Gen. Math. Notes, Vol. 4, No. 1, May 2011, pp.90-98
ISSN 2219-7184; Copyright ©ICSRS Publication, 2011
www.i-csrs.org
Available free online at http://www.geman.in

New Perspectives on CDPU Graphs

Beena Koshy[1] **and Germina K.A.**[2]

[1]Research Center and PG Department of Mathematics,
Mary Matha Arts & Science College,
Vemom P.O., Mananthavady - 670645, India
E-mail: beenakoshy1@yahoo.co.in
[2]Research Center and PG Department of Mathematics,
Mary Matha Arts & Science College,
Vemom P.O., Mananthavady - 670645, India
E-mail: srgerminaka@gmail.com

(Received: 24-2-11/Accepted: 6-4-11)

Abstract

A graph $G = (V, E)$ is Complementary Distance Pattern Uniform *if there exists $M \subset V(G)$ such that $f_M(u) = \{d(u, v) : v \in M\}$, for every $u \in V(G) - M$, is independent of the choice of $u \in V(G) - M$ and the set M is called the Complementary Distance Pattern Uniform Set (CDPU set). In this paper, we initiate a study on the CDPU sets of trees.*

Keywords: *Complementary Distance Pattern Uniform.*

1 Introduction

For all terminology and notation in graph theory, not defined specifically in this paper, we refer the reader to F. Harary [2]. Unless mentioned otherwise, all the graphs considered in this paper are simple, self-loop-free and finite.

B.D.Acharya define the M - distance pattern of a vertex as follows :

Definition 1.1. *[3] Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, each vertex $u \in G$ is associated with the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between the vertices u and v in G, is called the M- vertex distance pattern of u.*

Definition 1.2. *[1] If $f_M(u)$ is independent of the choice of $u \in V - M$, then G is called a Complementary Distance Pattern Uniform (CDPU) Graph. The set M is called the CDPU set. The least cardinality of CDPU set in G is called the* CDPU number *of G, denoted by $\sigma(G)$.*

Theorem 1.3. *[1] Every connected graph has a CDPU set.*

Theorem 1.4. *[1] A graph G has $\sigma(G) = 1$ if and only if G has atleast one vertex of full degree.*

Theorem 1.5. *[1] For any integer n, $\sigma(P_n) = n - 2$.*

Theorem 1.6. *Let G be a graph with n vertices and CDPU set M. Then the vertices in $V - M$ possess same eccentricity.*

Proof. Let $M = \{v_1, v_2, \ldots, v_j\}$ be the CDPU set of G. Let $V - M = \{z_1, z_2, \ldots, z_l\}$. Then $f_M(z_1) = f_M(z_2) = \cdots = f_M(z_l)$.
$\{d(z_1, M)\} = \{d(z_2, M)\} = \cdots = \{d(z_l, M)\}$.
Thus $z_1, z_2, \ldots, z_l$ have same eccentricities.

But the converse need not be true. That is if there exists a set of vertices which possess same eccentricity, then the remaining set of vertices need not be a CDPU set. In Figure 1, $\{v_2, v_6, v_7\}$ have same eccentricities, but $\{v_1, v_3, v_4, v_5, v_8\}$ is not a CDPU set.

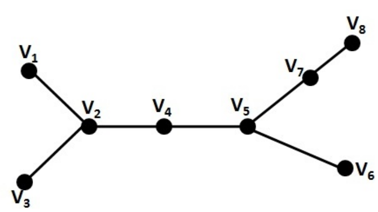

Figure 1:

Proposition 1.7. Let G be a non-self centered graph with vertex set V. Then $V - \{antipodal\ vertices\}$ and $V - \{central\ vertices\}$ are CDPU sets.

Proof. Let $\{v_{i1}, v_{i2}, \ldots, v_{ik}\}$ be the antipodal vertices. Then $f_M(v_{ij}) = \{1, 2, \ldots, d - 1\}, \forall j = 1, 2, \ldots, k$.

Also let $M = \{u, v\}$ be the central vertices of G. Then $f_M(u) = f_M(v) = \{1, 2, \ldots, r\}$.

Theorem 1.8. *Let G be a non-self centered graph having no full degree vertex. Then $\sigma(G) = 2$ if and only if G has exactly two different eccentricities $e_i < e_j$ and exactly two vertices corresponds to atleast one of the eccentricities.*

Proof. Let the vertices correspond to e_i be $\{v_{i1}, v_{i2}\}$ and the vertices correspond to e_j be $\{v_{j1}, v_{j2}, \ldots, v_{jq}\}$. Take $M = \{v_{i1}, v_{i2}\}$. Then $f_M(v_{j1}) = f_M(v_{j2}) = \cdots = f_M(v_{jq}) = \{1, 2\}$. Hence $\sigma(G) = 2$.

Conversely suppose that $\sigma(G) = 2$. Then there exists an M with $|M| = 2$ such that the vertices in $V - M$ should have the same eccentricity. since G is not self centered and $|M| = 2$, then the vertices in M also should have the same eccentricity. Hence G has two different eccentricities.

If G has more than two eccentricities, then $|M| > 2$, which is not possible.

Remark 1.9. Let G be a graph with no full degree vertices and exactly two different eccentricities. Then there are two CDPU sets M_1 and M_2 such that $M_1 \cap M_2 = \emptyset$.

2 CDPU Trees

In this section, we are characterizing the trees with $\sigma(T) = 1$, $\sigma(T) = 2$ *and* $\sigma(T) = 3$. Also through out this paper, in all the figures, the CDPU sets are represented by white circles.

Proposition 2.1. Let T be a tree. Then $\sigma(T) = 1$ if and only if $T \approx K_{1,n}$.

Proof. Suppose $T \approx K_{1,n}$. Then clearly $\sigma(T) = 1$. Conversely assume that $\sigma(T) = 1$. That is there exists a tree with atleast one full degree vertex. The only tree with a full degree vertex is $K_{1,n}$.

Proposition 2.2. $\sigma(B(m, n)) = 2$.

Proof. Let $T \approx B(m, n)$. Let u and v be the central vertices of T, $u_1, u_2, \ldots, u_m$ be the vertices attached to u and $v_1, v_2, \ldots, v_n$ be the vertices attached to v. Take $M = \{u, v\}$. Then $f_M(u_i) = \{1, 2\}, \forall\ i = 1, 2, \ldots, m$ and $f_M(v_j) = \{1, 2\}, \forall\ j = 1, 2, \ldots, n$. Hence $|M| \leq 2$. From Proposition 1, $|M| = 1$ is not possible for T. Hence $\sigma(B(m, n)) = 2$.

Theorem 2.3. *$\sigma(T) = 2$ if and only if $T \approx B(m, n)$.*

Proof. Suppose $T \approx B(m, n)$, Then from Proposition 2.2, $\sigma(T) = 2$.

Conversely suppose that $\sigma(T) = 2$. Then there are two vertices in M with same eccentricity or with different eccentricity.
Case 1: If the two vertices in M are of the different eccentricity, then there are three different eccentricities for T, since all the vertices in $V - M$ possess same eccentricity. Hence $\sigma(T) > 2$.
Case 2: If the two vertices in M are of same eccentricity and since all other vertices in T are of same eccentricity, $T \approx B(m, n)$.

Remark 2.4. Since stars and bistars have CDPU number 1 and 2 respectively, for all trees with p vertices, there exist trees with $\sigma(K_{1,p-1}) = 1$ and $\sigma(B(m,n)) = 2$.

Then naturally a question arises: when does the vertices in M possess same eccentricity for a tree?

Theorem 2.5. *Vertices in M have same eccentricity if and only if T have atmost two different eccentricities.*

Proof. Suppose the vertices in M have the same eccentricity. Since the vertices in $V-M$ possess same eccentricity, $f_M(v_j)$ is same for every $j \in V-M$. Since the vertices in M is having the same eccentricity,
if $|M| = 1$, then $G \approx K_{1,n}$
if $|M| = 2$, then $G \approx B(m,n)$
if $|M| = 3$, then $G \approx B(1,2)$
if $|M| = 4$, then $G \approx B(2,2)$ *or*$B(1,3)$
. . . .
if $|M| = s$, then $G \approx B(m,n)$, where $m + n = s$.

Proceeding like this we get that either T should be isomorphic to a star or a bistar. Hence T has atmost two different eccentricities.

Conversely suppose that T has atmost two different eccentricities. Then $T \approx K_{1,n}$ or $G \approx B(m,n)$. Then from Proposition 2.1 and Proposition 2.2, we get M should have the same eccentricity.

Theorem 2.6. *Let T be a tree with $p \leq 8$ vertices. Then there exists trees with $\sigma(T) = 1, 2, \ldots, p-2$, for every p.*

Figure 2: Trees with 2 *and* 3 vertices and $\sigma(T) = 1$

Proof. **Case 1:** When $p = 2$, then $T \approx K_{1,1}$. Clearly $\sigma(T) = 1$.
Case 2: $p = 3$, then $T \approx K_{1,2}$. In this case $\sigma(T) = 1$.
Case 3: $p = 4$, then $T \approx K_{1,3}$ *or* P_4.

Subcase 3.1: When $T \approx K_{1,3}$, then $\sigma(T) = 1$.
Subcase 3.2: When $T \approx P_4$, then $\sigma(T) = 2$.

Case 4: $p = 5$, then $T \approx K_{1,4}$ *or* P_5 *or* $B(1,2)$.

Subcase 4.1: When $T \approx K_{1,4}$, clearly $\sigma(T) = 1$.

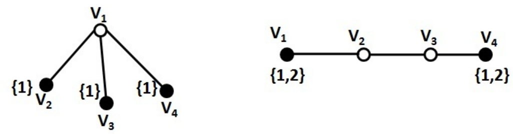

Figure 3: Trees on 4 vertices with $\sigma(T) = 1$ *and* 2

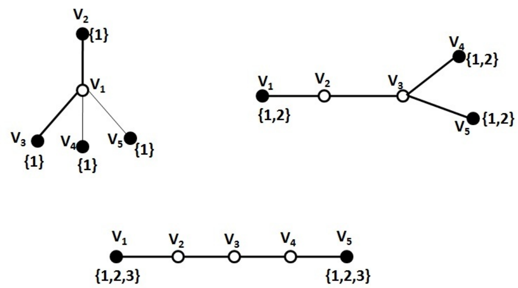

Figure 4: Trees on 5 vertices with $\sigma(T) = 1, 2$ *and* 3

Subcase 4.2: When $T \approx P_5$, then $\sigma(P_5) = 3$.

Subcase 4.3: When $T \approx B(1,2)$, then $\sigma(B(1,2)) = 2$.

Case 5: $p = 6$

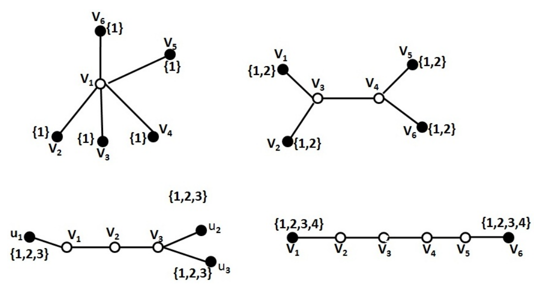

Figure 5: Trees on 6 vertices with $\sigma(T) = 1, 2, 3$ *and* 4

Subcase 5.1: $T \approx P_6$, then $\sigma(P_6) = 4$.

Subcase 5.2: $T \approx K_{1,5}$, then $\sigma(T) = 1$.

Subcase 5.3: $T \approx B(2,2)$ *or* $B(1,3)$. In both cases $\sigma(T) = 2$.

Subcase 5.4: T is isomorphic to a tree with one vertex is attached to a pendant vertex of P_3 and two vertices are attached to the other pendant vertex of P_3.

Let $\{v_1, v_2, v_3\}$ are the vertices of P_3 and u_1 be the vertex attached to v_1 and $\{u_2, u_3\}$ be the vertices attached to v_3. Take $M = \{v_1, v_2, v_3\}$. Then

$f_M(u_i) = \{1, 2, 3\}, \forall i = 1, 2, 3$. Hence $\sigma(T) = 3$.

Case 6: $p = 7$

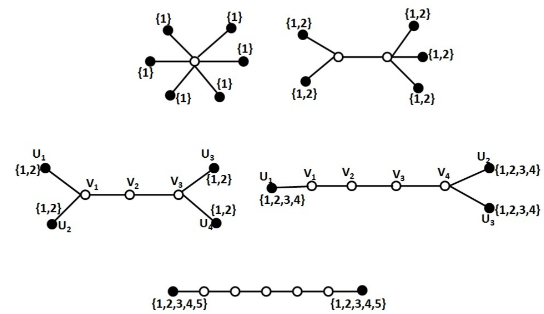

Figure 6: Trees on 7 vertices with $\sigma(T) = 1, 2, 3, 4$ *and* 5

Subcase 6.1: $T \approx P_7$, then $\sigma(T) = 5$.

Subcase 6.2: $T \approx K_{1,6}$, then clearly $\sigma(T) = 1$.

Subcase 6.3: $T \approx B(2, 3)$ *or* $B(1, 4)$, then $\sigma(T) = 2$.

Subcase 6.4: T is isomorphic to a tree with two vertices are attached to a pendant vertex of P_3 and two vertices are attached to the other pendant vertex of P_3.

Let $\{v_1, v_2, v_3\}$ are the vertices of P_3 and $\{u_1, u_2\}$ be the vertices attached to v_1 and $\{u_3, u_4\}$ be the vertices attached to v_3. Take $M = \{v_1, v_2, v_3\}$. Then $f_M(u_i) = \{1, 2, 3\}, \forall i = 1, 2, 3, 4$. Hence $\sigma(T) = 3$.

Subcase 6.5: T is isomorphic to a tree with one vertex is attached to a pendant vertex of P_4 and two vertices are attached to the other pendant vertex of P_4.

Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of P_4 and u_1 be the vertex attached to v_1 and $\{u_2, u_3\}$ be the vertices attached to v_4. Take $M = \{v_1, v_2, v_3, v_4\}$. Then $f_M(u_i) = \{1, 2, 3, 4\}, \forall i = 1, 2, 3$. Hence $\sigma(T) = 4$.

Case 7: $p = 8$

Subcase 7.1: $T \approx K_{1,7}$, then $\sigma(T) = 1$.

Subcase 7.2: $T \approx P_8$, then $\sigma(T) = 6$.

Subcase 7.3: $T \approx B(3, 3)$, then $\sigma(T) = 2$.

Subcase 7.4: T is isomorphic to a tree with two vertices are attached to a pendant vertex of P_3 and three vertices are attached to the other pendant vertex of P_3.

Let $\{v_1, v_2, v_3\}$ are the vertices of P_3 and $\{u_1, u_2\}$ be the vertices attached to v_1 and $\{u_3, u_4, u_5\}$ be the vertices attached to v_3. Take $M = \{v_1, v_2, v_3\}$. Then $f_M(u_i) = \{1, 2, 3\}, \forall i = 1, 2, 3, 4, 5$. Hence $\sigma(T) = 3$.

Subcase 7.5: T is isomorphic to a tree with two vertices are attached to a pendant vertex of P_4 and two vertices are attached to the other pendant vertex of P_4.

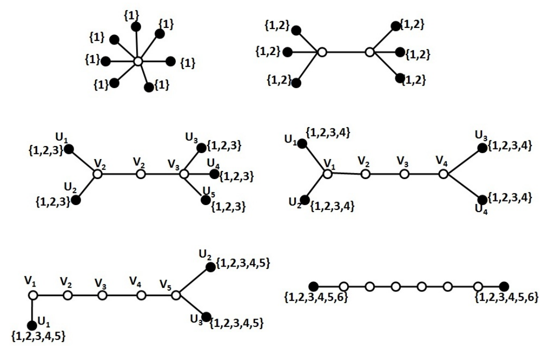

Figure 7: Trees on 8 vertices with $\sigma(T) = 1, 2, 3, 4, 5$ *and* 6

Let $\{v_1, v_2, v_3, v_4\}$ be the vertices of P_4 and $\{u_1, u_2\}$ be the vertices attached to v_1 and $\{u_3, u_4\}$ be the vertices attached to v_4. Take $M = \{v_1, v_2, v_3, v_4\}$. Then $f_M(u_i) = \{1, 2, 3, 4\}, \forall i = 1, 2, 3, 4$. Hence $\sigma(T) = 4$.

Subcase 7.6: T is isomorphic to a tree with one vertex is attached to a pendant vertex of P_5 and two vertices are attached to the other pendant vertex of P_5.

Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertices of P_5 and $\{u_1\}$ be the vertex attached to v_1 and $\{u_2, u_3\}$ be the vertices attached to v_5. Take $M = \{v_1, v_2, v_3, v_4, v_5\}$. Then $f_M(u_i) = \{1, 2, 3, 4, 5\}, \forall i = 1, 2, 3$. Hence $\sigma(T) = 5$.

Theorem 2.7. *For every trees on p vertices, there exists trees with* $\sigma(G) = 1, 2, \ldots, p - 2$.

Remark 2.8. For a tree T with $p < 5$ vertices, either $\sigma(T) = 1$ *or* 2.

Theorem 2.9. $\sigma(T) = 3$ *if and only if* T *is one among the following forms:*
(a). $T \approx P_5$
(b). To a path with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$*, pendant vertices should be attached to* v_2 *or* v_4 *or both*
(c). To a path with vertex set $\{v_1, v_2, v_3, v_4, v_5\}$*, pendant vertices should be attached to* v_3*.*

Proof. (a). From Theorem 1.5, $\sigma(P_5) = 3$.
(b). From Theorem 2.6, Case 6, subcase 6.4, $\sigma(T) = 3$.
(c). Let the vertex attach to v_3 be v_6. Take $M = \{v_1, v_3, v_5\}$. Then $f_M(v_i) = \{1, 2\}, \forall\ i = 2, 4, 6$. Hence $\sigma(T) = 3$.

Next we have to show that $\sigma(T) > 3$ for all other cases.
Case 1: P_5 with vertices attached to v_2, v_3 and v_4.
Let the vertices attached to v_2, v_3 and v_4 be v_6, v_7 and v_8 respectively. Take

$M = \{v_2, v_3, v_4, v_7\}$ giving $f_M(v_i) = \{1, 2, 3\}, \forall\ i = 1, 5, 6, 8$. Hence $\sigma(T) = 4$.
Case 2: P_5 with a path of length 2 is attached to the central vertex of P_5.
Let u_1 and u_2 be the vertices of P_2 attached to v_3. Take $M = \{v_2, v_3, v_4, u_1\}$ which implies $f_M(v_1) = f_M(v_5) = f_M(u_2) = \{1, 2, 3\}$. Hence $\sigma(T) = 4$.
Case 3: P_5 with a path of length 2 is attached to the vertex v_4 of P_5.
Let u_1 and u_2 be the vertices of P_2 attached to v_4. Take $M = \{v_2, v_3, v_4, v_5, u_1\}$ which implies $f_M(v_1) = f_M(u_2) = \{1, 2, 3, 4\}$. Hence $\sigma(T) = 5$.

For all other cases eccentricities should be greater than 5 and hence $\sigma(T) > 4$. Hence the proof.

Definition 2.10. *Olive tree T_k is a rooted tree consisting of k branches, the i^{th} branch is a path with a length k.*

Theorem 2.11. *Olive tree $T_k, k \geq 5$ has CDPU number $\frac{k^2-k+2}{2}$.*

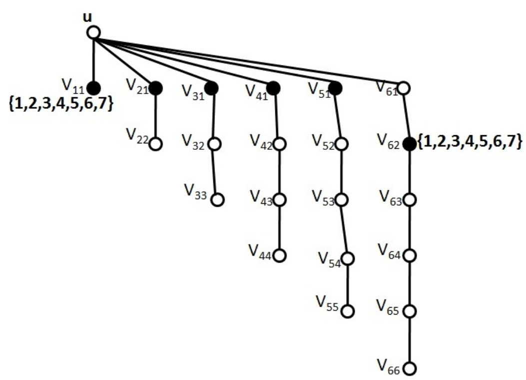

Figure 8: Olive Tree T_6 with $\sigma(T_6) = 16$ and $f_M(v_{i1}) = \{1, 2, 3, 4, 5, 6, 7\}$

Proof. Let the vertices of the i^{th} branch of T_k be $\{u, v_{i1}, v_{i2}, \ldots, v_{ii}\}$ and u be the central vertex. Take $V - M = \{v_{11}, v_{21}, v_{31}, \ldots, v_{k2}\}$. Take all other vertices inside M. Then $f_M(v_{i1}) = f_M(v_{k2}) = \{1, 2, \ldots, k+1\}, \forall i = 1, 2, \ldots, k-1$.
Hence
$\sigma(T_k) = (k-1) + (k-2) + (k-3) + \cdots + (k-(k-1)) + 1$
$= (k-1)k + (1 + 2 + \cdots + k - 1) + 1$
$= k(k-1) + \frac{(k-1)k}{2} + 1$
$= \frac{k(k-1)+2}{2}$
$= \frac{k^2-k+2}{2}$.

Acknowledgements

The authors are thankful to the Department of Science & Technology, Government of India for supporting this research under the Project No. SR/S4/MS:277/06.

References

[1] K.A. Germina and K. Beena, Complementary distance pattern uniform graphs, *International Journal on Contemporary Mathematical Sciences*, 5(55) 2010, 2745-2751.

[2] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, (1969).

[3] DST grant-in-aid project No.SR/S4/277/06, *Technical Report*, Funded by the Department of Science & Technology (DST), April, (2009).