paper:

(A_1) $\mu(r,t,x)$ is a measurable and $\mu(r,t,x) \ge 0$, $\mu(\cdot,t,x) \in L_{loc}[0,A)$, $\int_0^A \mu(r,t,x)dr = +\infty$;

(A_2) β is a measurable and $\beta \in L^\infty(Q)$, $0 \le \beta(r,t,x) \le \bar{\beta} < +\infty$, a.e. on $\bar{Q}$;

(A_3) $p_0 \in L^2(\Omega_A)$, $p_0(r,x) \ge 0$, $\int_0^A p_0(r,x)dr \le M_0 < +\infty$;

(A_4) $k > 0$, $\partial\Omega$ is smooth.

Let

$$\mathcal{U} \ = \ \text{closed, convex subset of } \ L^2(Q). \tag{6}$$

Clearly p depends on v and hence we write $p(r,t,x;v)$ or $p(v)$.

With every control $v \in \mathcal{U}$, we associate the cost:

$$I(v) = \|p(\cdot,T,\cdot\,;v) - z_d(\cdot,\ \cdot)\|^2_{\Omega_A} + \alpha\|v\|^2_Q, \quad \alpha > 0, \tag{7}$$

where z_d is a given element in Ω_A. Let

$$\|\cdot\|_{\Omega_A} = \|\cdot\|_{L^2(\Omega_A)}, \quad \|\cdot\|_Q = \|\cdot\|_{L^2(Q)}.$$

The control problem then is:

$$\text{Find } \ u \in \mathcal{U} \ \text{ satisfying } \ I(u) = \inf_{v \in \mathcal{U}} I(v). \tag{8}$$

The problem $(1)-(4)$, the cost function (7) and the minimization problem (8) constitute the mathematical model of the optimal distributed control for age-dependent population diffusion system. In (8), elment $u \in \mathcal{U}$ is termed the optimal distributed control of the system $(1)-(4)$.

2 Optimality Conditions and Optimality Systems

We state first the following existence and uniqueness theorem for the system $(1)-(4)$.
From Refs.3-5 we mat obtain the folloming Theorem 2.1.

Theorem 2.1. Assume that $(\mathrm{A}_1)-(\mathrm{A}_4)$ hold. Then the system $(1)-(4)$ admits a unique solution $p \in V = L^2(\Theta;\ H_0^1(\Omega))$, and bilinear mapping $(v, p_0) \to p$ is a continuous mapping of $L^2(Q) \times L^2(\Omega_A) \to V$.

From Theorem 2.1 and trace theorem (cf.Ref.3), we obtain:

Corollary 2.1. The mapping $v \to p\ (\cdot, T, \cdot\ ;\ v)$ is a continuous affine map of $L^2(Q) \to L^2(\Omega_A)$.

Theorem 2.2. Assume that $(\mathrm{A}_1)-(\mathrm{A}_4)$ hold. Let $v \in \mathcal{U}$, and $p(v) \in V$ be the solution of $(1)-(4)$. Then there exists a unique element $u \in \mathcal{U}$ such that

$$I(u) = \inf_{v \in \mathcal{U}} I(v)$$

and u is characterized by

$$(p(T;v) - p(T;u),\ p(T;u) - z_d)_{\Omega_A} + \rho(u, v-u)_Q \geq 0, \quad \forall v \in \mathcal{U}, \tag{9}$$

where

$$(\varphi, \psi)_{\Omega_A} = \int_{\Omega_A} \varphi\psi \mathrm{d}r\mathrm{d}x,\ (\varphi, \psi)_Q = \int_Q \varphi\psi \mathrm{d}r\mathrm{d}t\mathrm{d}x.$$

In other words, a necessary and sufficient condition for u to be optimal control is that u satisfies (9).

Proof. We set

$$p(r, T, x; v) = p(T; v).$$

According to definitions of $I(v)$ and definition (cf. Ref. 6) of Gâteaue differentiate $I'(u, v-u)$ an easy calculation shows that

$$\frac{1}{2} I'(u, v-u) = (p (T; v) - p (T; u), p (T, u) - z_d))_{\Omega_A} + \rho (v - u, u)_Q. \quad (10)$$

From Definition (7) and Corollary 2.1, we deduce that the functional $v \to I(v)$ is continuous from $L^2(Q)$ to R and it is actually a functional which is strictly convex. From the definition (7) of $I(v)$, we have $I(v) \geq \rho \|v\|_Q^2$, so that $I(v) \to +\infty$ if $\|v\|_Q \to +\infty$. Consequently, according to Ref. 7 there exists a unique element $u \in \mathcal{U}$ such that $I(u) = \inf\limits_{u \in \mathcal{U}} I(v)$ and u is characterized by

$$u \in \mathcal{U}, \qquad \frac{1}{2} I'(u, v-u) \geq 0 \qquad \forall v \in \mathcal{U}. \quad (11)$$

Form (11) and (10) we deduce (9). Thus, Theorem is proved.

We shall now transfrom (9) by utilizing the adjoint state. We define the adjoint state $q(u)$ by

$$L^* q \equiv -\frac{\partial q}{\partial r} - \frac{\partial q}{\partial t} - k \triangle q + \mu q - \beta(r, t, x) q(0, t, x) = 0, \quad \text{in } Q, \quad (12)$$
$$q(A, t, x) = 0, \quad \text{in } \Omega_T, \quad (13)$$
$$q(r, T, x) = p(T; u) - z_d, \quad \text{in } \Omega_A, \quad (14)$$
$$q(r, t, x) = 0, \quad \text{on } \Sigma. \quad (15)$$

Let

$$t = T - t', \ r = A - r', \ g(r', \ t', \ x) = q(A - r, \ T - t', \ x) \quad (16)$$

Then the problem (12)$-$(15) becomes down to a problem (1)$-$(4). From Theorem 2.1 and (16) we deduce that the problem (12)$-$(15) admits a unique solution $q \in V$.

Multiplying (12) by $(p(v) - p(u))$, applying Green′s Formula and (1)-

(4) and (13)-(15), and setting $p(A;v) = p(A,t,x;v)$, we have:

$$
\begin{aligned}
0 &= (p(v) - p(u), L^* q)_Q \\
&= (L(p(v) - p(u)), q)_Q - \int_{\Omega_T} [(p(A;v) - p(A;u))q(A,t,x) - (p(0,t,x;v) \\
&\quad - p(0,t,x;u))q(0,t,x;u)]dtdx - \int_{\Omega_A} [(p(T;v) - p(T;u))q(r,T,x) - (p(r,0,x;v) \\
&\quad - p(r,0,x;u))q(r,0,x)]\mathrm{d}r\mathrm{d}x + \int_\Sigma (p(v) - p(u))\frac{\partial q}{\partial \nu^*}\mathrm{d}\Sigma - \int_\Sigma q \frac{\partial}{\partial \nu}(p(v) - p(u))\mathrm{d}\Sigma \\
&\quad - (\int_0^A \beta(r,t,x)(p(r,t,x;v) - p(r,t,x;u))\mathrm{d}r, q(0,t,x))_{\Omega_T} \\
&= (v - u, q)_Q - 0 + ((\int_0^A (p(r,t,x;v) - p(r,t,x;u))\beta(r,t,x)\mathrm{d}r, q(0,t,x;u))_{\Omega_T} \\
&\quad - (p(T;v) - p(T;u), p(T;u) - z_d)_{\Omega_A} + 0 + 0 - (\int_0^A \beta(p(v) - p(u))\mathrm{d}r, q(0,t,x))_{\Omega_T} \\
&= (v - u, q)_Q - (p(T;v) - p(T;u), p(T;u) - z_d)_{\Omega_A} + 0,
\end{aligned}
$$

that is

$$
(p(T;v) - p(T;u), p(T;u) - z_d)_{\Omega_A} = (v - u, q)_Q. \tag{17}
$$

Thus, it follows from (17) that (9) becomes

$$
\int_Q (q(u) + \rho u)(v - u)dQ \geq 0, \qquad \forall v \in \mathcal{U}. \tag{18}
$$

Then we obtained:

Theorem 2.3. Assume that the state of the system $(\mathcal{P})$ is defined by $(1)-(4)$. Then the optimal control u to corresponding to cost functional (7) is determined by the optimality system consisting of the equation $(1)-(4)$ (where $v = u$) and the adjoint equation $(12)-(15)$ with the variation inequality (18).

3 Penalty Shifting Method for Numerical Approximation

The optimal control problem (8) can be written as the minimization problem $(\mathcal{P}_1)$:

$$
\begin{cases}
\text{with respect to } (p(v), v) \text{ under constraints } (1) - (4) \text{ and} \\
v \in \mathcal{U}, \text{ find } u \in \mathcal{U} \text{ satisfying the equality} \\
\inf_{v \in \mathcal{U}} I(p(v), v) = I(p(u), u), \text{ where } I(p(v), v) = I(v) \text{ in (7).}
\end{cases} \tag{19}
$$

We research the application of penalty shifting method that Di Pillo statemented for infinte dimensional systems (cf. Refs. 8, 9) to the approximate solution of the problem $(\mathcal{P}_1)$. We shall approximate the solution $(p(u), u)$ of the constrained minimization problem $(\mathcal{P}_1)$ by a family $\{(p_k, u_k)\}$ of solution of the non-constrained minimization problem in which p and v become the independent variables.

We introduce the set

$$Y = \{p \mid p \in V,\ p \geq 0,\ Lp \in L^2(Q),\ (Bp)(\cdot\ ,\ \cdot) \in L^2(\Omega_T)\}, \tag{20}$$

where

$$(Bp)(t,x) \equiv\ p(0,t,x) - \int_0^A \beta(r,t,x)p(r,t,x)\mathrm{d}r.$$

Endowed with the norm

$$\|p\|_Y = (\|p\|_V^2 + \|Lp\|_Q^2 + \|Bp\|_{\Omega_T}^2)^{1/2}, \tag{21}$$

Y is a closed convex subset in a Hilbert space,where $\|\cdot\|_E = \|\cdot\|_{L^2(E)}, E = Q, \Omega_T, \Omega_A, \Sigma.$

Now let $c \geq 0, \xi = (\lambda, \eta, \zeta)$ with $\lambda \in L^2(Q), \eta \in L^2(\Omega_T), \zeta \in L^2(\Omega_A)$, and define augmented Lagrangian:

$$\begin{aligned} J(p,v,c,\xi) = I(p,v) &+ c\ [\|\ Lp - v\|_Q^2 + \|Bp\|_{\Omega_T}^2 + \|p(\cdot\ ,0,\ \cdot) - p_0\|_{\Omega_A}^2] + (\lambda,\ Lp - v)_Q \\ &+ (\eta,\ Bp)_{\Omega_T} + (\zeta,\ p(\cdot, 0,\ \cdot) - p_0)_{\Omega_A} \end{aligned} \tag{22}$$

on the set $Y \times \mathcal{U}$, where $(\ \cdot,\ \cdot)_E$ denotes the scalar product in $L^2(E)$. In $J(p,v,c,\xi)$,
p and v are independent variables.

We state first the following result:

Theorem 3.1 For ang given ξ and $c \geq 0$, the minimization problem($\mathcal{P}_2$)

$$\inf_{p\in Y,\ v\in\mathcal{U}} J(p,v,c,\xi) \tag{23}$$

admits a unique solution $(\hat{p},\ \hat{v}) \in Y \times \mathcal{U}$.

proof. Set

$$w = (p,v), \quad w \in W = Y \times \mathcal{U}. \tag{24}$$

Then we have $J(w,c,\xi) = J(p,v,c,\xi)$. Thus,the minimization problem (3.5) comes down to a problem

$$\inf_{w\in W} J(w,c,\xi). \tag{25}$$

Clearly, W is a closed convex subset of a Hilbert space. We prove first that $J(w,c,\xi)$ is radially unbounded on W. We proceed by contradiction. Assume that there exists a sequence $\{(p_m, v_m, c, \xi)\}$ such that $(\|p_m\|_Y^2 + \|v_m\|_Q^2)^{1/2} \to +\infty$ and $J(p_m, v_m, c, \xi) \to l < +\infty$. It can be easily verified that this implies:

$$\|v_m\|_Q \leq C_1, \quad \|p_m(\cdot,0,\cdot)\|_{\Omega_A} \leq C_2, \quad \|Lp_m - v_m\|_Q \leq C_3, \quad \|Bp_m\|_{\Omega_T} \leq C_4. \tag{26}$$

Hence, in particular, by$(3.8)_1$, $(3.8)_3$, we have:

$$\|Lp_m\|_Q \leq C_5. \tag{27}$$

Taking into account the continuity of the map $(v,\ Bp,\ p_0) \to p$ defined by Theorem 2.1, we have by (26) and (27):

$$\|p_m\|_V \quad \leq C_6. \tag{28}$$

Then, from (28), (27), $(26)_4$ we get a contradiction with the original assumption. This proves that $J(w, c, \xi)$ is radially unbounded on W. Moreover, from the definition (22), (24) of J , it can be easily verified that J is also strictly convex and continuous; then from Ref. 7 (Remark 1.2, Chapter 1, p.8), we deduce that there exists a unique element $\hat{w} = (\hat{p},\ \hat{v})$ in $Y \times \mathcal{U} = W$ such that $J(\hat{w},\ c, \xi) = \inf\limits_{w \in W} J(w, c, \xi)$ i.e. $J(\hat{p},\ \hat{v},\ c, \xi) = \inf\limits_{p \in Y,\ v \in \mathcal{U}} J(p, v, c, \xi)$. Thus , the theorem 3.1 is proved.

Lemma 3.1. Let arbitrary point $(\bar{p}, \bar{v})$ be given in $Y \times \mathcal{U}$. Then for any given ξ and $c > 0$, we have

$$\begin{aligned} J(p, v, c, \xi) = {} & J(\bar{p}, \bar{v}, c, \xi) + \|p(\cdot, T, \cdot) - \bar{p}(\cdot, T, \cdot)\|^2_{\Omega_A} + \rho\|v - \bar{v}\|^2_Q \\ & + c[\ \|L(p - \bar{p}) - (v - \bar{v})\|^2_Q + \|B(p - \bar{p})\|^2_{\Omega_T} + \|p(\cdot, 0, \cdot) - \bar{p}(\cdot, 0, \cdot)\|^2_{\Omega_A}] \\ & + J'(\bar{p}, \bar{v}, c, \xi, p - \bar{p}, v - \bar{v}), \quad \forall p \in Y, \quad \forall v \in \mathcal{U}. \end{aligned} \tag{29}$$

Proof. According to definitions (22) and (7) of J and I and by noting that

$$\|y\|^2_E - \|\bar{y}\|^2_E = \|y - \bar{y}\|^2_E + 2(y - \bar{y},\ \bar{y})_E, \tag{30}$$

we have:

$$\begin{aligned} & J(p, v, c, \xi) - J(\bar{p}, \bar{v}, c, \xi) \\ & \ = \|p - \bar{p}\|^2_{\Omega_A} + \rho\|v - \bar{v}\|^2_Q + c[\ \|L(p - \bar{p}) - (v - \bar{v})\|^2_Q + \|B(p - \bar{p})\|^2_{\Omega_T} \\ & \quad + \|p(\cdot, 0, \cdot) - \bar{p}(\cdot, 0, \cdot)\|^2_{\Omega_A}] \qquad + \{2(p - \bar{p}, \bar{p} - z_d)_{\Omega_A} + 2\rho(v - \bar{v}, \bar{v})_Q \\ & \quad + 2c[(L(p - \bar{p}) - (v - \bar{v}), L\bar{p} - \bar{v})_Q + (B(p - \bar{p}), B\bar{p})_{\Omega_T} \\ & \quad + (p(\cdot, 0, \cdot) - \bar{p}(\cdot, 0, \cdot), \bar{p}(\cdot, 0, \cdot) - p_0)_{\Omega_A}] + (\lambda, L(p - \bar{p}) - (v - \bar{v}))_Q \\ & \quad + (\eta, B(p - \bar{p}))_{\Omega_T} + (\zeta, p(\cdot, 0, \cdot) - \bar{p}(\cdot, 0, \cdot))_{\Omega_A}\}. \end{aligned} \tag{31}$$

In order to prove (29), it suffices to prove that the part $\{\ldots\}$ in (31) is equal to $J'(\bar{p}, \bar{v}, c, \xi, p - \bar{p}, v - \bar{v})$. From the definition of Gâteaue differentiation and (30), we have:

$$\begin{aligned} & J'(\bar{p}, \bar{v}, c, \xi, p - \bar{p}, v - \bar{v}) = \lim_{\theta \to 0^+} \frac{1}{\theta}[J(\bar{p} + \theta(\bar{p} - p), \bar{v} + \theta(v - \bar{v}), c, \xi) - J(\bar{p}, \bar{v}, c, \xi)] \\ & = \lim_{\theta \to 0^+} \frac{1}{\theta}\{\|\bar{p} + \theta(p - \bar{p}) - z_d\|^2_{\Omega_T} + \rho\|\bar{v} + \theta(v - \bar{v})\|^2_Q + c[\ \|L(\bar{p} + \theta(p - \bar{p})) \\ & \quad - (\bar{v} + \theta(v - \bar{v}))\|^2_Q + \|B(\bar{p} + \theta(p - \bar{p}))\|^2_{\Omega_T} + \|(\bar{p} + \theta(p - \bar{p}))(\cdot, 0, \cdot) - p_0\|^2_{\Omega_A}] \end{aligned}$$

$$+(\lambda, L(\bar{p}+\theta(p-\bar{p}))-(\bar{v}+\theta(v-\bar{v})))_Q+(\eta, B(\bar{p}+\theta(p-\bar{p})))_{\Omega_T}$$
$$+(\zeta,(\bar{p}+\theta(p-\bar{p}))(\cdot,0,\cdot)-p_0)_{\Omega_A}-\|\bar{p}-z_d\|^2_{\Omega_T}-\rho\|\bar{v}\|^2_Q-c[\ \|L\bar{p}-\bar{v}\|^2_Q$$
$$+\|B\bar{p}\|^2_{\Omega_T}+\|\bar{p}(\cdot,0,\cdot)-p_0\|^2_{\Omega_A}]-(\lambda, L\bar{p}-\bar{v})_Q-(\eta, B\bar{p})_{\Omega_T}-(\zeta,\bar{p}(\cdot,0,\cdot)-p_0)_{\Omega_A}\}$$
$$=\{0+2(p-\bar{p},\bar{p}-z_d)_{\Omega_A}+0+2\rho(v-\bar{v},\bar{v})_Q+c[\ 0+2(L(p-\bar{p})-(v-\bar{v}),$$
$$L\bar{p}-\bar{v})_Q+0+2(B(p-\bar{p}),B\bar{p})_{\Omega_T}+0+2((p-\bar{p})(\cdot,0,\cdot),\bar{p}(\cdot,0,\cdot)-p_0)_{\Omega_A}]$$
$$+(\lambda, L(p-\bar{p})-(v-\bar{v}))_Q+(\eta, B(p-\bar{p}))_{\Omega_T}+(\zeta, p-\bar{p})_{\Omega_A}\}$$
$$=\{\ldots\}\text{ in (31).}$$

Lemma 3.1 is now proved.

Lemma 3.2. Let $\bar{w}=(\bar{p},\bar{u})$ be the minimizing point of $J(w,c,\xi)=J(p,v,c,\xi)$ in $W=Y\times\mathcal{U}$. Then, for any given ξ and $c>0$, we have

$$J(p,v,c,\xi)\geq J(\bar{p},\bar{u},c,\xi)+\|p(\cdot,T,\cdot)-\bar{p}(\cdot,T,\cdot)\|^2_{\Omega_A}+\rho\|v-\bar{u}\|^2_Q+c[\|L(p-\bar{p})$$
$$-(v-\bar{v})\|^2_Q+\|B(p-\bar{p})\|^2_{\Omega_T}+\|p(\cdot,0,\cdot)-\bar{p}(\cdot,0,\cdot)\|^2_{\Omega_A}]\quad\forall p\in Y,\ \ \forall v\in\mathcal{U}. \tag{32}$$

Proof. By setting $\bar{p}=\bar{p}$ and $\bar{v}=\bar{u}$ in (31), It follows from the necessary optimality condition in Ref. 7 (Theorem 1.3, Chapter 1, p.10) that

$$J'(\bar{w},c,\xi,w-\bar{w})=J'(\bar{p},\bar{u},c,\xi,p-\bar{p},v-\bar{u})\geq 0\quad\forall w=(p,v)\in Y\times\mathcal{U}=W. \tag{33}$$

From (31) and (33), we arrive at (32). Lemma 3.2 is proved.

Lemma 3.3. Let $(\tilde{p},u)$ be the optimal solution of the problem $(\mathcal{P}_1)$, where $\tilde{p}=p(r,t,x;u)$. Then there exists $\tilde{\xi}=(\tilde{\lambda},\tilde{\eta},\tilde{\zeta})$ such that

$$J(p,v,0,\tilde{\xi})\geq I(\tilde{p},u)+\|p(\cdot,T,\cdot)-\tilde{p}(\cdot,T,\cdot)\|^2_{\Omega_A}+\rho\|v-u\|^2_Q\quad\forall(p,v)\in Y\times\mathcal{U}. \tag{34}$$

Proof. Let $q(u)$ be the adjoint state given equations (22)$-$(25) (where $p(u)=\tilde{p}(u)$) and assume:

$$\tilde{\lambda}=-2q\ \text{ in } Q,\qquad \tilde{\eta}=-2q(0,t,x)\ \text{ on }\Omega_T,\qquad \tilde{\zeta}=-2q(r,0,x)\ \text{ on }\Omega_A. \tag{35}$$

Making use of the integration by parts and the Green′s formula in Ref. 7 yields:

$$\begin{aligned}\int_Q qLp\mathrm{d}Q=&\int_Q pL^*q\mathrm{d}Q+\int_{\Omega_T}[pq(A,t,x)-(pq)(0,t,x)]\mathrm{d}t\mathrm{d}x\\&+\int_{\Omega_A}[(pq)(r,T,x)-(pq)(q,0,x)]\mathrm{d}r\mathrm{d}x\\&+k\int_\Sigma(p\frac{\partial q}{\partial\nu^*}-q\frac{\partial p}{\partial\nu})d\Sigma+\int_Q(\beta p)(r,t,x)q(0,t,x)\mathrm{d}Q\end{aligned} \tag{36}$$

By applying (36) and noting that $q(u)$ satisfies (22)$-$(25) and $\tilde{p}(u)$ satisfies

(1)$-$(4), we have:

$$\begin{aligned}(q, L(p-\tilde{p}))_Q &= (L^*q, p-\tilde{p})_Q + (q(A,\cdot,\cdot), (p-\tilde{p})(A,\cdot,\cdot))_{\Omega_T}\\ &\quad - (q(0,\cdot,\cdot), (p-\tilde{p})(0,\cdot,\cdot))_{\Omega_T} + (q(\cdot,T,\cdot), (p-\tilde{p})(\cdot,T,\cdot))_{\Omega_A}\\ &\quad - (q(\cdot,0,\cdot), (p-\tilde{p})(\cdot,0,\cdot))_{\Omega_A} + (q(0,t,x), (\beta(p-\tilde{p}))(r,t,x))_Q\\ &= (\tilde{p}(\cdot,T,\cdot) - z_d, (p-\tilde{p})(\cdot,T,\cdot))_{\Omega_A} - (q(\cdot,0,\cdot), (p-\tilde{p})(\cdot,0,\cdot))_{\Omega_A}\\ &\quad - (q(0,\cdot,\cdot), (Bp)(0,\cdot,\cdot) - (B\tilde{p})(0,\cdot,\cdot))_{\Omega_A}.\end{aligned} \tag{37}$$

From (36) (set $\bar{p} = \tilde{p}$, $\bar{v} = u$), (22)(set $c = 0$), (35), (18) (30) and (37), we have:

$$\begin{aligned}&J'(\tilde{p}, u, 0, \tilde{\xi}, p-\tilde{p}, v-u)\\ &= J(p, v, 0, \tilde{\xi}) - J(\tilde{p}, u, 0, \tilde{\xi}) - \|p(\cdot,T,\cdot) - \tilde{p}(\cdot,T,\cdot)\|^2{}_{\Omega_A} - \rho\|v-u\|_Q^2\\ &= I(p,v) + (\tilde{\lambda}, Lp - v)_Q + (\tilde{\eta}, Bp)_{\Omega_T} + (\tilde{\zeta}, p(\cdot,0,\cdot) - p_0)_{\Omega_A} - I(\tilde{p}, u) + (\tilde{\lambda}, L\tilde{p} - u)_Q\\ &\quad - (\tilde{\eta}, B\tilde{p})_{\Omega_T} - (\tilde{\zeta}, \tilde{p}(\cdot,0,\cdot) - p_0)_{\Omega_A} - \|p(\cdot,T,\cdot) - \tilde{p}(\cdot,T,\cdot)\|_{\Omega_A}^2 - \rho\|v-u\|_Q^2\\ &= 2(p(\cdot,T,\cdot) - \tilde{p}(\cdot,T,\cdot),\ \tilde{p}(\cdot,T,\cdot) - z_d)_{\Omega_A} - 2(p(\cdot,T,\cdot) - z_d,\ p(\cdot,T,\cdot) - \tilde{p}(\cdot,T,\cdot))_{\Omega_A}\\ &\quad + 2(q(\cdot,0,\cdot),\ p(\cdot,0,\cdot) - \tilde{p}(\cdot,0,\cdot))_{\Omega_A} - 2(q,v\rangle_Q + 2\rho(v-u,u)_Q - 2(q(\cdot,0,\cdot), p(\cdot,0,\cdot)\\ &\quad - \tilde{p}(\cdot,0,\cdot))_{\Omega_A} + (2q,u)_Q - 2(q, B(p-\tilde{p}))_{\Omega_T} + 2(q, B(p-\tilde{p}))_{\Omega_A}\\ &= 2(q + \rho u, v-u)_Q \geq 0, \qquad \forall \in \mathcal{U},\end{aligned}$$

that is

$$J'(\tilde{p}, u, 0, \tilde{\xi}, p-\tilde{p}, v-u) \geq 0 \qquad \forall v \in \mathcal{U}. \tag{38}$$

On the other hand, since $(\tilde{p}, u)$ is a solution of the problem (1)$-$(4), the following equality holds:

$$J(\tilde{p}, u, 0, \tilde{\xi}) = I(\tilde{p}, u). \tag{39}$$

From (29) (where $c = 0$, $\bar{p} = \tilde{p}$, $\tilde{v} = u$), and (39), we have

$$\begin{aligned}J(p, v, 0, \tilde{\xi}) &= I(\tilde{p}, u) + \|p(\cdot,T,\cdot) - \tilde{p}(\cdot,T,\cdot)\|_{\Omega_A}^2 + \rho\|v-u\|_Q^2\\ &\quad + J'(\tilde{p}, u, 0, \tilde{\xi}, p-\tilde{p}, v-u).\ \ \forall p \in Y, \quad \forall v \in \mathcal{U}.\end{aligned} \tag{40}$$

Thus, (34) follows from (38), (40). Lemma 3.3 is proved.

Let now (p_m, u_m) be the minimizing point of $J(p, v, c, \xi)$ and consider the sequence $\{(p_m, u_m)\}$ obtained by employing the following multiplier adjustment rule:

$$\begin{cases}\lambda_{m+1} = \lambda_m + \alpha c(Lp_m - u_m) \quad \text{in } Q,\\ \eta_{m+1} = \eta_m + \alpha c B p_m \quad \text{on } \Omega_T,\\ \zeta_{m+1} = \zeta_m + \alpha c(p_m(\cdot,0,\cdot) - p_0) \quad \text{on } \Omega,\end{cases} \tag{41}$$

where $0 < \alpha \leq 2$ and $\xi_0 = (\lambda_0, \eta_0, \zeta_0)$ is any given initial value in $L^2(Q) \times L^2(\Omega_T) \times L^2(\Omega)$.

Then we can prove the following result:

Theorem 3.2. The sequence $\{(p_m, u_m)\}$ converges strongly in $Y \times L^2(Q)$ to the optimal solution (p, u) of the problem $(\mathcal{P}_1)$, where $p = p(u)$.

Proof. Let $\tilde{\xi} \equiv (\tilde{\lambda}, \tilde{\eta}, \tilde{\zeta})$ be the multiplier introduced in the proof of Lemma 3.3,
i.e. (35), in order to write in pithy style, $\tilde{\xi}$ be still denoted by $\xi \equiv (\lambda, \eta, \zeta)$. we have from (41) that

$$\begin{cases} \|\lambda_m - \lambda\|_Q^2 = \|\lambda_{m+1} - \lambda\|_Q^2 - \alpha^2 c^2 \|Lp_m - u_m\|_Q^2 - 2\alpha c(\lambda_m - \lambda, Lp_m - u_m)_Q, \\ \|\eta_m - \eta\|_{\Omega_T}^2 = \|\eta_{m+1} - \eta\|_{\Omega_T}^2 - \alpha^2 c^2 \|Bp_m\|_{\Omega_T}^2 - 2\alpha c(\eta_m - \eta, Bp_m)_{\Omega_T}, \\ \|\zeta_m - \zeta\|_{\Omega_A}^2 = \|\zeta_{m+1} - \zeta\|_{\Omega_A}^2 - \alpha^2 c^2 \|p_m(\cdot, 0, \cdot) - p_0\|_{\Omega_A}^2 \\ \qquad\qquad - 2\alpha c(\zeta_m - \zeta, p_m(\cdot, 0, \cdot) - p_0)_{\Omega_A}. \end{cases} \tag{42}$$

From Lemma 3.2 and (39), let $p = p(u)$, $v = u$, $\bar{p} = p$, $\bar{u} = u_m$, $\zeta = \zeta_m$ in (32), we get:

$$I(p, u) \geq J(p_m, u_m, c, \xi_m) + \|p - p_m\|_{\Omega_A}^2 + \rho\|u - u_m\|_Q^2 + c[\ \|L(p - p_m) - (u - u_m)\|_Q^2 + \|B(p - p_m)\|_{\Omega_T}^2 + \|p(\cdot, 0, \cdot) - p_m(\cdot, 0, \cdot)\|_{\Omega_A}^2]. \tag{43}$$

From Lemma 3.3 and let $p = p_m$, $v = u_m$, $\tilde{p} = p(u)$, $u = u$ in (34), we get:

$$J(p_m, u_m, 0, \xi) \geq I(p, u) + \|p(\cdot, T, \cdot) - p_m(\cdot, T, \cdot)\|_{\Omega_A}^2 + \rho\|u - u_m\|_Q^2. \tag{44}$$

Adding (43) to (44) and noting $Lp = u$, $Bp|_{\Omega_T} = 0$, we obtain:

$$J(p_m, u_m, 0, \xi) \geq J(p_m, u_m, c, \xi_m) + 2\|p(\cdot, T, \cdot) - p_m(\cdot, T, \cdot)\|_{\Omega_A}^2 + 2\rho\|u - u_m\|_Q^2 + c[\ \|Lp_m - u_m\|_Q^2 + \|Bp_m\|_{\Omega_T}^2 + \|p(\cdot, 0, \cdot) - p_m(\cdot, 0, \cdot)\|_{\Omega_A}^2]. \tag{45}$$

Noting the definition (22) of J, we obtain from (45) that

$$\begin{aligned} & I(p_m, u_m) + (\lambda, Lp_m - u_m)_Q + (\eta, Bp_m)_{\Omega_T} + (\zeta, p_m(\cdot, 0, \cdot) - p_0)_{\Omega_A} \\ & \geq I(p_m, u_m) + c[\|Lp_m - u_m\|_Q^2 + \|Bp_m\|_{\Omega_T}^2 + \|p_m(\cdot, 0, \cdot) - p_0\|_{\Omega_A}^2] + (\lambda_m, Lp_m - u_m)_Q \\ & +(\eta_m, Bp_m)_{\Omega_T} + (\zeta_m, p_m(\cdot, 0, \cdot) - p_0)_{\Omega_A} + 2\|p(\cdot, T, \cdot) - p_m(\cdot, T, \cdot)\|_{\Omega_A}^2 \\ & +2\rho\|u - u_m\|_Q^2 + c[\ \|Lp_m - u_m\|_Q^2 + \|Bp - Bp_m\|_{\Omega_T}^2 + \|p(\cdot, 0, \cdot) - p_m(\cdot, 0, \cdot)\|_{\Omega_A}^2]. \end{aligned} \tag{46}$$

Rearranging terms and noting that $\|p(\cdot, T, \cdot) - p_m(\cdot, T, \cdot)\|_{\Omega_A} \geq 0$ in (46) we obtain:

$$\begin{aligned} & (\lambda, Lp_m - u_m)_Q + (\eta, Bp_m)_{\Omega_T} + (\zeta, p_m(\cdot, 0, \cdot) - p_0)_{\Omega_A} \\ & \geq (\lambda_m, Lp_m - u_m)_Q + (\eta_m, Bp_m)_{\Omega_T} + (\zeta, p_m(\cdot, 0, \cdot) - p_0)_{\Omega_A} + 2c[\ \|Lp_m - u_m\|_Q^2 \\ & +\|Bp_m\|_{\Omega_T}^2] + c[\ \|p(\cdot, 0, \cdot) - p_m(\cdot, 0, \cdot)\|_{\Omega_A}^2 + \|p_m(\cdot, 0, \cdot) - p_0\|_{\Omega_A}^2] + 2\rho\|u - u_m\|_Q^2. \end{aligned} \tag{47}$$

Recalling that $0 < \alpha \leq 2$, and hence $-2\alpha c^2 \leq -\alpha^2 c^2$.

Thus, we have from (41) that

$$\begin{cases} \|\lambda_m - \lambda\|_Q^2 \geq \|\lambda_{m+1} - \lambda\|_Q^2 - 2\alpha c^2 \|Lp_m - u_m\|_Q^2 - 2\alpha c(\lambda_m, Lp_m - u_m)_Q \\ \qquad\qquad + 2\alpha c(\lambda, Lp_m - u_m)_Q, \\ \|\eta_m - \eta\|_{\Omega_T}^2 \geq \|\eta_{m+1} - \eta\|_{\Omega_T}^2 - 2\alpha c^2 \|Bp_m\|_{\Omega_T}^2 - 2\alpha c(\eta_m, Bp_m)_{\Omega_T} + 2\alpha c(\eta, Bp_m)_{\Omega_T}, \\ \|\zeta_m - \zeta\|_{\Omega_A}^2 \geq \|\zeta_{m+1} - \zeta\|_{\Omega_A}^2 - 2\alpha c^2 \|p_m(\cdot,0,\cdot) - p_0\|_{\Omega_A}^2 - 2\alpha c(\zeta_m, p_m(\cdot,0,\cdot) - p_0)_{\Omega_A} \\ \qquad\qquad + 2\alpha c(\zeta, p_m(\cdot,0,\cdot) - p_0)_{\Omega_A}. \end{cases} \tag{48}$$

From (48), (47), we obtain:

$$\begin{aligned} &\|\lambda_m - \lambda\|_Q^2 + \|\eta_m - \eta\|_{\Omega_T}^2 + \|\zeta_m - \zeta\|_{\Omega_A}^2 \\ &\geq \|\lambda_{m+1} - \lambda\|_Q^2 + \|\eta_{m+1} - \eta\|_{\Omega_T}^2 + \|\zeta_{m+1} - \zeta\|_{\Omega_A}^2 - 2\alpha c^2 \|Lp_m - u_m\|_Q^2 \\ &\quad - 2\alpha c(\lambda_m, Lp_m - u_m)_Q + 2\alpha c(\lambda, Lp_m - u_m)_Q - 2\alpha c^2 \|Bp_m\|_{\Omega_T}^2 \\ &\quad - 2\alpha c(\eta_m, Bp_m)_{\Omega_T} + 2\alpha c(\eta, Bp_m)_T - 2\alpha c^2 \|p_m(\cdot,0,\cdot) - p_0\|_{\Omega_A}^2 \\ &\quad - 2\alpha c(\zeta_m, p_m(\cdot,0,\cdot) - p_0)_{\Omega_A} + 2\alpha c(\zeta, p_m(\cdot,0,\cdot) - p_0)_{\Omega_A} \\ &\geq 2\alpha c[(\lambda_m, Lp_m - u_m)_Q + (\eta_m, Bp_m)_{\Omega_T} + (\zeta_m, p_m(\cdot,0,\cdot) - p_0)_{\Omega_A}] \\ &\quad + 4\alpha c^2 [\ \|Lp_m - u_m\|_Q^2 + \|Bp_m\|_{\Omega_T}^2] + 2\alpha c^2 [\|p(\cdot,0,\cdot) - p_m(\cdot,0,\cdot)\|_{\Omega_A}^2 \end{aligned}$$

$$\begin{aligned} &+ \|p_m(\cdot,0,\cdot) - p_0\|_{\Omega_A}^2] + 4\rho\alpha c \|u - u_m\|_Q^2 + \|\lambda_{m+1} - \lambda\|_Q^2 + \|\eta_{m+1} - \eta\|_{\Omega_T}^2 \\ &+ \|\zeta_{m+1} - \zeta\|_{\Omega_A}^2 - 2\alpha c^2 \|Lp_m - u_m\|_Q^2 - 2\alpha c(\lambda_m, Lp_m - u_m)_Q - 2\alpha c^2 \|Bp_m\|_{\Omega_T}^2 \\ &- 2\alpha c(\eta_m, Bp_m)_{\Omega_T} - 2\alpha c^2 \|p_m(\cdot,0,\cdot) - p_0\|_{\Omega_A}^2 - 2\alpha c(\zeta_m, p_m(\cdot,0,\cdot) - p_0)_{\Omega_A} \\ = &\|\lambda_{m+1} - \lambda\|_Q^2 + \|\eta_{m+1} - \eta\|_{\Omega_T}^2 + \|\zeta_{m+1} - \zeta\|_{\Omega_A}^2 + 2\alpha c^2 \|Lp_m - u_m\|_Q^2 \\ &+ 2\alpha c^2 \|Bp_m\|_{\Omega_T}^2 + 2\alpha c^2 \|p_m(\cdot,0,\cdot) - p(\cdot,0,\cdot)\|_{\Omega_A}^2 + 4\rho\alpha c \|u - u_m\|_Q^2. \end{aligned}$$

that is

$$\begin{aligned} &\|\lambda_m - \lambda\|_Q^2 + \|\eta_m - \eta\|_{\Omega_T}^2 + \|\zeta_m - \zeta\|_{\Omega_A}^2 \\ &\geq \|\lambda_{m+1} - \lambda\|_Q^2 + \|\eta_{m+1} - \eta\|_{\Omega_T}^2 + \|\zeta_{m+1} - \zeta\|_{\Omega_A}^2 + 2\alpha c^2 \|Lp_m - u_m\|_Q^2 \\ &\quad + 2\alpha c^2 \|Bp_m\|_{\Omega_T}^2 + 2\alpha c^2 \|p_m(\cdot,0,\cdot) - p(\cdot,0,\cdot)\|_{\Omega_A}^2 + 4\rho\alpha c \|u - u_m\|_Q^2. \end{aligned} \tag{49}$$

From it follows that the sequence $\{\|\lambda_m - \lambda\|_Q^2 + \|\eta_m - \eta\|_{\Omega_T}^2 + \|\zeta_m - \zeta\|_{\Omega_A}^2\}$ is nonincreasing and therefore it admits a limit. This implies: as $m \to +\infty$,

$$\|u_m - u\|_Q \to 0,\ \|Lp_m - u_m\|_Q^2 \to 0,\ \|Bp_m\|_{\Omega_T}^2 \to 0,\ \|p_m(\cdot,0,\cdot) - p(\cdot,0,\cdot)\|_{\Omega_A} \to 0. \tag{50}$$

Since $Lp = u$ and $\|L\,(p_m - p)\|_Q \leq \|Lp_m - u_m\|_Q + \|u_m - u\|_Q$, from $(50)_1$, $(50)_2$ we deduce: as m $\to +\infty$,

$$\|L\ (p_m - p)\|_Q \to 0. \tag{51}$$

According to $(Bp)(t,x) = 0$, it follows from $(50)_3$ that as $m \to +\infty$,

$$\|B\ (p_m - p)\|_{\Omega_T} = \|Bp_m\|_{\Omega_T} \to 0. \tag{52}$$

Taking into account the continuity of the solution of equations $(1) - (4)$ with respect to (v, Bp, p_0) in Theorem 2.1, from (50) and (52) it can be easily deduce that $\{(p_m, u_m)\}$ converges strongly in $Y \times \mathcal{U}$ to (p, u) as $m \to +\infty$. Theorem 3.2 is proved.

4 Conclusions

In this paper, we investigate the optimal distributed control problem for age-dependent population diffusion system. For the cost as a quadratic functional, the existence and uniqueness of the optimal control for the system is proved, and the necessary and sufficient condition for a control to be optimal is obtained. The optimality system determining the optimal control is deduced. The application of penalty shifting method to the approximate solution of the control problem for the population system is researched, and the convergence of method in appropriate Hilbert spaces is derived. These results may significantly provide theoretical reference for the practical research of the control problem in population systems.

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