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On Summability of the Series Involving Exton's Quadruple Hypergeometric Function K_{12}

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Abstract

In this paper, we derive a general theorem on summability and then make its application to obtain summation of the series involving Exton's quadruple hypergeometric function K_{12}. This work may be useful in the theory of approximation and for computation work of damped oscillatory problems.

Keywords: *Summability of the series, quadruple hypergeometric function K_{12}, bilinear generating relation, summability theorem.*

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1 Introduction

Exton [4] has derived a quadruple convergent hypergeometric representation of a solution of a Schroedinger equation of an inverse eighth degree one dimensional anharmonic oscillator. The normalized Dirichlet integral in the dimensional space $R^n \subset C^{n+1}$ has introduced due to Mathai and Houbold [7] in the form

$$I(\mu) = \int_{R^n} d_{\mu(x)}$$
$$= \frac{1}{B(\mu)} \int_0^1 .(n). \int_0^1 x_1^{\mu_1 - 1} \dots x_n^{\mu_n - 1} (1 - x_1 - \dots - x_n)^{\mu_{n+1} - 1} dx_1 \dots dx_n$$

(1.1)

where, x = $(x_1, \dots, x_n) \in R^n$ such that $0 \le (x_1 + \dots + x_n) \le 1$,

$\mu = (\mu_1, \dots, \mu_n, \mu_{n+1}) \in C,^{n+1}$ $B(\mu) = \frac{\Gamma(\mu_1)\dots\Gamma(\mu_n)\Gamma(\mu_{n+1})}{\Gamma(\mu_1 + \dots + \mu_n + \mu_{n+1})}$

$, Re(\mu_i) > 0, \forall\, i = 1, 2, \dots n, n+1.$

Exton ([2], [3]) had defined following complete qudruple hypergemetric function out of twenty one quadruple hypergeometric functions

$$K_{12}(a, a, a, a, b_1, b_2, b_3, b_4; c_1, c_1, c_2, c_2, x, y, z, t) = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b_1)_m(b_2)_n(b_3)_p(b_4)_q}{(c_1)_{m+n}(c_2)_{p+q}} \frac{x^m}{m!} \frac{y^n}{n!} \frac{z^p}{p!} \frac{t^q}{q!}$$

(1.2)

Basanquet and Kastelman [3] Theorem 1.1 Suppose that $f_n(x)$ in measurable in (a, b), where b-a $\le \infty$, for n = 1, 2, … , then a necessary and sufficient condition that, for every function $g(x)$ integrable (L) over (a,b), the function $f_n(x)$ $g(x)$ be integrable L in (a,b) such that $\sum_{n=0}^{\infty} \left| \int_a^b f_n(x) g(x) dx \right| \le K'$

(1.3)

and $\sum_{n=0}^{\infty} |f_n(x)| \le K'$, (1.4)

where K' is an absolute constant for almost every x in (a, b).

We have presented following extension of the theorem 1.1

Theorem 1.2 Suppose that $f_n(x_1, \cdots, x_r)$ is measurable in the region $x_1 \in (0, \alpha_1), \dots, x_r \in (0, \alpha_r), \alpha_i > 0, \forall\, i = 1,2, \dots, r,$ and for n = 0, 1, 2, …, then a necessary and sufficient condition that for every probability density function $g(x_1, \cdots, x_r)$ defined in the region $x_1 \in (0, \alpha_1), \dots, x_r \in (0, \alpha_r)$ there exists

$$\sum_{n=0}^{\infty} \left| \int_0^{\alpha_1} \cdots \int_0^{\alpha_r} g(x_1, \cdots x_r) f_n(x_1, \cdots, x_r) dx_1 \cdots dx_r \right| \le K,$$

(1.5)

$and\ \sum_{n=0}^{\infty}|f_n(x_1,\cdots,x_r)| \le K,$
(1.6)
where K is an absolute constant for almost every
$x_1 \in (0,\alpha_1),\ldots, x_r \in (0,\alpha_r)$.

Proof

In both sides of (1.6) multiply $g(x_1,\cdots x_r)$, we get

$$\sum_{n=0}^{\infty}|f_n(x_1,\cdots,x_r)|g(x_1,\cdots x_r) \le Kg(x_1,\cdots x_r)$$

(1.7)
Now integrate equation (1.7) with respect to $x_1,\cdots,x_r$ from $0\ to\ \alpha_1,\cdots,0\ to\ \alpha_r$ respectively, then we find that

$$\int_0^{\alpha_1}\cdots\int_0^{\alpha_r} g(x_1,\cdots x_r)|f_n(x_1,\cdots,x_r)|dx_1\cdots dx_r$$
$$\le K\int_0^{\alpha_1}\cdots\int_0^{\alpha_r} g(x_1,\cdots x_r)dx_1\cdots dx_r \qquad (1.8)$$

But $g(x_1,\cdots,x_r)$ is a probability density function in the region $x_1 \in (0,\alpha_1),\cdots,x_r \in (0,\alpha_r)$ so that

$$\int_0^{\alpha_1}\cdots\int_0^{\alpha_r} g(x_1,\cdots x_r)dx_1\cdots dx_r = 1 \qquad (1.9)$$

Hence, taking mode value of (1.8) and then using (1.9), we find the inequality (1.5).

Recently, Kumar, Pathan and Yadav [6] have presented following theorem:

Theorem 1.3

For
$\alpha > 0,\beta > 0,\gamma > 0,\delta > 0, Re(\sigma_i) > Re(-\mu_i), Re(\mu_i) > 0, \forall i = 1,2,3,4\ and\ Re(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)) > 0, a, b = (b_1,b_2,b_3,b_4), c = (c_1,c_2), \mu = (\mu_1,\mu_2,\mu_3,\mu_4), h_1,h_2,h_3\ and\ h_4 \in C$
, a function due to a weighted Dirichlet type integral formula exists

$$F^{a,b,c,\mu,\sigma}(h_1\alpha,h_2\beta,h_3\gamma,h_4\delta) = \frac{\Gamma(a)(\alpha)^{-(\mu_1+\sigma_1)}(\beta)^{-(\mu_2+\sigma_2)}(\gamma)^{-(\mu_3+\sigma_3)}(\delta)^{-(\mu_4+\sigma_4)}}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma\left(a-\sum_{i=1}^{4}(\mu_i+\sigma_i)\right)}$$

$$\times\int_0^{\alpha}\int_0^{\beta}\int_0^{\gamma}\int_0^{\delta}(1-x\alpha^{-1}-y\beta^{-1}-z\gamma^{-1}-t\delta^{-1})^{a-\sum_{i=1}^{4}(\mu_i+\sigma_i)-1}$$
$$\times x^{\mu_1+\sigma_1-1}y^{\mu_2+\sigma_2-1}z^{\mu_3+\sigma_3-1}t^{\mu_4+\sigma_4-1}$$
$$\times K_{12}(a,a,a,a,b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;h_1x,h_2y.h_3z,h_4t)dxdydzdt$$

(1.10)

provided that $0 \leq x\alpha^{-1} + y\beta^{-1} + z\gamma^{-1} + t\delta^{-1} \leq 1$.

Then, for $max\{|h_1\alpha|,|h_2\beta|\} < 1 \; and \; max\{|h_3\gamma|,|h_4\delta|\} < 1$, there holds the degeneration formula

$$F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)$$
$$= F_3[\mu_1+\sigma_1, \mu_2+\sigma_2, b_1, b_2; c_1; h_1\alpha, h_2\beta]$$
$$\times F_3[\mu_3+\sigma_3, \mu_4+\sigma_4, b_3, b_4; c_2; h_3\gamma, h_4\delta]$$
(1.11)

Here, in our investigation, first we obtain some inequalities of the function $F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)$. Then make their applications to obtain summability formulae of quadruple hypergeometric function K_{12}.

2 Inequalities

In this section we evaluate some inequalities which are useful for finding out the summability formulae of quadruple hypergeometric function $\mathbf{K_{12}}$.

Theorem 2.1

For $0 < h_1\alpha < 1, 0 < h_2\beta < 1 \; and \; 0 < h_3\gamma < 1, 0 < h_4\delta < 1$, $c_1 > \mu_1 + \sigma_1 > c_1 - b_1 > 0$, $c_1 > \mu_2 + \sigma_2 > c_1 - b_2 > 0$, $c_2 > \mu_3 + \sigma_3 > c_2 - b_3 > 0$, and $c_2 > \mu_4 + \sigma_4 > c_2 - b_4 > 0$, then there holds an inequality

$$\left|F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)\right|$$
$$< \frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+b_2-c_1)\Gamma(\mu_3+\sigma_3+b_3-c_2)\left(\Gamma(c_1)\right)^2\left(\Gamma(c_2)\right)^2}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}$$
$$\times \Gamma(\mu_4+\sigma_4+b_4-c_2)(1-h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}(1-h_2\beta)^{c_1-\mu_2-\sigma_2-b_2}$$
$$\times (1-h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}(1-h_4\delta)^{c_2-\mu_4-\sigma_4-b_4}$$
$$\times \; {}_2F_1\left[\begin{matrix} c_1-1, \frac{(c_1+1)}{2}; \\ \frac{(c_1-1)}{2}; \end{matrix} -h_1h_2\alpha\beta\right] {}_2F_1\left[\begin{matrix} c_2-1, \frac{(c_2+1)}{2}; \\ \frac{(c_2-1)}{2}; \end{matrix} -h_3h_4\gamma\delta\right]$$
(2.1)

Proof

Under the restrictions
$c > a_1 > c - b_1 > 0, c > a_2 > c - b_2 > 0,$ 0 < x < 1, 0 < y < 1, Joshi and Arya [5] have derived the inequality

$$F_3[a_1, a_2, b_1, b_2; c; x, y] < \frac{\Gamma(a_1 + b_1 - c)\Gamma(a_2 + b_2 - c)\left(\Gamma(c)\right)^2}{\Gamma(a_1)\Gamma(b_1)\Gamma(a_2)\Gamma(b_2)}$$
$$\times (1-x)^{c-a_1-b_1}(1-y)^{c-a_2-b_2} \; {}_2F_1\left[\begin{matrix} c-1, {}^{(c+1)}/_2; \\ {}^{(c-1)}/_2; \end{matrix} -xy\right]$$

(2.2)

In the right hand side of the equation (1.11) for both F_3 [.] functions under the restrictions $0 < h_1\alpha < 1, 0 < h_2\beta < 1 \; and \; 0 < h_3\gamma < 1, 0 < h_4\delta < 1,$
$\mu_1 + \sigma_1 > c_1 - b_1 > 0, \mu_2 + \sigma_2 > c_1 - b_2 > 0,$
$\mu_3 + \sigma_3 > c_2 - b_3 > 0,$ and $\mu_4 + \sigma_4 > c_2 - b_4 > 0, c_1 > 0, c_2 > 0,$
apply the formula (2.2), we find the inequality (2.1).

Theorem 2.2

For
$0 < Re(c_1) < 1, 0 < Re(c_2) < 1, 0 < h_1\alpha < 1, 0 < h_2\beta < 1 \; and \; 0 < h_3\gamma < 1, 0 < h_4\delta < 1, \; \mu_1 + \sigma_1 > c_1 - b_1 > 0, \mu_2 + \sigma_2 > c_1 - b_2 > 0, \mu_3 + \sigma_3 > c_2 - b_3 > 0,$ and $\mu_4 + \sigma_4 > c_2 - b_4 > 0,$
there holds an inequality

$$\left|F^{\mathrm{a,b,c,\mu,\sigma}}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)\right|$$
$$< \frac{\Gamma(\mu_1 + \sigma_1 + b_1 - c_1)\Gamma(\mu_2 + \sigma_2 + b_2 - c_1)\Gamma(\mu_3 + \sigma_3 + b_3 - c_2)\left(\Gamma(c_1)\right)^2\left(\Gamma(c_2)\right)^2}{\Gamma(\mu_1 + \sigma_1)\Gamma(\mu_2 + \sigma_2)\Gamma(\mu_3 + \sigma_3)\Gamma(\mu_4 + \sigma_4)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}$$
$$\times \Gamma(\mu_4 + \sigma_4 + b_4 - c_2)\Gamma(c_1 - 1)\Gamma(c_2 - 1)(1 - h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}$$
$$\times (1 - h_2\beta)^{c_1-\mu_2-\sigma_2-b_2}(1 - h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}(1 - h_4\delta)^{c_2-\mu_4-\sigma_4-b_4}$$
$$\times \; {}_2F_0\left[\tfrac{c_1+1}{2}, c_1 - 1; -; \tfrac{-2h_1h_2\alpha\beta}{c_1-1}\right] {}_2F_0\left[\tfrac{c_2+1}{2}, c_2 - 1; -; \tfrac{-2h_3h_4\gamma\delta}{c_2-1}\right]$$

(2.3)

Proof

For the well known Pochhammer symbol we have the inequality

$(\lambda)_n \geq (\lambda)^n, n \in N \cup \{0\}, \lambda > 0$

(2.4)

The Laplacian integral formula of Gaussian hypergeometric function ${}_2F_1$ [.] is given by (See, Exton [3])

$${}_2F_1 \left[\begin{matrix} a, b; \\ c; \end{matrix} x\right] = \int_0^\infty e^{-r} r^{a-1} \; {}_1F_1 \left[\begin{matrix} b; \\ c; \end{matrix} xr\right] dr$$

(2.5)

In the right hand side of the inequality (2.1) for both Gaussian hypergeometric functions ${}_2F_1$ [.] apply the formula (2.5) and then expand ${}_1F_1$ functions in the series form and then change the order of summation and integration and make the application of the inequality (2.4) in the denominator the series and on solving the integration finally on defining ${}_pF_q$ [.] generalized hypergeometric function (Rainville [8])), we get the inequality (2.3).

Theorem 2.3

For $0 < h_1\alpha < 1, 0 < h_2\beta < 1$ *and* $0 < h_3\gamma < 1, 0 < h_4\delta < 1$, and

$\mu_1 + \sigma_1 > c_1 - b_1 > 0, \mu_2 + \sigma_2 > c_1 - b_2 > 0,$

$\mu_3 + \sigma_3 > c_2 - b_3 > 0,$ and $\mu_4 + \sigma_4 > c_2 - b_4 > 0,$

$c_1 > 0, c_2 > 0,$ then there holds an inequality

$$\left|F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)\right|$$
$$< \frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+b_2-c_1)\Gamma(\mu_3+\sigma_3+b_3-c_2)\left(\Gamma(c_1)\right)^2\left(\Gamma(c_2)\right)^2}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)\Gamma(b_4)}$$
$$\times \Gamma(\mu_4+\sigma_4+b_4-c_2)(1-h_1\alpha)^{c_1-\mu_1-\sigma_1-b_1}(1-h_2\beta)^{c_1-\mu_2-\sigma_2-b_2}$$
$$\times (1-h_3\gamma)^{c_2-\mu_3-\sigma_3-b_3}(1-h_4\delta)^{c_2-\mu_4-\sigma_4-b_4}(1+h_1h_2\alpha\beta)^{-c_1}(1+h_3h_4\gamma\delta)^{-c_2}$$
$$\times ((1-h_1h_2\alpha\beta)(1-h_3h_4\gamma\delta)).$$

(2.6)

Proof

The contiguous function relation for Gaussian hypergeometric function ${}_2F_1$ (.) in the notations (See, Rainville ([8], p. 53))

F = ${}_2F_1$ (a, b; c; x), F (a+) = ${}_2F_1$ (a+1, b; c; x), F (a-) = ${}_2F_1$ [a-1, b; c; x], is given by

(1-x)F = F (b-) – c^{-1} (c-a) x F(c+)

(2.7)

Multiply $((1 + h_1h_2\alpha\beta)(1 + h_3h_4\gamma\delta))$ in both sides of the inequality (2.1) and then in its right hand side use the contiguous function relation (2.7) and again solving it we obtain the inequality (2.6).

3 Summability of Quadruple Hypergeometric Function K_{12}

In this section, we use the inequalities obtained in the section 2 and obtain that quadruple hypergeometric function K_{12} is summable.

Theorem 3.1 If $x \in (0,\alpha), y \in (0,\beta), z \in (0,\gamma), and\ t \in (0,\delta)$, such that $\alpha > 0, \beta > 0, \gamma > 0, \delta > 0$,
$0 \le x\alpha^{-1} + y\beta^{-1} + z\gamma^{-1} + t\delta^{-1} \le 1$, then for $\lambda > 0$,
$0 < h_1\alpha < 1, 0 < h_2\beta < 1\ and\ 0 < h_3\gamma < 1, 0 < h_4\delta < 1$,
$\mu_1 + \sigma_1 > c_1 - b_1 > 0, \mu_2 + \sigma_2 > c_1 - \lambda > 0$,
$\mu_3 + \sigma_3 > c_2 - b_3 > 0$, and $\mu_4 + \sigma_4 > c_2 - \lambda > 0$,
$c_1 > 0, c_2 > 0, |T| < 1$, following summability relation of quadruple hypergeometric function K_{12} holds:

$$\sum_{n=0}^{\infty} \left| \frac{(\lambda)_n}{n!} K_{12}(a,a,a,a,b_1,-n,b_3,-n;c_1,c_1,c_2,c_2;h_1x,h_2y.h_3z,h_4t)T^n \right|$$

$$< (1-T)^{-\lambda} \frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+\lambda-c_1)\Gamma(\mu_3+\sigma_3+b_2-c_2)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)}$$
$$\times \frac{\Gamma(\mu_4+\sigma_4+\lambda-c_2)\{\Gamma(c_1)\}^2\{\Gamma(c_2)\}^2}{\Gamma(b_1)\Gamma(b_2)\{\Gamma(\lambda)\}^2} \left\{1 - \frac{h_1h_2\alpha\beta T}{T-1}\right\}\left\{1 - \frac{h_3h_4\gamma\delta T}{T-1}\right\}$$
$$\times \left\{1 + \frac{h_1h_2\alpha\beta T}{T-1}\right\}^{-c_1} \left\{1 + \frac{h_3h_4\gamma\delta T}{T-1}\right\}^{-c_2} \left\{1 - \frac{h_2\beta T}{T-1}\right\}^{c_1-\mu_2-\sigma_2-\lambda}$$
$$\times \left\{1 - \frac{h_4\delta T}{T-1}\right\}^{c_2-\mu_4-\sigma_4-\lambda} \{1-h_1\alpha\}^{c_1-\mu_1-\sigma_1-b_1}\{1-h_3\gamma\}^{c_2-\mu_3-\sigma_3-b_2}$$
$$\times\ {}_4F_3 \left[\begin{matrix} c_1, c_2, \mu_2+\sigma_2+\lambda-c_1, \mu_4+\sigma_4+\lambda-c_2; \\ \lambda, 1+c_1-\mu_1-\sigma_1-b_1, 1+c_2-\mu_3-\sigma_3-b_2; \end{matrix} \right.$$
$$\left. \frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2h_4\beta\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}} \right]$$

(3.1)

provided that

$$\left| \frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2h_4\beta\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}} \right| < 1.$$

Proof

To prove this theorem 3.1, we consider the bilinear generating relation of Kumar, Pathan and Yadav [6] given by, when $|T| < 1$,

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} F^{a,(b_1,-n,b_3,-n),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)T^n$$

$$= (1-T)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n(\mu_2+\sigma_2)_n(\mu_4+\sigma_4)_n}{n!(c_1)_n(c_2)_n}\left\{\frac{h_2h_4\beta\delta T}{(1-T)^2}\right\}^n$$

$$\times F_3\left[\mu_1+\sigma_1, \mu_2+\sigma_2+n, b_1, \lambda+n; c_1+n; h_1\alpha, \frac{h_2\beta T}{T-1}\right]$$

$$\times F_3\left[\mu_3+\sigma_3, \mu_4+\sigma_4+n, b_2, \lambda+n; c_2+n; h_3\gamma, \frac{h_4\delta T}{T-1}\right]$$

(3.2)

Then, we follow the theorem 2.1 and theorem 2.3 in (3.2) and find that

$$\sum_{n=0}^{\infty} \left|\frac{(\lambda)_n}{n!} F^{a,(b_1,-n,b_3,-n),c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)T^n\right|$$

$$< (1-T)^{-\lambda} \frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+\lambda-c_1)\Gamma(\mu_3+\sigma_3+b_2-c_2)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)}$$

$$\times \frac{\Gamma(\mu_4+\sigma_4+\lambda-c_2)\{\Gamma(c_1)\}^2\{\Gamma(c_2)\}^2}{\Gamma(b_1)\Gamma(b_2)\{\Gamma(\lambda)\}^2}\left\{1-\frac{h_1h_2\alpha\beta T}{T-1}\right\}\left\{1-\frac{h_3h_4\gamma\delta T}{T-1}\right\}$$

$$\times \left\{1+\frac{h_1h_2\alpha\beta T}{T-1}\right\}^{-c_1}\left\{1+\frac{h_3h_4\gamma\delta T}{T-1}\right\}^{-c_2}\left\{1-\frac{h_2\beta T}{T-1}\right\}^{c_1-\mu_2-\sigma_2-\lambda}$$

$$\times \left\{1-\frac{h_4\delta T}{T-1}\right\}^{c_2-\mu_4-\sigma_4-\lambda} \{1-h_1\alpha\}^{c_1-\mu_1-\sigma_1-b_1}\{1-h_3\gamma\}^{c_2-\mu_3-\sigma_3-b_2}$$

$$\times {}_4F_3\left[\begin{matrix} c_1, c_2, \mu_2+\sigma_2+\lambda-c_1, \mu_4+\sigma_4+\lambda-c_2; \\ \lambda, 1+c_1-\mu_1-\sigma_1-b_1, 1+c_2-\mu_3-\sigma_3-b_2; \end{matrix} \frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2h_4\beta\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right]$$

(3.3)

provided that

$$\left|\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2h_4\beta\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right| < 1.$$

Now in left hand side of (3.3) define the function $F^{a,b,c,\mu,\sigma}(h_1\alpha, h_2\beta, h_3\gamma, h_4\delta)$, by the theorem 1.3 and then make an appeal to the theorem 1.2, we get the summability relation (3.1) of quadruple hypergeometric function K_{12} .

4 Examples

Let in the region, $x \in (0,\alpha), y \in (0,\beta), z \in (0,\gamma), and\ t \in (0,\delta)$, such that $\alpha > 0, \beta > 0, \gamma > 0, \delta > 0,$

$0 \leq x\alpha^{-1} + y\beta^{-1} + z\gamma^{-1} + t\delta^{-1} \leq 1,$ the position of the particle is given by the sequence of function

$f_n(x,y,z,t) = K_{12}(a,a,a,a,b_1,-n,b_3,-n;c_1,c_1,c_2,c_2;h_1x,h_2y.h_3z,h_4t)$,

$\forall n \in N_0 = \{0,1,2,\dots\}$, Then there exists a convergent function $g(x,y,z,t)$ defined for $\lambda > 0, and\ |T| < 1,$ such that

$g(x,y,z,t) = \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} f_n(x,y,z,t)T^n$, then

$$|g(x,y,z,t)| < (1-T)^{-\lambda} \frac{\Gamma(\mu_1+\sigma_1+b_1-c_1)\Gamma(\mu_2+\sigma_2+\lambda-c_1)\Gamma(\mu_3+\sigma_3+b_2-c_2)}{\Gamma(\mu_1+\sigma_1)\Gamma(\mu_2+\sigma_2)\Gamma(\mu_3+\sigma_3)\Gamma(\mu_4+\sigma_4)}$$
$$\times \frac{\Gamma(\mu_4+\sigma_4+\lambda-c_2)\{\Gamma(c_1)\}^2\{\Gamma(c_2)\}^2}{\Gamma(b_1)\Gamma(b_2)\{\Gamma(\lambda)\}^2} \left\{1-\frac{h_1h_2\alpha\beta T}{T-1}\right\}\left\{1-\frac{h_3h_4\gamma\delta T}{T-1}\right\}$$
$$\times \left\{1+\frac{h_1h_2\alpha\beta T}{T-1}\right\}^{-c_1}\left\{1+\frac{h_3h_4\gamma\delta T}{T-1}\right\}^{-c_2}\left\{1-\frac{h_2\beta T}{T-1}\right\}^{c_1-\mu_2-\sigma_2-\lambda}$$
$$\times \left\{1-\frac{h_4\delta T}{T-1}\right\}^{c_2-\mu_4-\sigma_4-\lambda} \{1-h_1\alpha\}^{c_1-\mu_1-\sigma_1-b_1}\{1-h_3\gamma\}^{c_2-\mu_3-\sigma_3-b_2}$$

$$\times\ {}_4F_3\left[\begin{matrix} c_1,c_2,\mu_2+\sigma_2+\lambda-c_1,\mu_4+\sigma_4+\lambda-c_2; \\ \lambda,1+c_1-\mu_1-\sigma_1-b_1,1+c_2-\mu_3-\sigma_3-b_2; \end{matrix} \frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2h_4\beta\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right]$$

(4.1)

provided that

$\left|\frac{(1-T)^2(1-h_1\alpha)(1-h_3\gamma)(h_2h_4\beta\delta T)}{\{T(1+h_1h_2\alpha\beta)-1\}\{T(1+h_3h_4\gamma\delta)-1\}\{T(1-h_2\beta)-1\}\{T(1-h_4\delta)-1\}}\right| < 1.$

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