Gen. Math. Notes, Vol. 23, No. 2, August 2014, pp.71-78
ISSN 2219-7184; Copyright ©ICSRS Publication, 2014
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Revised Szeged Index of Product Graphs

S. Nagarajan[1], K. Pattabiraman[2] and M. Chendrasekharan[3]

[1]Department of Mathematics
Kongu Arts and Science College
Erode - 638 107, India
E-mail: profnagarajan@rediffmail.com
[2]Department of Mathematics
Faculty of Engineering and Technology
Annamalai University, Annamalainagar - 608 002, India
E-mail: pramank@gmail.com
[3]Department of Mathematics
Erode Arts and Science College
Erode - 638 009, India
E-mail: mcbrindha@gmail.com

(Received: 13-3-14 / Accepted: 3-5-14)

Abstract

The Szeged index of a graph G is defined as $Sz(G) = \sum\limits_{uv=e\in E(G)} n_u(e)n_v(e)$, *where* $n_u(e)$ *is number of vertices of* G *whose distance to the vertex* u *is less than the distance to the vertex* v *in* G. *Similarly, the revised Szeged index of* G *is defined as* $Sz^*(G) = \sum\limits_{uv=e\in E(G)} \left(n_u(e) + \frac{n_G(e)}{2}\right)\left(n_v(e) + \frac{n_G(e)}{2}\right)$, *where* $n_G(e)$ *is the number of equidistant vertices of* e *in* G. *In this paper, the revised Szeged index of Cartesian product of two connected graphs is obtained. Using this formula, the revised Szeged indices of the hypercube of dimension* n, *Hamming graph, grid,* C_4 *nanotubes and nanotorus are computed.*

Keywords: *Cartesian product, Szeged index, revised Szeged index.*

1 Introduction

All the graphs considered in this paper are connected and simple. The *Cartesian product,* $G \square H$, of graphs G and H has the vertex set $V(G \square H) = V(G) \times V(H)$ and

$(u, x)(v, y)$ is an edge of $G \square H$ if $u = v$ and $xy \in E(H)$ or, $uv \in E(G)$ and $x = y$, that is, to each vertex $u \in V(G)$, there is an isomorphic copy of H in $G \square H$ and to each vertex $v \in V(H)$, there is an isomorphic copy of G in $G \square H$, see Fig.1.

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, $d_G(u, v)$ denotes the distance between u and v in G. The *Wiener index* of G is defined as $W(G) = \frac{1}{2} \sum\limits_{u,v \in V(G)} d_G(u, v)$.

This topological index has been extensively studied in the mathematical literature; see [4, 5]. A vertex $x \in V(G)$ is said to be *equidistant* from the edge $e = uv$ of G if $d_G(u, x) = d_G(v, x)$, where $d_G(u, x)$ denotes the distance between u and x in G. For an edge $uv = e \in E(G)$, the number of vertices of G whose distance to the vertex u is smaller than the distance to the vertex v in G is denoted by $n_u(e)$; analogously, $n_v(e)$ is the number of vertices of G whose distance to the vertex v in G is smaller than the distance to the vertex u; the vertices equidistant from both the ends of the edge $e = uv$ are not counted. Similarly, the number of equidistant vertices of e is denoted by $n_G(e)$.

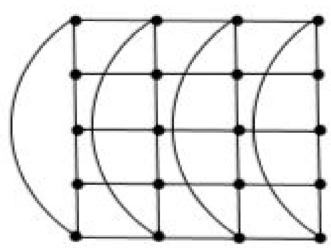

Fig.1. Cartesian Product of C_5 and P_4

A long time known property of the Wiener index is the formula [7, 15], $W(G) = \sum\limits_{e=uv \in E(G)} n_u(e) n_v(e)$, which is applicable for trees. Motivated by the above formula, Gutman [6] introduced a graph invariant, named as the *Szeged index*, as an extension of the Wiener index and defined by $Sz(G) = \sum\limits_{e=uv \in E(G)} n_u(e) n_v(e)$.

Randić [14] observed that the Szeged index does not take into account the contributions of the vertices at equal distances from the endpoints of an edge, and so he conceived a modified version of the Szeged index which is named as the revised Szeged index. The *revised Szeged index* of a connected graph G is defined as $Sz^*(G) = \sum\limits_{e=uv \in E(G)} \left(n_u(e) + \frac{n_G(e)}{2}\right)\left(n_v(e) + \frac{n_G(e)}{2}\right)$.

The Szeged index studied by Gutman [**?**], Gutman et. al. [9] and Khadikar et. al. [10] is closely related to the Wiener index of a graph. Basic properties of Szeged index and its analogy to the Wiener index are discussed by Klavžar et al. [11] . It is proved that for a tree T the Wiener index of T is equal to its Szeged index. The mathematical properties and chemical applications of Szeged and revised Szeged indices are well studied in [3, 8, 2, 12, 13]. In [1], Aouchiche and Hansen showed that for a connected graph G of order n and size m, an upper bound of the revised Szeged index of G is $\frac{n^2m}{4}$. In [16], Xing and Zhou determined the unicyclic graphs of order n with the smallest and the largest revised Szeged indices for $n \geq 5$, and they also determined the unicyclic graphs of order n with the unique cycle of length $r(3 \leq r \leq n)$, with the smallest and the largest revised Szeged indices. In [12], Li and Liu have identified those graphs whose revised Szeged index is maximal among bicyclic graphs. In this paper, the revised Szeged index of Cartesian product of two connected graphs is obtained. Using this formula, the revised Szeged indices of the hypercube of dimension n, Hamming graph, grid, C_4 nanotubes and nanotorus are computed.

2 Revised Szeged Index of $G\Box H$

The proof of the following lemma is left to the reader as it follows easily from the structure of $G\Box H$. The lemma is used in the proof of the main theorem of this paper.

Lemma 2.1. *Let G and H be two graphs. Then*
(i) $|V(G\Box H)| = |V(G)|\,|V(H)|\,,\, |E(G\Box H)| = |E(G)|\,|V(H)| + |E(H)|\,|V(G)|\,.$
(ii) $d_{G\Box H}((g,h)(g',h')) = d_G(g,g') + d_H(h,h').$ □

For an edge $e = uv \in E(G)$, let $T_G(e,u)$ be the set of vertices closer to u than v and $T_G(e,v)$ be the set of vertices closer to v than u. That is,

$$\begin{aligned} T_G(e,u) &= \{x \in V(G) | d_G(u,x) < d_G(v,x)\} \\ T_G(e,v) &= \{x \in V(G) | d_G(u,x) > d_G(v,x)\}. \end{aligned}$$

Theorem 2.2. *Let G and H be two connected graphs. Then* $Sz^*(G\Box H) = |V(G)|^3$ $Sz^*(H) + |V(H)|^3\ Sz^*(G)$.

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$, $V(H) = \{v_1, v_2, \ldots, v_m\}$. For our convenience, we partition the edge set of $G\Box H$ into two sets, $E_1 = \{(u_r, v_i)(u_r, v_k) \,|\, u_r \in V(G),\ v_iv_k \in E(H)\}$ and $E_2 = \{(u_r, v_i)(u_s, v_i) \,|\, u_ru_s \in E(G),\ v_i \in V(H)\}$, that is, $E_1 = \cup_{u_i \in V(G)} E(\langle X_i \rangle)$ and $E_2 = \cup_{j=1}^{m} E(\langle Y_j \rangle)$.

Let $e = v_iv_k \in E(H)$ and let v_j be equidistant from e in H. Then, for $u_r \in V(G)$ and $e' = (u_r, v_i)(u_r, v_k) \in E(G\Box H)$, $d_{G\Box H}((u_r, v_i),(u_r, v_j)) = d_{G\Box H}((u_r, v_k),(u_r, v_j))$. Further, both (u_r, v_i) and (u_r, v_k) are equidistant to all the vertices of Y_j; so, if $(u_s, v_j) \in Y_j$, then

$$\begin{aligned} d_{G\Box H}((u_r, v_i),(u_s, v_j)) &= d_G(u_r, u_s) + d_H(v_i, v_j), \text{ by Lemma 2.1,} \\ &= d_G(u_r, u_s) + d_H(v_k, v_j), \\ &\quad \text{since } v_j \text{ is equidistant from the edge } v_iv_k, \\ &= d_{G\Box H}((u_r, v_k),(u_s, v_j)), \text{ by Lemma 2.1.} \end{aligned}$$

Thus to each edge $e = v_iv_k \in E(H)$ and a vertex v_j equidistant from e in H, there correspond $|V(G)|$ edges $e' \in E(Y_i, Y_k) \subseteq G\Box H$ such that all the vertices of Y_j are equidistant from e'. If v_j is not equidistant from $e = v_iv_k$ in H, then we can observe that each of the corresponding $|V(G)|$, edges $e' \in E(Y_i, Y_k)$ are not equidistant to any of the vertices of Y_j. Hence

$$n_{G\Box H}(e') = |V(G)|\, n_H(e). \tag{2.1}$$

Thus we have computed the number of equidistant vertices of the edges of $E_1 \subseteq E(G\Box H)$.

Let $e = v_iv_k \in E(H)$ and let $v_j \in T_H(e; v_i)$. Then, for any $u_r \in V(G)$ and $e' \in E_1 \subset E(G\Box H)$, the distance of (u_r, v_i) to each vertex of Y_j, is less than its distance to the vertex (u_r, v_k) in $G\Box H$. It can be observed that if some vertex $v_s \notin T_H(e, v_i)$, then all the vertices of the column Y_s are not in $T_{G\Box H}(e'; (u_r, v_i))$ in $G\Box H$. Also if v_r is equidistant to e in H, then every vertex of Y_r is equidistant to e'. Consequently, for the edge $e' \in E_1$ (of $G\Box H$) corresponding to e (in H),

$$n_{(u_r,v_i)}(e') = |V(G)|\; n_{v_i}(e) \tag{2.2}$$

and similarly,

$$n_{(u_r,v_k)}(e') = |V(G)|\; n_{v_k}(e). \tag{2.3}$$

Hence for E_1 defined as above,

$$\begin{aligned}
&\sum_{(u_r,v_i)(u_r,v_k)\,=\,e'\,\in\,E_1} \left(n_{(u_r,v_i)}(e') + \tfrac{n_{G\Box H}(e')}{2}\right)\left(n_{(u_r,v_k)}(e') + \tfrac{n_{G\Box H}(e')}{2}\right)\\
&= \sum_{(u_r,v_i)(u_r,v_k)\,=\,e'\,\in\,E_1} \left(|V(G)|\, n_{v_i}(e) + |V(G)|\,\frac{n_H(e)}{2}\right)\left(|V(G)|\; n_{v_k}(e) + |V(G)|\,\frac{n_H(e)}{2}\right),\\
&\quad \text{by (2.1) (2.2) and (2.3), where } e = v_iv_k \in E(H),\\
&= |V(G)| \sum_{v_iv_k\,=\,e\,\in\,E(H)} |V(G)|^2 \left(n_{v_i}(e) + \frac{n_H(e)}{2}\right)\left(n_{v_k}(e) + \frac{n_H(e)}{2}\right),\\
&\quad \text{since } |E_1| = |V(G)|\,|E(H)|,\\
&= |V(G)|^3\; Sz^*(H).
\end{aligned} \tag{2.4}$$

Since Cartesian product is commutative, for any edge $e' = (u_r, v_i)(u_s, v_i) \in E_2 \subset E(G\Box H)$,

$$\begin{aligned}
n_{G\Box H}(e') &= |V(H)|\, n_G(e).\\
n_{(u_r,v_i)}(e') &= |V(H)|\; n_{u_r}(e)\\
n_{(u_s,v_i)}(e') &= |V(H)|\; n_{u_s}(e).
\end{aligned} \tag{2.5}$$

Hence for E_2 defined as above,

$$\sum_{(u_r,v_i)(u_s,v_i)=e'\in E_2}\left(n_{(u_r,v_i)}(e')+\tfrac{n_{G\square H}(e')}{2}\right)\left(n_{(u_s,v_i)}(e')+\tfrac{n_{G\square H}(e')}{2}\right)$$

$$\begin{aligned}
&= \sum_{(u_r,v_i)(u_s,v_i)=e'\in E_1}\left(|V(H)|\ n_{u_r}(e)+|V(H)|\,\frac{n_G(e)}{2}\right)\left(|V(H)|\ n_{u_s}(e)+|V(H)|\,\frac{n_G(e)}{2}\right),\\
&\quad\text{by (2.5), where } e=u_ru_s\in E(G),\\
&= |V(H)|\sum_{u_ru_s=e\in E(G)}|V(H)|^2\left(n_{u_r}(e)+\frac{n_G(e)}{2}\right)\left(n_{u_s}(e)+\frac{n_G(e)}{2}\right),\\
&\quad\text{since } |E_2|=|V(H)|\,|E(G)|,\\
&= |V(H)|^3\ Sz^*(G).
\end{aligned}\tag{2.6}$$

Now we shall obtain the $Sz^*(G\square H)$. By the definition,

$$\begin{aligned}
Sz^*(G\square H) &= \sum_{(u_r,v_i)(u_s,v_k)=e'\in E(G\square H)}\left(n_{(u_r,v_i)}(e')+\frac{n_{G\square H}(e')}{2}\right)\left(n_{(u_s,v_k)}(e')+\frac{n_{G\square H}(e')}{2}\right)\\
&= \sum_{(u_r,v_i)(u_r,v_k)=e'\in E_1}\left(n_{(u_r,v_i)}(e')+\frac{n_{G\square H}(e')}{2}\right)\left(n_{(u_r,v_k)}(e')+\frac{n_{G\square H}(e')}{2}\right)\\
&\quad+\sum_{(u_r,v_i)(u_s,v_i)=e'\in E_2}\left(n_{(u_r,v_i)}(e')+\frac{n_{G\square H}(e')}{2}\right)\left(n_{(u_s,v_i)}(e')+\frac{n_{G\square H}(e')}{2}\right)\\
&= |V(G)|^3\ Sz^*(H)+|V(H)|^3\ Sz^*(G),\ \text{by (2.4) and (2.6).}
\end{aligned}$$

□

Denote by $\square_{i=1}^n G_i$ the Cartesian product of graphs $G_1, G_2, \ldots, G_n$. In [11], Klavžar et al. have proved $Sz(\square_{i=1}^n G_i) = \sum_{i=1}^{n} Sz(G_i) \prod_{j=1, j\neq i}^{n} |V(G_i)|^3$.

Using Theorem 2.2, we have the following corollaries.

Corollary 2.3. *Let $G_1, G_2, \ldots, G_n$ be connected graphs. Then $Sz^*(\square_{i=1}^n G_i) = \sum_{i=1}^{n} Sz^*(G_i) \prod_{j=1, j\neq i}^{n} \left|V(G_j)\right|^3$.* □

Corollary 2.4. *Let G be a connected graph. Then $Sz^*(\square G^n) = Sz^*(\square_{i=1}^n G) = n\,|V(G)|^{3(n-1)}\ Sz^*(G)$.* □

Example 2.5. *Suppose Q_n denotes a hypercube of dimension n. Then by Corollary 2.4, $Sz^*(Q_n) = Sz^*(K_2^n) = n2^{3(n-1)}$.*

Let us consider the graph G whose vertices are the $N-$tuples $b_1b_2\ldots b_N$ with $b_i \in \{0, 1, \ldots, n_i - 1\}, n_i \geq 2$, and two vertices be adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a *Hamming graph*. It is well-known fact that a graph G is a Hamming graph if and only if it can be written in the form $G = \square_{i=1}^N K_{n_i}$ and so the hamming graph is usually denoted as $H_{n_1n_2\ldots n_N}$. In the following lemma, the revised Szeged index of a Hamming graph is computed.

Lemma 2.6. *Let G be a hamming graph with above parameter. Then* $Sz^*(H_{n_1n_2\ldots n_N}) = \prod_{j=1}^{N} {n_i}^3 \left((\sum_{i=1}^{N} \frac{n_i}{8}) - \frac{N}{8} \right)$.

Proof. It is easy to see that $Sz^*(K_n) = \frac{n^3(n-1)}{8}$. Since Hamming graph is a product of complete graphs, by Corollary 2.3,

$$\begin{aligned} Sz^*(H_{n_1n_2\ldots n_N}) &= Sz^*(\square_{i=1}^{N} K_{n_i}) \\ &= \sum_{i=1}^{N} Sz^*(K_{n_i}) \prod_{j=1, j\neq i}^{N} {n_j}^3 \\ &= \sum_{i=1}^{N} \frac{{n_i}^3(n_i-1)}{8} \prod_{j=1, j\neq i}^{N} {n_j}^3 \\ &= \prod_{j=1}^{N} {n_i}^3 \sum_{i=1}^{N} \frac{(n_i-1)}{8} \\ &= \prod_{j=1}^{N} {n_i}^3 \left((\sum_{i=1}^{N} \frac{n_i}{8}) - \frac{N}{8} \right). \end{aligned}$$

□

Let C_n and P_n denote the cycle and path on n vertices, respectively. It can be easily verified that $Sz^*(C_n) = \frac{n^3}{4}$ and $Sz^*(P_n) = \binom{n+1}{3}$.

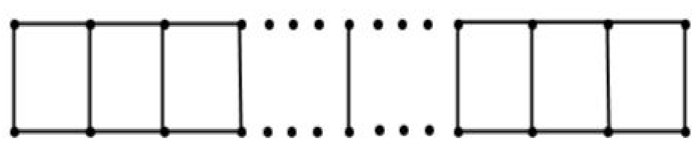

Fig.2. Ladder graph of $2n$ vertices

Example 2.7. *Using Corollary 2.3, we obtain the exact revised Szeged index of the grid graph* $P_{n_1} \square P_{n_2} \square \ldots \square P_{n_k}$.

$Sz^*(P_{n_1} \square P_{n_2} \square \ldots \square P_{n_k}) = \frac{1}{6}\Big(\prod_{i=1}^{k} n_i^3\Big)\Big(\sum_{i=1}^{k}(1+\frac{1}{n_i})(1-\frac{1}{n_i})\Big)$.

If each $n_i = n$, *then* $Sz^*(\square P_n^k) = \frac{kn^{3k-2}}{6}(n+1)(n-1)$.

Example 2.8. *Using Corollary 2.3, we obtain the exact revised Szeged index of the graph* $C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}$.

$Sz^*(C_{n_1} \square C_{n_2} \square \ldots \square C_{n_k}) = \frac{k}{4} \prod_{i=1}^{k} n_i^3$.

If each $n_i = n$, *then* $Sz^*(\square C_n^k) = \frac{kn^{3k}}{4}$.

Example 2.9. *The graphs* $L_n = P_n \Box K_2$, $R = P_n \Box C_m$, $S = C_m \Box C_n$ *and* $T = P_m \Box P_n$ *are known as ladder,* C_4 *nanotubes,* C_4 *nanotorus and grid, respectively. The exact revised Szeged indices of these graphs are given below.*

1. $Sz^*(L_n) = \frac{n(9n^2+4)}{3}$.
2. $Sz^*(R) = \frac{nm^3(5n^2-2)}{12}$.
3. $Sz^*(S) = \frac{n^3}{8}$.
4. $Sz^*(T) = \frac{1}{6}(2n^3m^3 - nm(n^2 - m^2))$.

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