Gen. Math. Notes, Vol. 23, No. 1, July 2014, pp. 79-88
ISSN 2219-7184; Copyright © ICSRS Publication, 2014
www.i-csrs.org
Available free online at http://www.geman.in

Treatment for High-Order Linear Fredholm Integro-Differential Equations of Degenerated Kernel

Aloko MacDonald D[1] and Abdulmalik Ibrahim Onuwe[2]

[1]Department of Mathematics, University of Lagos, Lagos- Nigeria
E-mail: mac_aloko@yahoo.com
[2]National Agency for Science and Engineering Infrastructure (NASENI), Abuja
Federal Ministry of Science and Technology, Nigeria
E-mail: Onuwe64@gmail.com

(Received: 2-4-14 / Accepted: 2-7-14)

Abstract

In this work, projection method with degenerated Kernel and its properties are used to solve linear Fredholm Integro-Differential integral equations (LFIDE). The kernel in the equation are degenerated and convert to matrix in which each equations are integrated over the interval [0, 1] to produce the elements for obtaining the Eigen value. It approaches linear Fredholm Integro-Differential equations in a manner that gives the solution in an exact form and not in a series form, the algorithm is simple and effective, and could also provide an accurate approximate solution.

Keywords: *Fredholm Integro-Differential Equations, Projection Method.*

1.0 Introduction

Fredholm Integro-Differential equations (FIDE) arise in many scientific applications. It was also shown that FIDE can be derived from boundary value problems. Converting boundary value problems to Integro-Differential Fredholm integral equations and converting Fredholm Integro-Differential equations (FIDE) to equivalent boundary value problems are rarely used. Erik Ivar Fredholm (1866–1927) is best remembered for his work on integral equations and spectral theory. However, Adomian decomposition method (ADM) for solving integral equations has been presented by Adomian [1-2] and then this has been extended by Wazwaz to Volterra integral equation [13] and to boundary value problems for higher-order integro-differential equations. A Fredholm Integro-Differential equations (FIDE) equation of the second kind may be written as

$$u^{n}(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{1.0}$$

Subject to initial conditions $u^{k}(0) = c_k, 0 \le k \le (n-1)$ (1.1)

2.0 Formulation and Implementation Method

Definition 1: *A kernel $k(x,t)$is called degenerate if it can be expressed as the sum of a finite number of terms, each of which is the product of a function of x only and a function of t only* ***[16]****: If we can write the kernel of (1) in the form:*

$$k(x,t) = \sum_{i=1}^{N} \alpha_i(x)\beta_i(t) \tag{2.0}$$

(or $k = \sum_{i=1}^{n} |\alpha_i\beta_i|$) We may assume that the$\alpha_i(x)$are linearly independent. Any continuous function $k(x,t)$ can be uniformly approximated by polynomials in a closed interval.

To illustrate the efficiency of the proposed numerical methods in studying the model equation (1.0) subject to the initial conditions, and given the kernel in (2.0), the general second kind Integro-Differential Fredholm equation,

$$u^{n}(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt, u^{k}(0) = c_k, 0 \le k \le (n-1),$$

c_k are the initial conditions

Equation (1.0) can be rewritten as

$$u^{n}(x) = f(x) + \lambda \int_a^b \sum_{i=1}^{m} \alpha_i(x)\beta_i(t)\,u(t)dt \tag{2.1}$$

$$u^{n}(x) = f(x) + \lambda \sum_{i=1}^{m} \alpha_i(x) \int_a^b \beta_i(t)u(t)dt \tag{3.0}$$

To discretized (2.1) let us denote

$$b_i = \int_a^b \beta_i(t)u(t)dt \tag{4.0}$$

and since our focus is to solve for f, we need to find our $f_{new}(x)$which is depending on the initial $f(x)$

$$u(x) = f_{new}(x) + \lambda \sum_{i=1}^{m} b_i \alpha_{inew}(x) \tag{5.0}$$

from (3.0), we have

$$u^n(x) = f(x) + \lambda \sum_{i=1}^{m} b_i \alpha_i(x) \tag{6.0}$$

$$L^n(x) = f(x) + \lambda \sum_{i=1}^{m} b_i \alpha_i(x) \tag{6.1}$$

Operating with L^{-n} on both sides of (6.1) and with the application of (1.1), it follows

$$u(x) - \sum_{i=1}^{n} \frac{x^{(i-1)}}{(i-1)!} u^{(i-1)}(0) = L^{-n}[f(x) + \lambda \sum_{i=1}^{m} b_i \alpha_i(x)]$$

$$u(x) = \sum_{i=1}^{n} \frac{x^{(i-1)}}{(i-1)!} u^{(i-1)}(0) + L^{-n}[f(x) + \lambda \sum_{i=1}^{m} b_i \alpha_i(x)] \tag{6.2}$$

Let us multiply (6.2) by $\beta_j(x), j = 1, 2, 3..n$integrate from a to b over x

$$u(x)\beta_j(x) = \sum_{i=1}^{n} \frac{x^{(i-1)}}{(i-1)!} u^{(i-1)}(0)\beta_j(x) + (L^{-n}[f(x) + \lambda \sum_{i=1}^{m} b_i \alpha_i(x)])\beta_j(x)$$

$$\int_a^b u(x)\beta_j(x)dx = \sum_{i=1}^{n} \int_a^b \frac{x^{(i-1)}}{(i-1)!} u^{(i-1)}(0)\beta_j(x)dx +$$

$$(\left[\int_a^b (L^{-n}f(x))\beta_j(x)dx + \lambda \sum_{i=1}^{m} b_i \int_a^b (L^{-n}\alpha_i(x))\beta_j(x)dx\right]) \tag{6.3}$$

where the differential operators $L^n = \frac{d^n}{dx^n}, L^{n-1} = \frac{d^{n-1}}{dx^{n-1}}$.

The inverse operator L^{-n} is therefore considered an n−fold integral operator defined by $L^{-n}(.) = \int(.)dx$

Let

$$\left.\begin{aligned} b_j &= \int_a^b u(x)\beta_j(x)dx \\ f_j &= \int_a^b (L^{-n}f(x))\beta_j(x)dx \\ a_{ij} &= \int_a^b (L^{-n}\alpha_i(x))\beta_j(x)dx \end{aligned}\right\} \tag{7.0}$$

Putting (7.0) into (6.3), we have

$$\left.\begin{aligned} &b_j = (f_j + \lambda \sum_{i=1}^{m} b_i a_{ij}) \\ &b_j - \lambda \sum_{i=1}^{m} b_i a_{ij} = f_j \\ &(I - \lambda \sum_{i=1}^{m} a_{ij}) b_{ij} = f_j \\ &(I - \lambda A)B = F \end{aligned}\right\} \tag{8.0}$$

There exist a unique solution for (8.0) if the determinant$|I - \lambda A| \neq 0$ and either no solution or infinitely many solutions when $|I - \lambda A| = 0$.

This is a system of n linear equations in the n unknowns f_j. Suppose that there is a unique solution $f_1, f_2, f_3 \ldots . f_n$to this system.This system can be solved by finding allb_i, $i = 1, 2, 3, \ldots . n$; and substituting into

$$u(x) = f_{new}(x) + \lambda \sum_{i=1}^{m} b_i \alpha_{inew}(x) \tag{8.1}$$

to find $u(x)$

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \qquad F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix} \tag{9.0}$$

From (7.0)

$$A = \begin{bmatrix} \int_a^b L^{-n}\alpha_{1new}(x)\beta_1(x)dx & \int_a^b L^{-n}\alpha_{2new}(x)\beta_1(x)dx & . & . & \int_a^b L^{-n}\alpha_{nnew}(x)\beta_1(x)dx \\ \int_a^b L^{-n}\alpha_{1new}(x)\beta_2(x)dx & \int_a^b L^{-n}\alpha_{2new}(x)\beta_2(x)dx & . & . & \int_a^b L^{-n}\alpha_{nnew}(x)\beta_2(x)dx \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \int_a^b L^{-n}\alpha_{1new}(x)\beta_n(x)dx & \int_a^b L^{-n}\alpha_{2new}(x)\beta_n(x)dx & . & . & \int_a^b L^{-n}\alpha_{nnew}(x)\beta_n(x)dx \end{bmatrix}$$

3.0 Application and Numerical Results

To illustrate the efficiency of the proposed Algorithm in studying the model equation (1.0) subject to the initial conditions, we solve three examples of Fredholm integro-differential equations by the method of solution above.

Example 3.1 Solve the following Fredholm integro-differential equation

$$u'''(x) = 2 + \sin(x) - \int_0^{\pi} (x - t)u(t)\,dt, u(0) = 1, u'(0) = 0, u''(0) = -1 \tag{10.0}$$

The kernel is $K(x, t) = (x - t) = \sum_{i=1}^{2} \alpha_i(x)\beta_i(t)$

$\alpha_1(x) = x$
$\alpha_2(x) = -1$
$\beta_1(t) = 1$
$\beta_2(t) = t$
$n = 3$
$\lambda = -1$

From (6.3), $u(x) - 1 + \frac{x^2}{2} = \frac{\mathbf{x^3}}{\mathbf{3}} + \mathbf{cos}(\boldsymbol{x}) - (\frac{x^4}{4!}b_1 - \frac{x^3}{3!}b_2)$ (11.0)

$f_{new}(x) = \frac{x^3}{3} + \cos(x)$

$f_1 = \int_0^{\pi} \beta_1(t) f_{new}(t)dt = \int_0^{\pi} \left(\frac{t^3}{3} + \cos(t)\right) dt = \frac{\pi^4}{12}$

$f_2 = \int_0^{\pi} \beta_2(t) f_{new}(t)dt = \int_0^{\pi} t(\frac{t^3}{3} + \cos(t))dt = \frac{\pi^5}{15} - 2$

$$F = \begin{bmatrix} \frac{\pi^4}{12} \\ \frac{\pi^5}{15} - 2 \end{bmatrix}$$

$\alpha_{1new}(x) = \frac{x^4}{4!}$

$\alpha_{2new}(x) = -\frac{x^3}{3!}$

$\sum_{i=1}^{2} a_{ij} = \int_0^{\pi} \alpha_{inew}(x)\beta_j(x)dx$

$a_{11} = \int_0^{\pi} \alpha_{1new}(x)\beta_1(x)dx = \int_0^{\pi} \frac{t^4}{4!} dt = \frac{\pi^5}{5!}$

$a_{12} = \int_0^{\pi} \alpha_{2new}(x)\beta_1(x)dx = -\int_0^{\pi} \frac{t^3}{3!} dt = -\frac{\pi^4}{4!}$

$a_{21} = \int_0^{\pi} \alpha_{1new}(x)\beta_2(x)dx = \int_0^{\pi} \frac{t^5}{4!} dt = \frac{\pi^6}{144}$

$a_{22} = \int_0^{\pi} \alpha_{2new}(x)\beta_2(x)dx = \int_0^{\pi} -\frac{t^4}{3!} dt = -\frac{\pi^5}{30!}$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi^4}{12} \\ \frac{\pi^5}{15} - 2 \end{bmatrix} - \begin{bmatrix} \frac{\pi^5}{5!} & -\frac{\pi^4}{4!} \\ \frac{\pi^6}{144} & -\frac{\pi^5}{30} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad (12.0)$$

The determinant is non-zero and solving (12.0) to obtain

$b_1 = 0,\ b_2 = -2$

From (5.0)

$u(x) = f_{new}(x) + \lambda \sum_{i=1}^{n} b_i \alpha_{inew}(x)$

$u(x) = \frac{x^3}{3} + \cos(x) - (\frac{x^4}{4!} * 0 - \frac{x^3}{3!} * -2)$

$u(x) = \cos(x)$

Example 3.2 Solve the following Fredholm integro-differential equation

$u^{iv}(x) = (2x - \pi) + \sin(x) + \cos(x) - \int_0^{\frac{\pi}{2}} (x - 2t)u(t)\,dt,$

$u(0) = u'(0) = 1,\ u''(0) = u'''(0) = -1$ (13.0)

The kernel is $K(x,t) = (x - 2t) = \sum_{i=1}^{2} \alpha_i(x)\beta_i(t)$

$\alpha_1(x) = x$
$\alpha_2(x) = -2$
$\beta_1(t) = 1$
$\beta_2(t) = t$
$n = 4$
$\lambda = -1$

From (6.3), $u(x) - u(0) - xu'(0) - \frac{x^2}{2!}u''(0) - \frac{x^3}{3!}u'''(0) =$

$\frac{\mathbf{x^5}}{\mathbf{60}} - \frac{\boldsymbol{\pi}\mathbf{x^4}}{\mathbf{24}} + \mathbf{sin(x)} + \mathbf{cos}(\boldsymbol{x}) - (\frac{x^5}{120}b_1 - \frac{x^4}{12}b_2)$ (14.0)

$f_{new}(x) = \frac{\mathbf{x^5}}{\mathbf{60}} - \frac{\boldsymbol{\pi}\mathbf{x^4}}{\mathbf{24}} + \mathbf{sin(x)} + \mathbf{cos}(\boldsymbol{x})$

$f_1 = \int_0^{\frac{\pi}{2}} \beta_1(t) f_{new}(t)dt = \int_0^{\frac{\pi}{2}} \left(\frac{\mathbf{t^5}}{\mathbf{60}} - \frac{\boldsymbol{\pi}\mathbf{t^4}}{\mathbf{24}} + \mathbf{sin(t)} + \mathbf{cos}(\boldsymbol{t})\right) dt = 2 - \frac{\pi^6}{4608}$

$f_2 = \int_0^{\frac{\pi}{2}} \beta_2(t) f_{new}(t)dt = \int_0^{\frac{\pi}{2}} t(\frac{\mathbf{t^5}}{\mathbf{60}} - \frac{\boldsymbol{\pi}\mathbf{t^4}}{\mathbf{24}} + \mathbf{sin(t)} + \mathbf{cos}(\boldsymbol{t}))dt = \frac{\pi}{2} - \frac{29\pi^7}{322560}$

$$F = \begin{bmatrix} 2 - \frac{\pi^6}{4608} \\ \frac{\pi}{2} - \frac{29\pi^7}{322560} \end{bmatrix}$$

$\alpha_{1new}(x) = \frac{x^5}{5!}$
$\alpha_{2new}(x) = -\frac{x^4}{12}$

$\beta_1(t) = 1$
$\beta_2(t) = t$

$\sum_{i=1}^{2} a_{ij} = \int_0^{\frac{\pi}{2}} \alpha_{inew}(x)\beta_j(x)dx$

$a_{11} = \int_0^{\frac{\pi}{2}} \alpha_{1new}(t)\beta_1(t)dt = \int_0^{\frac{\pi}{2}} \frac{t^5}{5!} dt = \frac{\pi^6}{46080}$

$a_{12} = \int_0^{\frac{\pi}{2}} \alpha_{2new}(t)\beta_1(t)dt = -\int_0^{\frac{\pi}{2}} \frac{t^4}{12} dt = -\frac{\pi^5}{1920}$

$a_{21} = \int_0^{\frac{\pi}{2}} \alpha_{1new}(t)\beta_2(t)dx = \int_0^{\frac{\pi}{2}} \frac{t^6}{5!} dt = \frac{\pi^7}{107520}$

$a_{22} = \int_0^{\frac{\pi}{2}} \alpha_{2new}(t)\beta_2(t)dx = \int_0^{\frac{\pi}{2}} -\frac{t^5}{12} dt = -\frac{\pi^6}{4608}$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 2 - \frac{\pi^6}{4608} \\ \frac{\pi}{2} - \frac{29\pi^7}{322560} \end{bmatrix} - \begin{bmatrix} \frac{\pi^6}{46080} & -\frac{\pi^5}{1920} \\ \frac{\pi^7}{107520} & -\frac{\pi^6}{4608} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \quad (15.0)$$

The determinant is non-zero and solving (15.0) to obtain

$b_1 = 2,\ b_2 = \frac{\pi}{2}$

From (5.0)

$u(x) = f_{new}(x) + \lambda \sum_{i=1}^{n} b_i \alpha_{inew}(x)$

$u(x) = \mathbf{\frac{x^5}{60}} - \mathbf{\frac{\pi x^4}{24}} + \mathbf{sin}(\mathbf{x}) + \mathbf{cos}(\boldsymbol{x}) - (\frac{x^5}{5!} * 2 - -\frac{x^4}{12} * \frac{\pi}{2})$

$u(x) = \mathbf{sin}(\mathbf{x}) + \mathbf{cos}(\boldsymbol{x})$

Example 3.3: Solve the following Fredholm integro-differential equation

$$u^{i\prime}(x) = 2x - x\sin(x) + 2\cos(x) - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (t - x)u(t)\,dt,$$
$$u(0) = u'(0) = 0 \quad (16.0)$$

The kernel is $K(x, t) = (t - x) = \sum_{i=1}^{2} \alpha_i(x)\beta_i(t)$

$\alpha_1(x) = 1$

$\alpha_2(x) = x$

$\beta_1(t) = \mathrm{t}$

$\beta_2(t) = -1$

$n = 2$

$\lambda = -1$

From (6.3), $u(x) - u(0) - xu'(0) =$

$$\frac{\mathbf{x^3}}{\mathbf{3}} + \boldsymbol{x}\mathbf{sin}(\mathbf{x}) - (\frac{\mathrm{x}^2}{2!}b_1 - \frac{\mathrm{x}^3}{3!}b_2) \quad (17.0)$$

$f_{new}(x) = \frac{\mathbf{x^3}}{\mathbf{3}} + \boldsymbol{x}\mathbf{sin}(\mathbf{x})$

$f_1 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_1(t) f_{new}(t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t\left(\frac{\mathbf{t^3}}{\mathbf{3}} + \boldsymbol{t}\mathbf{sin}(\mathbf{t})\right)dt = \frac{\pi^5}{240}$

$f_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \beta_2(t) f_{new}(t)dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -(\frac{\mathbf{t^3}}{\mathbf{3}} + \boldsymbol{t}\mathbf{sin}(\mathbf{t}))dt = -2$

$F = \begin{bmatrix} \frac{\pi^5}{240} \\ -2 \end{bmatrix}$

$\alpha_{1new}(x) = \frac{\mathrm{x}^2}{2!}$
$\alpha_{2new}(x) = -\frac{\mathrm{x}^3}{3!}$

$\beta_1(t) = \mathrm{t}$
$\beta_2(t) = -1$

$\sum_{i=1}^{2} a_{ij} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_{inew}(x)\beta_j(x)dx$

$a_{11} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_{1new}(t)\beta_1(t)dt = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{t}^3}{2!}dt = 0$

$a_{12} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_{2new}(t)\beta_1(t)dt = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{t}^4}{3!}dt = -\frac{\pi^5}{480}$

$$a_{21} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_{1new}(t)\beta_2(t)dx = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t^2}{2!}dt = -\frac{\pi^3}{24}$$

$$a_{22} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \alpha_{2new}(t)\beta_2(t)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{t^3}{3!}dt = 0$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{\pi^5}{240} \\ -2 \end{bmatrix} - \begin{bmatrix} 0 & -\frac{\pi^5}{480} \\ -\frac{\pi^3}{24} & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{18.0}$$

The determinant is non-zero and solving (15.0) to obtain

$b_1 = 0,\ b_2 = -2$

From (5.0)

$u(x) = f_{new}(x) + \lambda \sum_{i=1}^{n} b_i \alpha_{inew}(x)$

$u(x) = \frac{\mathbf{x^3}}{\mathbf{3}} + \boldsymbol{x}\mathbf{sin}(\mathbf{x}) - (\frac{x^2}{2!} * 0 - -\frac{x^4}{3!} * -2)$

$u(x) = \mathbf{xsin}(\mathbf{x})$

4.0 Conclusion

The present method reduces the computational difficulties of other traditional methods and all the calculation can be made simple. The accuracy of the obtained solution can be improved by taking more terms in the solution.

References

[1] G. Adomian, A review of the decomposition method in applied mathematics, *J. Math.Anal. Appl.*, 135(1988), 501-544.

[2] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, (1994).

[3] H.T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover, New York, (1962).

[4] E. Deeba, S.A. Khuri and S. Xie, An algorithm for solving a nonlinear integro-differential equation, *Appl. Math. Comput.*, 115(2000), 123-131.

[5] H. Jaradat, O. Alsayyed and S. Al-Shara, Numerical solution of linear integro-differential equations, *Journal of Mathematics and Statistics*, 4(4) (2008), 250-254.

[6] Y.A. Kuznetssov, *Elements of Applied Bifurcation Theory*, Springer-Verlag, New York, (1995).

[7] K. Maleknejad and Y. Mohmoudi, Taylor polynomial solution of high-order nonlinear Volterra-Fredholm integro-differential equations, *Applied Mathematics and Computation*, 145(2003), 641-653.

[8] M. Masujima, *Applied Mathematical Methods in Theoretical Physics*, Wiley-VCH, Weinheim, (2005).

[9] G. Micula and P. Pavel, *Differential and Integral Equations through Practical Problems and Exercises*, Kluwer, Boston, (1992).

[10] R.K. Miller, *Nonlinear Volterra Integral Equations*, W.A. Benjamin, Menlo Park, CA, (1967).

[11] D.L. Phillips, A technique for the numerical solution of certain integral equations of the first kind, *J. Asso. Comput. Mach*, 9(1962), 84-96.

[12] A. Shidfar, A. Molabahrami, A. Babaei and A. Yazdanian, A series solution of the nonlinear Volterra and Fredholm integro-differential equations, *Communications in Nonlinear Science and Numerical Simulation*, 15(2) (2010), 205-215.

[13] A. Wazwaz, A reliable treatment for mixed Volterra-Fredholm integral equations, *Appl. Math. Comput.*, 127(2-3) (2002), 405-414.

[14] S. Yalcinbas, Taylor polynomial solutions of nonlinear Volterra–Fredholm integral equations, *Appl. Math. Comput.*, 127(2002), 195-206.

[15] L.N. Sretenskii, Ivar Fredholm: Biography and scientific research (Russian), in *History Methodology Natur. Sci. V Math.*, Moscow, (1966), 150-156.

[16] M.D. Raisinghania, *Integral Equations and Boundary Value Problems*, S. Chand & Company LTD, India, (2007).