On fuzzy h-ideals in h-regular Γ-hemiring and h-duo Γ-hemiring

S. K. Sardar[1] and D. Mandal[2]

[1,2]Department of Mathematics,Jadavpur University, Kolkata, India
E-mail:[1]sksardarjumath@gmail.com, [2]dmandaljumath@gmail.com

Abstract

In this paper, fuzzy h-bi-ideals, fuzzy h-quasi-ideals and fuzzy h-interior ideals of a Γ-hemiring are studied and some related properties are investigated. The notions of h-intra-hemiregularity and h-quasi-hemiregularity of a Γ-hemiring are introduced and studied along with h-hemiregularity and their characterizations in terms of fuzzy h-ideals are also obtained. The concept of fuzzy h-duo Γ-hemiring is introduced and some of its characterizations are obtained.

Keywords: *Γ-hemiring, h-Duo Γ-hemiring, Fuzzy h-ideal, Fuzzy h-bi-ideal, Fuzzy h-quasi-ideal, Fuzzy h-interior ideal, Fuzzy h-duo, h-hemiregularity, h-quasi-hemiregularity, h-intra-hemiregularity.*

2000 MSC No: 08A72,16Y99

1 Introduction

Semiring is a well known universal algebra. This is a generalization of an associative ring (R,+,.). If (R, +) becomes a semigroup instead of a group then (R,+,.) reduces to a semiring. Semiring has been found very useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems

for semiring. To ammend this gap Henriksen[4] defined a more restricted class of ideals, which are called k-ideals. A still more restricted class of ideals in hemirings are given by Iizuka[6], which are called h-ideals. LaTorre[9], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring and to amend the gap between ring ideals and semiring ideals. The theory of Γ-semiring was introduced by Rao[12]. These concepts are extended by Dutta and Sardar[2].

The theory of fuzzy sets, proposed by Zadeh[15], has provided a useful mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. The study of fuzzy algebraic structure has started by Rosenfeld[13]. Since then many researchers developed this ideas.

Various relationship between the fuzzy sets and semirings have been considered by Dutta and Biswas[1]. The concept of fuzzy h-ideals in hemiring was studied by Jun et al[7]. Recently Zhan et al[16], Yin et al[14], Huang et al[5] introduced and studied the concepts of h-hemiregularity, h-intra-hemiregularity of a hemiring and gave some of their characterization using fuzzy h-ideals. Ma et al [10] extended some of these results in more general setting of hemiring i.e. Γ-hemiring . In this paper we study some of these properties by using *o*-composition instead of Γ-composition thereby encompassing some results of Ma et al [10]. We also study h-intra-hemiregularity and h-quasi-hemiregularity in Γ-hemirings and h-duo Γ-hemirings.

2 Preliminaries

We recall the following preliminaries for subsequent use.

Definition 2.1. *[3] A hemiring [respectively semiring] is a nonempty set S on which operations addition and multiplication have been defined such that the following conditions are satisfied:*
(i) $(S,+)$ is a commutative monoid with identity 0.
(ii) $(S,.)$ is a semigroup [respectively monoid with identity 1_S].
(iii) Multiplication distributes over addition from either side.
(iv) $0s=0=s0$ for all $s \in S$.
(v) $1_S \neq 0$

Definition 2.2. *[11] Let S and Γ be two additive commutative semigroups with zero. Then S is called a $\Gamma-$hemiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ ($(a,\alpha,b) \mapsto a\alpha b$) satisfying the following conditions:*
(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$
for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.
For simplification we write 0 instead of 0_S and 0_Γ.

Example 2.3. *Let S be the set of all $m \times n$ matrices over $\mathbf{Z_0^-}$ (the set of all non-positive integers) and Γ be the set of all $n \times m$ matrices over $\mathbf{Z_0^-}$, then S forms a Γ-hemiring with usual addition and multiplication of matrices.*

Example 2.4. *Let S=Γ= a non-positive cone of a totally ordered ring. Then S is a $\Gamma-$hemiring where ternary operation is the multiplication of the ring. (It may be noted here that though non-negative cones of totally ordered rings form semirings, non-positive cones do not as the induced multiplication is not a binary composition in this case).*

Throughout this paper, unless otherwise mentioned S denotes a $\Gamma-$hemiring.

A subset A of a Γ-hemiring S is called a left(resp. right) ideal of S if A is closed under addition and $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A subset A of a hemiring S is called an ideal if it is both left and right ideal of S.
A subset A of a Γ-hemiring S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$.
A subset A of a Γ-hemiring S is called a bi-ideal(resp. interior ideal) if A is closed under addition and $A\Gamma S\Gamma A \subseteq A (resp. S\Gamma A\Gamma S \subseteq A)$.
A left ideal A of S is called a left h-ideal if x,z∈S, a,b∈A and x+a+z=b+z implies x∈A. A right h-ideal is defined analoguesly.
The h-closure $\overline{A}$ of A in S is defined as $\overline{A} = \{x \in S \mid x+a+z=b+z$, for some a,b∈ A and z∈S}.
Now if A is a left (right) ideal of S, then $\overline{A}$ is the smallest left (right) h-ideal containing A.
A quasi-ideal(bi-ideal,interior-ideal) A of S is called an h-quasi-ideal(resp. h-bi-ideal, h-interior-ideal) of S if $\overline{S\Gamma A} \cap \overline{A\Gamma S}(resp. \overline{A\Gamma S\Gamma A}, \overline{S\Gamma A\Gamma S}) \subseteq A$ and x+a+z=b+z implies x∈A for all x,z∈S and a,b∈A.

Lemma 2.5. *For a Γ-hemiring S, we have*
i) $A \subseteq \overline{A}$ for all A⊆S.
ii) $A \subseteq B \subseteq S \Rightarrow \overline{A} \subseteq \overline{B}$.
iii) $\overline{\overline{A}} = \overline{A}$ for all A⊆S.
iv) $\overline{A\Gamma B} = \overline{\overline{A}\Gamma\overline{B}}$
v) For any left(resp. right) h-ideal, h-bi-ideal, h-quasi-ideal A of S, we have $A = \overline{A}$.

Note that every h-ideal of S is an h-quasi-ideal and h-interior ideal; Every h-quasi-ideal of S is an h-bi-ideal of S. However the converse of these properties do not hold in general.

Example 2.6. *Let $S=\Gamma = M_2(Z_0^-)$, be the set of all 2×2 matrices over Z_0^-, the set of all non-positive integers. Let $Q=\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in Z_0^-\}$. Then Q is a h-quasi-ideal of S, but not a (left,right) h-ideal of S.*

Definition 2.7. *[15] A fuzzy subset of a nonempty set X is defined as a function $\mu : X \rightarrow [0,1]$.*

Definition 2.8. *[11]Let μ be a non empty fuzzy subset of a $\Gamma-$hemiring S (i.e. $\mu(x) \neq 0$ for some $x \in S$). Then μ is called a fuzzy left ideal [fuzzy right ideal] of S if*
(i) $\mu(x + y) \geq \min[\mu(x), \mu(y)]$ and
(ii) $\mu(x\gamma y) \geq \mu(y)$ [respectively $\mu(x\gamma y) \geq \mu(x)$] for all $x, y \in S$, $\gamma \in \Gamma$.
A fuzzy ideal of a $\Gamma-$hemiring S is a non empty fuzzy subset of S which is a fuzzy left ideal as well as a fuzzy right ideal of S.
Note that if μ is a fuzzy left or right ideal of a $\Gamma-$hemiring S, then $\mu(0) \geq \mu(x)$ for all $x \in$ S.

Definition 2.9. *[11] A fuzzy left ideal μ of a $\Gamma-$hemiring S is called a fuzzy left h-ideal if for all a, b, x, z $\in$ S, x + a + z = b + z implies*
$\mu(x) \geq min \{\mu(a), \mu(b)\}$.

A fuzzy right h-ideal is defined similarly.
By a fuzzy h-ideal μ, we mean that μ is both fuzzy left and fuzzy right h-ideal.

Example 2.10. *[11] Let S be the additive commutative semigroup of all non positive integers and Γ be the additive commutative semigroup of all non positive even integers. Then S is a $\Gamma-$hemiring if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let μ be a fuzzy subset of S, defined as follows* $\mu(x) = \begin{cases} 1 & if \ x = 0 \\ 0.7 & if \ x \ is \ even \\ 0.1 & if \ x \ is \ odd \end{cases}$
The fuzzy subset μ of S is both fuzzy ideal and fuzzy h-ideal of S.

Definition 2.11. *[10] Let μ and θ be two fuzzy sets of a $\Gamma-$hemiring S. Define h-product of μ and θ by*
$$\mu\Gamma_h\theta(x) = \sup_{x+a_1\gamma b_1+z=a_2\delta b_2+z}[\min\{\mu(a_1), \mu(a_2), \theta(b_1), \theta(b_2)\}]$$
$$= 0, \text{ if } x \text{ cannot be expressed as } x + a_1\gamma b_1 + z = a_2\delta b_2 + z$$
for $x, z, a_1, a_2, b_1, b_2 \in S$ and $\gamma, \delta \in \Gamma$.

Definition 2.12. *Let μ and θ be two fuzzy sets of a $\Gamma-$hemiring S. Define a generalized h-product of μ and θ by*

$$\mu o_h \theta(x) = \sup_i [\min\{\min\{\mu(a_i), \mu(c_i), \theta(b_i), \theta(d_i)\}\}]$$

$$x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

= 0, if x cannot be expressed as above

where $x, z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for i=1,...,n.

Lemma 2.13. *Let μ_1, μ_2 be two fuzzy h-ideal of a $\Gamma-$hemiring S. Then $\mu_1 \Gamma_h \mu_2 \subseteq \mu_1 o_h \mu_2 \subseteq \mu_1 \cap \mu_2 \subseteq \mu_1, \mu_2$.*

Lemma 2.14. *Let S be a Γ-hemiring and $A, B \subseteq S$. Then we have*
i) $A \subseteq B$ if and only if $\chi_A \subseteq \chi_B$.
ii) $\chi_A \cap \chi_B = \chi_{A \cap B}$
iii) $\chi_A o_h \chi_B = \chi_{A \Gamma B}$

For more preliminaries of semirings(hemirings) and $\Gamma-$semirings we refer to [3] and [2] respectively.

3   Fuzzy h-ideals in Γ-hemiring

Definition 3.1. *[10] A fuzzy subset μ of a Γ-hemiring S is called fuzzy h-bi-ideal if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have*
(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(x \alpha y) \geq \min\{\mu(x), \mu(y)\}$
(iii) $\mu(x \alpha y \beta z) \geq \min\{\mu(x), \mu(z)\}$
(iv) $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$

Definition 3.2. *A fuzzy subset μ of a Γ-hemiring S is called fuzzy h-quasi-ideal if for all $x, y, z, a, b \in S$ we have*
(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $(\mu o_h \chi_S) \cap (\chi_S o_h \mu) \subseteq \mu$
(iii) $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$

Definition 3.3. *[10] A fuzzy subset μ of a Γ-hemiring S is called fuzzy h-interior-ideal if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have*
(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\mu(x \alpha y) \geq \min\{\mu(x), \mu(y)\}$
(iii) $\mu(x \alpha y \beta z) \geq \mu(y)$
(iv) $x + a + z = b + z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$

Example 3.4. *Let $S = \Gamma = Z^-$, the set of all non-positive integers. Then S is a Γ-hemiring. Define $\mu : S \to [0,1]$ by $\mu(x) = \begin{cases} t & if \ x \in 2S \\ 0 & otherwise \end{cases}$ Let*

B=2S. Then for any t∈[0,1], $\mu_t = \{2S\}$. Since $\{2S\}$ is a h-bi-ideal(resp. h-quasi-ideal) of Z^-, μ_t is a h-bi-ideal of Z^- for all t. Hence μ is a fuzzy h-bi-ideal(resp. h-quasi-ideal)of Z^-.

For any fuzzy subset in a set X and any t∈[0,1], define $\mu_t = \{x \in X \mid \mu(x) \geq t\}$, which is called a level subset of μ. In [8],Kondo et al. introduced the Transfer Principle in fuzzy set theory, from which a fuzzy set can be characterized by its level subsets. For any algebraic system $\mathcal{U} = (X, F)$, where F is a family of operations defined on X, the Transfer Principle can be formulated as follows:

Lemma 3.5. *A fuzzy subset defined on $\mathcal{U}$ has the property $\mathcal{P}$ if and only if all non-empty level subset μ_t have the property $\mathcal{P}$.*

As a direct consequence of the above Lemma, the following results can be obtained.

Lemma 3.6. *[10] Let S be a Γ-hemiring. Then the following conditions hold:*
(i) μ is a fuzzy left(resp. right) h-ideal of S if and only if all non-empty level subsets μ_t are left (resp. right) h-ideals of S.
(ii) μ is a fuzzy h-bi-ideal of S if and only if all non-empty level subsets μ_t are h-bi-ideals of S.
(iii) μ is a fuzzy h-quasi-ideal of S if and only if all non-empty level subsets μ_t are h-quasi-ideals of S.

Lemma 3.7. *Let S be a Γ-hemiring and A⊆S. Then the following conditions hold:*
(i) A is a left(resp. right) h-ideal of S if and only if χ_A is a fuzzy left (resp. right) h-ideal of S.
(ii) A is an h-bi-ideal of S if and only if χ_A is a fuzzy h-bi-ideal of S.
(iii) A is an h-quasi-ideal of S if and only if χ_A is a fuzzy h-quasi-ideal of S.

Lemma 3.8. *A fuzzy subset μ of a Γ-hemiring S is a fuzzy left(resp. right) h-ideal of S if and only if for all x,y,z,a,b∈S, we have*
(i) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$
(ii) $\chi_S o_h \mu \subseteq \mu (resp. \mu o_h \chi_S \subseteq \mu)$
(iii) $x+a+z=b+z \Rightarrow \mu(x) \geq \min\{\mu(a), \mu(b)\}$.

Proof. Assume that μ is a fuzzy left h-ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let x∈S. If $(\chi_S o_h \mu)(x) = 0$, it is clear that $(\chi_S o_h \mu)(x) \leq \mu(x)$. Otherwise, there exist elements $z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for i=1,...,n such that $x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$. Then we

have

$$(\chi_S o_h \mu)(x) \;=\; \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\min_i\{\min\{\chi_S(a_i), \chi_S(c_i), \mu(b_i), \mu(d_i)\}\}]$$

$$=\; \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\min_i\{\min\{\mu(b_i), \mu(d_i)\}\}]$$

$$\leq\; \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\min_i\{\min\{\mu(a_i\gamma_i b_i), \mu(c_i\delta_i d_i)\}\}]$$

$$\leq\; \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\min\{\mu(\sum_{i=1}^{n} a_i\gamma_i b_i), \mu(\sum_{i=1}^{n} c_i\delta_i d_i)\}]$$

$$\leq\; \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} \mu(x) \;=\; \mu(x).$$

This implies that $\chi_S o_h \mu \subseteq \mu$.

Conversely, assume that the given conditions hold. Then it is sufficient to show the 2nd condition of the definition of h-ideal. Let x,y$\in$ S and $\gamma \in \Gamma$. Then we have

$$\mu(x\gamma y) \;\geq\; (\chi_S o_h \mu)(x\gamma y) \;=\; \sup_{x\gamma y+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}[\min_i\{\min\{\mu(b_i), \mu(d_i)\}\}]$$

$$\geq \mu(y)(since\; x\gamma y + 0\gamma y + 0 = x\gamma y + 0).$$

Hence μ is a fuzzy left h-ideal of S. The case for fuzzy right h-ideal can be proved similarly. $\square$

Lemma 3.9. *Let μ and ν be a fuzzy right h-ideal and a fuzzy left h-ideal of a Γ-hemiring S, respectively. Then $\mu \cap \nu$ is a fuzzy h-quasi-ideal of S.*

Proof. Let x,y be any element of S. Then

$$\begin{aligned}
(\mu \cap \nu)(x + y) \;&=\; \min(\mu(x+y), \nu(x+y)) \\
&\geq\; \min(\min(\mu(x), \mu(y)), \min(\nu(x), \nu(y))) \\
&=\; \min(\min(\mu(x), \nu(x)), \min(\mu(y), \nu(y))) \\
&=\; \min((\mu \cap \nu)(x), (\mu \cap \nu)(y)).
\end{aligned}$$

Now let a,b,x,z∈S such that x+a+z=b+z. Then

$$\begin{aligned}
(\mu \cap \nu)(x) &= \min(\mu(x), \nu(x)) \\
&\geq \min(\min(\mu(a), \mu(b)), \min(\nu(a), \nu(b))) \\
&= \min(\min(\mu(a), \nu(a)), \min(\mu(b), \nu(b))) \\
&= \min((\mu \cap \nu)(a), (\mu \cap \nu)(b))
\end{aligned}$$

On the other hand, we have

$$((\mu \cap \nu) o_h \chi_S) \cap (\chi_S o_h (\mu \cap \nu)) \subseteq (\mu o_h \chi_S) \cap (\chi_S o_h \nu) \subseteq \mu \cap \nu.$$

This completes the proof. □

Lemma 3.10. *Any fuzzy h-quasi-ideal of S is a fuzzy h-bi-ideal of S.*

Proof. Let μ be any fuzzy h-quasi-ideal of S. It is sufficient to show that $\mu(x\alpha y\beta z) \geq \min\{\mu(x), \mu(z)\}$ and $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$ for all x,y,z∈ S and $\alpha, \beta \in \Gamma$.

In fact, by the assumption,we have
$$\mu(x\alpha y\beta z) \geq ((\mu o_h \chi_S) \cap (\chi_S o_h \mu))(x\alpha y\beta z)$$
$$= \min\{(\mu o_h \chi_S)(x\alpha y\beta z), (\chi_S o_h \mu)(x\alpha y\beta z)\}$$
$$= \min\{ \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}(\min(\mu(a_i), \mu(c_i))) , \sup_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z}(\min(\mu(b_i), \mu(d_i))) \}$$
$$\geq \min\{\min(\mu(0), \mu(x)), \min(\mu(0), \mu(z))\}(\text{since } x\alpha y\beta z + 0\gamma 0 + 0 = x\alpha y\beta z + 0)$$
$$= \min\{\mu(x), \mu(z)\}.$$
Similarly, we can show that $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$ for all x,y∈S and for all $\alpha \in \Gamma$. □

4 h-hemiregularity

In this section we study the concept of h-hemiregularity in Γ-hemiring by using h-ideal, h-bi-ideal, h-quasi-ideal.

Definition 4.1. *[10] A Γ-hemiring S is said to be h-hemiregular if for each x∈S, there exist a,b∈S and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.*

Definition 4.2. *A Γ-hemiring S is said to be h-interior-regular if for each x∈S, there exist $a, a', b, b' \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + a\alpha x\beta a' + z = b\gamma x\delta b' + z$.*

Lemma 4.3. *[10] A Γ-hemiring S is h-hemiregular if and only if for any right h-ideal R and any left h-ideal L of S we have $R\Gamma L = R \cap L$.*

Theorem 4.4. *A Γ-hemiring S is h-hemiregular if and only if for any fuzzy right h-ideal μ and any fuzzy left h-ideal ν of S we have $\mu o_h \nu = \mu \cap \nu$.*

Proof. The proof of this theorem follows from simple generalization of the proof of Theorem 5.5 of [10]. □

Proposition 4.5. *Let S be a Γ-hemiring. Then if S is h-interior-regular, $I = \overline{S\Gamma I\Gamma S}$ for every h-interior-ideal I of S.*

Proof. Let S be a Γ-hemiring and x$\in$I. Since S is h-interior-regular, $x + a\alpha x\beta a' + z = b\gamma x\delta b' + z$. for some $a, a', b, b', z \in$S and $\alpha, \beta, \gamma, \delta \in \Gamma$. Now $a\alpha x\beta a', b\gamma x\delta b' \in S\Gamma I\Gamma S$. So, x$\in \overline{S\Gamma I\Gamma S}$. On the other hand, since I is an h-interior-ideal of S, we have $S\Gamma I\Gamma S \subseteq I$ and so $\overline{S\Gamma I\Gamma S} \subseteq \overline{I} = I$. Therefore $I = \overline{S\Gamma I\Gamma S}$. □

We need the following Lemma which was stated in [10] without proof.

Lemma 4.6. *[10] Let S be a Γ-hemiring. Then the following conditions are equivalent.*
(i) S is h-hemiregular.
(ii) $B = \overline{B\Gamma S\Gamma B}$ for every h-bi-ideal B of S.
(iii) $Q = \overline{Q\Gamma S\Gamma Q}$ for every h-quasi-ideal Q of S.

Proof. (i)$\Rightarrow$(ii) Assume that (i) holds. Let B be any h-bi-ideal of S and x be any element of B. Then there exist a,b,z$\in$ S and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x+x\alpha a\beta x+z = x\gamma b\delta x+z$. Then it is easy to see that $x\alpha a\beta x, x\gamma b\delta x \in B\Gamma S\Gamma B$ and so $x \in \overline{B\Gamma S\Gamma B}$. Hence $B \subseteq \overline{B\Gamma S\Gamma B}$. On the other hand, since B is an h-bi-ideal of S, we have $B\Gamma S\Gamma B \subseteq B$ and so $\overline{B\Gamma S\Gamma B} \subseteq \overline{B} = B$. Therefore $B = \overline{B\Gamma S\Gamma B}$.
(ii)$\Rightarrow$(iii) This is straightforward.
(iii)$\Rightarrow$(i) Assume that (iii) holds. Let R and L be any right and left h-ideal of S respectively. Then we have $\overline{(R \cap L)\Gamma S} \cap \overline{S\Gamma(R \cap L)} \subseteq \overline{R\Gamma S} \cap \overline{S\Gamma L} \subseteq \overline{R} \cap \overline{L} = R \cap L$ and thus $R\cap L$ is a h-quasi-ideal of S. Also, $R\cap L = \overline{(R \cap L)\Gamma S\Gamma(R \cap L)} \subseteq \overline{R\Gamma S\Gamma L} \subseteq \overline{R\Gamma L} \subseteq \overline{R \cap L} \subseteq \overline{R} \cap \overline{L} = R \cap L$. Therefore $\overline{R\Gamma L} = R \cap L$. Hence from Lemma 4.3 S is h-hemiregular. □

Now we obtain the following characterizations of h-hemiregular Γ-hemirings.

Theorem 4.7. *Let S be a Γ-hemiring. Then the following conditions are equivalent.*
(i) S is h-hemireglar.
(ii) $\mu \subseteq \mu o_h \chi_S o_h \mu$ for every fuzzy h-bi-ideal μ of S.
(iii) $\mu \subseteq \mu o_h \chi_S o_h \mu$ for every fuzzy h-quasi-ideal μ of S.

Proof. (i)⇒(ii) Assume that (i) holds. Let μ be any fuzzy h-bi-ideal of S and x be any element of S. Since S is h-hemiregular there exist a,b,z∈ S and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$.

$(\mu o_h \chi_S o_h \mu)(x)$

$= \sup(\min\{(\mu o_h \chi_S)(a_i), (\mu o_h \chi_S)(c_i), \mu(b_i), \mu(d_i)\})$

$$x+ \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$\geq \min\{(\mu o_h \chi_S)(x\alpha a), (\mu o_h \chi_S)(x\gamma b), \mu(x)\}$

$= \min\{ \quad \sup(\min\{(\mu(a_i), \mu(c_i))\}) \quad , \quad \sup(\min\{(\mu(a_i), \mu(c_i))\}) \quad ,$

$$x\alpha a+ \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z \quad x\gamma b+ \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$\mu(x)\}$

$\geq \min\{\mu(x), \mu(x), \mu(x)\}$ (*since* $x\alpha a+x\alpha a\beta x\alpha a+z\alpha a = x\gamma b\delta x\alpha a+z\alpha a$ *and* $x\gamma b+ x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b$).

$= \mu(x)$

This implies that $\mu \subseteq \mu o_h \chi_S o_h \mu$.

(ii)⇒(iii) This is straight forward from Lemma 3.10

(iii)⇒(i) Assume that (iii) holds and Q be any h-quasi-ideal of S. Then the characteristic function χ_Q of Q is a fuzzy h-quasi-ideal of S. Now $\chi_Q \subseteq \chi_Q o_h \chi_S o_h \chi_Q = \chi_{\overline{Q\Gamma S\Gamma Q}}$. Therefore $Q \subseteq \overline{Q\Gamma S\Gamma Q}$. On the other hand, since Q is any h-quasi-ideal of S, we have $\overline{Q\Gamma S\Gamma Q} \subseteq \overline{S\Gamma Q} \cap \overline{Q\Gamma S} \subseteq Q$ and so $\overline{Q\Gamma S\Gamma Q} = Q$. Therefore S is h-hemiregular by Lemma 4.6. $\square$

Theorem 4.8. *Let S is a Γ-hemiring. Then the following conditions are equivalent.*

(i) S is h-hemiregular.

(ii) $\mu \cap \nu \subseteq \mu o_h \nu o_h \mu$ *for every fuzzy h-bi-ideal μ and every fuzzy h-ideal ν of S.*

(iii) $\mu \cap \nu \subseteq \mu o_h \nu o_h \mu$ *for every fuzzy h-quasi-ideal μ and every fuzzy h-ideal ν of S.*

Proof. (i)⇒(ii) Assume that (i) holds. Let μ and ν be any fuzzy h-bi-ideal and fuzzy h-ideal of S, respectively and x be any element of B. Since S is h-hemiregular, there exist a,b,z∈ S and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x+x\alpha a\beta x+z = x\gamma b\delta x + z$.

$(\mu o_h \nu o_h \mu)(x)$

$= \sup(\min\{(\mu o_h \nu)(a_i), (\mu o_h \nu)(c_i), \mu(b_i), \mu(d_i)\})$

$$x+ \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$\geq \min\{(\mu o_h \nu)(x\alpha a), (\mu o_h \nu)(x\gamma b), \mu(x)\}$

$$= \min\{\sup(\min\{(\mu(a_i), \mu(c_i), \nu(b_i), \nu(d_i))\}), \sup(\min\{(\mu(a_i), \mu(c_i), \nu(b_i), \nu(d_i))\}),$$

$$\underset{x\alpha a + \sum\limits_{i=1}^{n} a_i\gamma_i b_i + z = \sum\limits_{i=1}^{n} c_i\delta_i d_i + z}{} \quad \underset{x\gamma b + \sum\limits_{i=1}^{n} a_i\gamma_i b_i + z = \sum\limits_{i=1}^{n} c_i\delta_i d_i + z}{}$$

$$\mu(x)\}$$

$\geq \min\{\min\{\mu(x), \nu(a\beta x\alpha a), \nu(b\delta x\alpha a)\}, \min\{\mu(x), \nu(a\beta x\gamma b), \nu(b\delta x\gamma b)\}, \mu(x)\}$
(*since* $x\alpha a + x\alpha a\beta x\alpha a + z\alpha a = x\gamma b\delta x\alpha a + z\alpha a$ *and* $x\gamma b + x\alpha a\beta x\gamma b + z\gamma b = x\gamma b\delta x\gamma b + z\gamma b$)

$\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x)$.

(ii)$\Rightarrow$(iii) This is straight forward from Lemma 3.10.

(iii)$\Rightarrow$(i) Assume that (iii) holds. Let μ be any fuzzy h-quasi-ideal of S. Then since χ_S is a fuzzy h-ideal of S, we have $\mu = \mu \cap \chi_S \subseteq \mu o_h \chi_S o_h \mu$. Therefore S is h-hemiregular by Theorem 4.7. $\square$

Theorem 4.9. *Let S is a Γ-hemiring. Then the following conditions are equivalent.*

(i) S is h-hemiregular.

(ii) $\mu \cap \nu \subseteq \mu o_h \nu$ for every fuzzy h-bi-ideal μ and every fuzzy left h-ideal ν of S.

(iii) $\mu \cap \nu \subseteq \mu o_h \nu$ for every fuzzy h-quasi-ideal μ and every fuzzy left h-ideal ν of S.

(iv) $\mu \cap \nu \subseteq \mu o_h \nu$ for every fuzzy right h-ideal μ and every fuzzy h-bi-ideal ν of S.

(v) $\mu \cap \nu \subseteq \mu o_h \nu$ for every fuzzy right h-ideal μ and every fuzzy h-quasi-ideal ν of S.

(vi) $\mu \cap \nu \cap \omega \subseteq \mu o_h \nu o_h \omega$ for every fuzzy right h-ideal μ, for every fuzzy h-bi-ideal ν and for every fuzzy left h-ideal ω of S.

(vii) $\mu \cap \nu \cap \omega \subseteq \mu o_h \nu o_h \omega$ for every fuzzy right h-ideal μ, for every fuzzy h-quasi-ideal ν and for every fuzzy left h-ideal ω of S.

Proof. The proof follows by simple verification. $\square$

Proposition 4.10. *If a Γ-hemiring S is h-hemiregular then any right h-ideal R and left h-ideal L are idempotent and $\overline{R\Gamma L}$ is an h-quasi-ideal of S.*

Proof. Assume that S is h-hemiregular and R, L be any right and left h-ideal of S respectively. Now $\overline{R\Gamma R} \subseteq \overline{R} = R$. Let x$\in$R. Since S is h-hemiregular, there exist a,b$\in$S and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$. Since R is a right h-ideal of S $x\alpha a, x\gamma b \in$R and so $x\alpha a\beta x, x\gamma b\delta x \in R\Gamma R$. Hence $x \in \overline{R\Gamma R}$ that is $R \subseteq \overline{R\Gamma R}$. Thus $R = \overline{R\Gamma R}$ and so R is idempotent. Similarly we can show that L is idempotent. Now since S is h-hemiregular we have $R \cap L = \overline{R\Gamma L}$, and so $\overline{R\Gamma L}$ is an h-quasi-ideal of S. $\square$

Proposition 4.11. *If a Γ-hemiring S is h-hemiregular then any fuzzy right h-ideal μ and fuzzy left h-ideal ν are idempotent and $\mu o_h \nu$ is an quasi-ideal of S.*

Proof. Suppose S is h-hemiregular and μ be any fuzzy right h-ideal of S. Then $\mu o_h \mu \subseteq \mu o_h \chi_S \subseteq \mu$. Moreover, since S is h-hemiregular, $\mu \subseteq \mu o_h \mu$ and so $\mu = \mu o_h \mu$. Hence μ is idempotent. Similarly, we can prove for fuzzy left h-ideals.

Now since S is h-hemiregular $\mu o_h \nu = \mu \cap \nu$, which implies $\mu o_h \nu$ is a fuzzy h-quasi-ideal of S by Lemma 3.9. □

5   h-intra-hemiregularity

In this section we introduce the concept of h-intra-hemiregularity in Γ-hemiring and characterize some of its properties using h-ideal, h-bi-ideal and h-quasi-ideal.

Definition 5.1. *A Γ-hemiring S is said to be h-intra-hemiregular if for each $x \in S$, there exist $z, a_i, a_i', b_i, b_i' \in S$, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, $i \in \mathbf{N}$, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$.*

Example 5.2. *Let $S = \Gamma = \{0, a, b\}$ with addition(+) and multiplication(.) as follows:*

+	0	a	b
0	0	a	b
a	a	a	a
b	b	a	b

and

.	0	a	b
0	0	0	0
a	0	a	a
b	0	a	b

.

Then S is a Γ-hemiring where the ternary composition is defined as $x \alpha y = x.\alpha.y$. Here S is both h-hemiregular and h-intra-hemiregular.

Lemma 5.3. *Let S be a Γ-hemiring. Then the following conditions are equivalent*
(i) S is h-intra-hemiregular.
(ii) $L \cap R \subseteq \overline{L\Gamma R}$ for every left h-ideal L and every right h-ideal R of S.

Proof. (i)$\Rightarrow$(ii). Assume that (i) holds. Let L and R be any left and right h-ideal of S respectively. Since S is h-intra-hemiregular, we have $L \cap R \subseteq \overline{S\Gamma(L \cap R)\Gamma(L \cap R)\Gamma S} \subseteq \overline{S\Gamma L\Gamma R\Gamma S} \subseteq \overline{L\Gamma R}$.
(ii)$\Rightarrow$(i). Assume that (ii) holds. Let $x \in S$. Then $\overline{S\Gamma x + Nx}$ and $\overline{x\Gamma S + xN}$ where $N = \{0,1,2...\}$ are the principal left h-ideal and principal right h-ideal of S generated by x, respectively. Then $x = 0\gamma x + 1x = x\gamma 0 + x1 \in \overline{S\Gamma x + Nx} \cap \overline{x\Gamma S + xN} \subseteq \overline{(S\Gamma x + Nx)\Gamma(x\Gamma S + xN)}$
$= \overline{S\Gamma x\Gamma x\Gamma S + S\Gamma x\Gamma xN + Nx\Gamma x\Gamma S + Nx\Gamma xN}$.
Thus we have $x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i' + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i' + z$, for some $z, a_i, a_i', b_i, b_i' \in S$, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, $i \in \mathbf{N}$, the set of natural numbers. This implies S is h-intra-hemiregular. □

Theorem 5.4. *Let S be a Γ-hemiring. Then S is h-intra-hemiregular if and only if $\mu \cap \nu \subseteq \mu o_h \nu$ for every fuzzy left h-ideal μ and every fuzzy right h-ideal ν of S.*

Proof. Suppose S is h-intra-hemiregular. Let μ and ν be any fuzzy left h-ideal and fuzzy right h-ideal of S respectively. Now let x∈S. Then by hypothesis there exist $z, a_i, a_i^{'}, b_i, b_i^{'}$ ∈S, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, i∈ **N**, the set of natural numbers, such that $x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i^{'} + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i^{'} + z$. Therefore

$$
\begin{aligned}
(\mu o_h \nu)(x) \quad &= \sup_{i}[\min\{\min\{\mu(a_i), \mu(c_i), \theta(b_i), \theta(d_i)\}\}] \\
& \quad\quad x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z \\
&\geq \min_{i}[\min\{\mu(a_i \alpha_i x), \mu(b_i \gamma_i x), \nu(x \beta_i a_i^{'}), \nu(x \delta_i b_i^{'})\}] \\
&\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x).
\end{aligned}
$$

Conversely, Suppose the condition holds. Let L and R be any left h-ideal and right h-ideal of S, respectively. Then the characteristic function χ_L and χ_R are fuzzy left h-ideal and fuzzy right h-ideal of S, respectively. So, from Lemma 2.14 we obtain $\chi_{L \cap R} = \chi_L \cap \chi_R \subseteq \chi_L o_h \chi_R = \chi_{\overline{L\Gamma R}}$. From which it follows that $L \cap R \subseteq \overline{L\Gamma R}$, whence S is h-intra-hemiregular by Lemma 5.3. □

Theorem 5.5. *Let S be a Γ-hemiring and x∈S. Then S is h-intra-hemiregular if and only if $\mu(x) = \mu(x\gamma x)$, for all fuzzy h-ideal(resp. fuzzy h-interior ideal) μ of S and for all $\gamma \in \Gamma$.*

Proof. Suppose S is h-intra-hemiregular Γ-hemiring and μ be any fuzzy h-ideal of S. Let x∈S. Then by hypothesis there exist $z, a_i, a_i^{'}, b_i, b_i^{'}$ ∈S, and $\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, i∈ **N**, the set of natural numbers, such that
$x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i^{'} + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i^{'} + z.$
Then $\mu(x) \geq \min_{i}[\min\{\mu(a_i \alpha_i x \eta x \beta_i a_i^{'}), \mu(b_i \gamma_i x \eta x \delta_i b_i^{'})\}] \geq \mu(x \eta x) \geq \mu(x).$
Therefore $\mu(x) = \mu(x \gamma x)$.
Conversely, suppose the condition holds.
Now $\overline{M x \eta x + S\Gamma x \eta x + x \eta x \Gamma S + S\Gamma x \eta x \Gamma S}$, where M={0,1,2... } is the principal h-ideal of S generated by $x \eta x$. So, $\chi_{\overline{M x \eta x + S\Gamma x \eta x + x \eta x \Gamma S + S\Gamma x \eta x \Gamma S}}$, the characteristic function of $\overline{M x \eta x + S\Gamma x \eta x + x \eta x \Gamma S + S\Gamma x \eta x \Gamma S}$, is a fuzzy h-ideal of S. Since $x \eta x = 1 x \eta x + 0_S \Gamma x \eta x \in \overline{M x \eta x + S\Gamma x \eta x + x \eta x \Gamma S + S\Gamma x \eta x \Gamma S}$, we have $\chi_{\overline{M x \eta x + S\Gamma x \eta x + x \eta x \Gamma S + S\Gamma x \eta x \Gamma S}}(x) = \chi_{\overline{M x \eta x + S\Gamma x \eta x + x \eta x \Gamma S + S\Gamma x \eta x \Gamma S}}(x \eta x) = 1.$
Hence $x \in \overline{M x \eta x + S\Gamma x \eta x + x \eta x \Gamma S + S\Gamma x \eta x \Gamma S}$. Thus we have
$x + \sum_{i=1}^{n} a_i \alpha_i x \eta x \beta_i a_i^{'} + z = \sum_{i=1}^{n} b_i \gamma_i x \eta x \delta b_i^{'} + z$, for some $z, a_i, a_i^{'}, b_i, b_i^{'}$ ∈S, and

$\alpha_i, \beta_i, \gamma_i, \delta_i, \eta \in \Gamma$, i$\in$ **N**, the set of natural numbers. Therefore S is h-intra-hemiregular. $\square$

Lemma 5.6. *Let S be a Γ-hemiring. Then the following conditions are equivalent.*
(i) S is both h-hemiregular and h-intra-hemiregular.
(ii) $B=\overline{B\Gamma B}$ for every h-bi-ideal B of S.
(iii) $Q=\overline{Q\Gamma Q}$ for every h-quasi-ideal B of S.

Proof. (i)$\Rightarrow$(ii) Assume that S is both h-hemiregular and h-intra-hemiregular. Let x$\in$B. Then x can be expressed as

$$x+\sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x\eta x\beta_i' b_i \beta_i x+z = \sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x\eta x\delta_i' d_i \delta_i x+z \text{ where } z, a_i, b_i, c_i, d_i \in \text{S}$$

and $\alpha_i, \beta_i, \alpha_i', \beta_i', \gamma_i, \delta_i, \gamma_i', \delta_i', \eta \in \Gamma$. Since B is a bi-ideal $\sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x\eta x\beta_i' b_i \beta_i x$,

$\sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x\eta x\delta_i' d_i \delta_i x \in B\Gamma B$, which implies $x \in \overline{B\Gamma B}$ that is B$\subseteq \overline{B\Gamma B}$. Since
B is a h-bi-ideal we have $\overline{B\Gamma B} \subseteq B$. Therefore $B = \overline{B\Gamma B}$
(ii)$\Rightarrow$(iii) follows easily.
(iii)$\Rightarrow$(i) Suppose L and R be left and right h-ideal of S respectively. Then $L\cap R$ is a h-quasi-ideal of S. So, $L \cap R = \overline{(L \cap R)\Gamma(L \cap R)} \subseteq \overline{R\Gamma L} \subseteq \overline{R} \cap \overline{L} = R \cap L$ and $L \cap R = \overline{(L \cap R)\Gamma(L \cap R)} \subseteq \overline{L\Gamma R}$. Therefore S is both h-hemiregular and h-intra-hemiregular. $\square$

Theorem 5.7. *Let S be a Γ-hemiring. Then the following conditions are equivalent.*
(i) S is both h-hemiregular and h-intra-hemiregular.
(ii) $\mu = \mu o_h \mu$ for every h-bi-ideal μ of S.
(iii) $\mu = \mu o_h \mu$ for every h-quasi-ideal μ of S.

Proof. (i)$\Rightarrow$(ii) Assume that (i) holds. Let x$\in$ S and μ be any fuzzy h-bi-ideal of S. Since S is both h-hemiregular and h-intra-hemiregular there exist $z, a_i, b_i, c_i, d_i \in$ S and $\alpha_i, \beta_i, \alpha_i', \beta_i', \gamma_i, \delta_i, \gamma_i', \delta_i', \eta \in \Gamma$, i$\in$ **N** such that

$$x + \sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x\eta x\beta_i' b_i \beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x\eta x\delta_i' d_i \delta_i x + z. \text{ Therefore}$$

$(\mu o_h \mu)(x)$
$= \sup[\min_i \{\min\{\mu(a_i), \mu(c_i), \mu(b_i), \mu(d_i)\}\}]$

$$x + \sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$$\geq \min_i[\min\{\mu(x\alpha_i a_i \alpha_i' x), \mu(x\beta_i' b_i \beta_i x), \mu(x\gamma_i c_i \gamma_i' x), \mu(x\delta_i' d_i \delta_i x)\}]$$

$$x+\sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x\eta x\beta_i' b_i \beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x\eta x\delta_i' d_i \delta_i x + z$$

$$\geq \mu(x).$$

Now $\mu o_h \mu \subseteq \mu o_h \chi_S \subseteq \mu$. Hence $\mu o_h \mu = \mu$ for every fuzzy h-bi-ideal μ of S.

(ii)$\Rightarrow$(iii) This is straightforward from the Lemma 3.10.

(iii)$\Rightarrow$(i) Assume that (iii) holds. Let Q be any h-quasi-ideal of S. Then the characteristic function χ_Q of Q is a fuzzy h-quasi ideal of S. Now $\chi_Q = \chi_Q o_h \chi_Q = \chi_{\overline{Q\Gamma Q}}$, which implies Q=$\overline{Q\Gamma Q}$. Hence S is both h-hemiregular and h-intra-hemiregular by Lemma 5.6. $\square$

Theorem 5.8. *Let S be a Γ-hemiring. Then the following conditions are equivalent.*

(i) S is both h-hemiregular and h-intra-hemiregular.

(ii) $\mu \cap \nu \subseteq \mu o_h \nu$ for all fuzzy h-bi-ideals μ and ν of S.

(iii) $\mu \cap \nu \subseteq \mu o_h \nu$ for every fuzzy h-bi-ideals μ and every fuzzy h-quasi-ideal ν of S.

(iv) $\mu \cap \nu \subseteq \mu o_h \nu$ for every fuzzy h-quasi-ideals μ and every fuzzy h-bi-ideal ν of S.

(v) $\mu \cap \nu \subseteq \mu o_h \nu$ for all fuzzy h-quasi-ideals μ and ν of S.

Proof. (i)$\Rightarrow$(ii) Assume that (i) holds. Let x$\in$ S and μ and ν be any fuzzy h-bi-ideals of S. Since S is both h-hemiregular and h-intra-hemiregular there exist there exist z,$a_i, b_i, c_i, d_i \in$ S and $\alpha_i, \beta_i, \alpha_i', \beta_i', \gamma_i, \delta_i, \gamma_i', \delta_i' \in \Gamma$ i$\in$ **N** such that $x+\sum_{i=1}^{n} x\alpha_i a_i \alpha_i' x\eta x\beta_i' b_i \beta_i x + z = \sum_{i=1}^{n} x\gamma_i c_i \gamma_i' x\eta x\delta_i' d_i \delta_i x + z$. Therefore

$(\mu o_h \nu)(x)$

$= \sup_i[\min\{\min\{\mu(a_i), \mu(c_i), \nu(b_i), \nu(d_i)\}\}]$

$$x+\sum_{i=1}^{n} a_i \gamma_i b_i + z = \sum_{i=1}^{n} c_i \delta_i d_i + z$$

$\geq \min_i[\min\{\mu(x\alpha_i a_i \alpha_i' x), \mu(x\gamma_i c_i \gamma_i' x), \nu(x\beta_i' b_i \beta_i x), \nu(x\delta_i' d_i \delta_i x)\}]$

$\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x)$, which implies $\mu \cap \nu \subseteq \mu o_h \nu$.

(ii)$\Rightarrow$(iii)$\Rightarrow$(v) and (ii)$\Rightarrow$(iv)$\Rightarrow$(v) are obvious from Lemma 3.10.

Now for (v)$\Rightarrow$(i), suppose Q be any h-quasi-ideal of S. Then the characteristic function χ_Q of Q is a fuzzy h-quasi ideal of S. Now $\chi_Q = \chi_Q \cap \chi_Q \subseteq \chi_Q o_h \chi_Q = \chi_{\overline{Q\Gamma Q}}$, which implies Q$\subseteq \overline{Q\Gamma Q}$. Since the converse inclusion always holds; Therefore Q= $\overline{Q\Gamma Q}$. Hence S is both h-hemiregular and h-intra-hemiregular, by Lemma 5.6. $\square$

6   h-quasi-hemiregularity

Definition 6.1. *A Γ-hemiring S is called (left, right) h-quasi hemiregular if every (left, right) h-ideal of S is idempotent.*
Then it is easily seen that, a Γ-hemiring S is left(right) h-quasi-hemiregular if and only if $a \in \overline{S\Gamma a\Gamma S\Gamma a}(resp.\ a \in \overline{a\Gamma S\Gamma a\Gamma S})$.

Theorem 6.2. *A Γ-hemiring S is left(right) h-quasi-hemiregular if and only if every fuzzy left(right) h-ideal is idempotent.*

Proof.   Let S be a left h-quasi-hemiregular Γ-hemiring and μ be any fuzzy left h-ideal of S. Let $a \in$ S. Since S is left h-quasi-hemiregular $a \in \overline{(S\Gamma a)\Gamma(S\Gamma a)}$. So, there exist $z, x_i, y_i \in$ S and $\alpha_i, \beta_i, \gamma_i \in \Gamma$, i=1,2 such that
$a + (x_1\alpha_1 a)\beta_1(y_1\gamma_1 a) + z = (x_2\alpha_2 a)\beta_2(y_2\gamma_2 a) + z$. Therefore

$$(\mu o_h \mu)(a) \ = \sup_{a+\sum\limits_{i=1}^{n} a_i\gamma_i b_i + z = \sum\limits_{i=1}^{n} c_i\delta_i d_i + z}[\min_i\{\min\{\mu(a_i), \mu(c_i), \mu(b_i), \mu(d_i)\}\}]$$

$$\geq \min_{a+(x_1\alpha_1 a)\beta_1(y_1\gamma_1 a)+z=(x_2\alpha_2 a)\beta_2(y_2\gamma_2 a)+z}\{\mu(x_1\alpha_1 a), \mu(x_2\alpha_2 a), \mu(y_1\gamma_1 a), \mu(y_2\gamma_2 a)\}$$

$$\geq \min\{\mu(a), \mu(a), \mu(a), \mu(a)\} = \mu(a).$$

Hence $\mu \subseteq \mu o_h \mu$. Since μ is a fuzzy left h-ideal of S, $\mu o_h \mu \subseteq \mu$. Therefore $\mu = \mu o_h \mu$.
Conversely, assume that every fuzzy left h-ideal is idempotent. Let $a \in$ S. Then $\overline{< a|}$ is a left h-ideal of S. So, $\chi_{\overline{<a|}}$ is a fuzzy left h-ideal of S. Now, $\chi_{\overline{<a|}\Gamma\overline{<a|}}(a) = (\chi_{\overline{<a|}} o_h \chi_{\overline{<a|}})(a) = \chi_{\overline{<a|}}(a) = 1$. So, $a \in \overline{< a|\Gamma< a|} = \overline{S\Gamma a\Gamma S\Gamma a} \subseteq \overline{S\Gamma a\Gamma S\Gamma a} = \overline{S\Gamma a\Gamma S\Gamma a}$. Thus S is left h-quasi-hemiregular. □

Theorem 6.3. *Let S be a Γ-hemiring. Then the following conditions are equivalent*
(i) S is h-quasi-hemiregular
(ii) $\mu = (\mu o_h \chi_s)^2 \cap (\chi_s o_h \mu)^2$ for every fuzzy h-quasi-ideal μ of S.

Proof.   (i)⇒(ii)
Assume that (i) holds. Let μ be a fuzzy h-quasi-ideal of S. Since S is h-quasi-hemiregular, the fuzzy right h-ideal $(\mu o_h \chi_s)$ and the fuzzy left h-ideal $(\chi_S o_h \mu)$ are idempotent. Therefore $(\mu o_h \chi_s)^2 \cap (\chi_s o_h \mu)^2 = (\mu o_h \chi_s) \cap (\chi_s o_h \mu) \subseteq \mu$.
To prove the reverse inclusion, let $a \in$ S. Since S is left h-quasi-hemiregular $a \in \overline{(S\Gamma a)\Gamma(S\Gamma a)}$. So there exist $z, x_i, y_i \in$ S and $\alpha_i, \beta_i, \gamma_i \in \Gamma$, i=1,2 such that
$a + (x_1\alpha_1 a)\beta_1(y_1\gamma_1 a) + z = (x_2\alpha_2 a)\beta_2(y_2\gamma_2 a) + z....(1)$. Therefore
$(\chi_s o_h \mu)^2(a) = ((\chi_s o_h \mu)o_h(\chi_s o_h \mu))(a)$

$$= \sup_i [\min\{\min\{(\chi_s o_h \mu)(a_i), (\chi_s o_h \mu)(c_i), (\chi_s o_h \mu)(b_i), (\chi_s o_h \mu)(d_i)\}\}]$$

$$a + \sum_{i=1}^n a_i \gamma_i b_i + z = \sum_{i=1}^n c_i \delta_i d_i + z$$

$$\geq \min\{(\chi_s o_h \mu)(x_1\alpha_1 a), (\chi_s o_h \mu)(x_2\alpha_2 a), (\chi_s o_h \mu)(y_1\gamma_1 a), (\chi_s o_h \mu)(y_2\gamma_2 a)\}....(2)$$
$$a+(x_1\alpha_1 a)\beta_1(y_1\gamma_1 a)+z=(x_2\alpha_2 a)\beta_2(y_2\gamma_2 a)+z$$

Now from (1) we have, $x_1\alpha_1 a + (x_1\alpha_1 x_1\alpha_1 a\beta_1 y_1)\gamma_1 a + x_1\alpha_1 z = (x_1\alpha_1 x_2\alpha_2 a\beta_2 y_2)\gamma_2 a + x_1\alpha_1 z$, $x_2\alpha_2 a + (x_2\alpha_2 x_1\alpha_1 a\beta_1 y_1)\gamma_1 a + x_2\alpha_2 z = (x_2\alpha_2 x_2\alpha_2 a\beta_2 y_2)\gamma_2 a + x_2\alpha_2 z$, $y_1\gamma_1 a + (y_1\gamma_1 x_1\alpha_1 a\beta_1 y_1)\gamma_1 a + y_1\gamma_1 z = (y_1\gamma_1 x_2\alpha_2 a\beta_2 y_2)\gamma_2 a + y_1\gamma_1 z$ and $y_2\gamma_2 a + (y_2\gamma_2 x_1\alpha_1 a\beta_1 y_1)\gamma_1 a + y_2\gamma_2 z = (y_2\gamma_2 x_2\alpha_2 a\beta_2 y_2)\gamma_2 a + y_2\gamma_2 z$. So from (2) we have $(\chi_s o_h \mu)^2 \geq \min\{\chi_s(x_1\alpha_1 x_1\alpha_1 a\beta_1 y_1), \chi_s(x_1\alpha_1 x_2\alpha_2 a\beta_2 y_2), \chi_s(x_2\alpha_2 x_1\alpha_1 a\beta_1 y_1), \chi_s(x_2\alpha_2 x_2\alpha_2 a\beta_2 y_2), \chi_s(y_1\gamma_1 x_1\alpha_1 a\beta_1 y_1), \chi_s(y_1\gamma_1 x_2\alpha_2 a\beta_2 y_2), \chi_s(y_2\gamma_2 x_1\alpha_1 a\beta_1 y_1), \chi_s(y_2\gamma_2 x_2\alpha_2 a\beta_2 y_2), \mu(a), \mu(a), \mu(a), \mu(a), \mu(a), \mu(a), \mu(a), \mu(a)\} = \mu(a)$.
Therefore $(\chi_s o_h \mu)^2 \supseteq \mu$. Similarly we obtain $(\mu o_h \chi_S)^2 \supseteq \mu$.
So $(\mu o_h \chi_s)^2 \cap (\chi_s o_h \mu)^2 \supseteq \mu$. Hence $(\mu o_h \chi_s)^2 \cap (\chi_s o_h \mu)^2 = \mu$.
(ii)$\Rightarrow$(i)
Conversely, assume that (ii) holds. Let μ be any fuzzy right h-ideal of S. Then since μ is a fuzzy h-quasi-ideal of S, we have
$\mu = (\mu o_h \chi_s)^2 \cap (\chi_s o_h \mu)^2 \subseteq (\mu o_h \chi_s)^2 \subseteq \mu o_h \mu \subseteq \mu o_h \chi_s \subseteq \mu$.
So, $\mu = \mu o_h \mu$. Therefore it follows from Theorem 6.2 that S is right h-quasi-hemiregular. Similarly, it can be prove that S is left h-quasi-hemiregular. Thus (ii)$\Rightarrow$(i). $\square$

Theorem 6.4. *Let S be a Γ-hemiring. Then the following conditions are equivalent*
(i) S is both h-intra-hemiregular and left h-quasi-hemiregular
(ii) $\mu \cap \nu \cap \omega \subseteq \mu o_h \nu o_h \omega$, for every fuzzy h-bi-ideal ω, every fuzzy left h-ideal μ and every right h-ideal ν of S.
(ii) $\mu \cap \nu \cap \omega \subseteq \mu o_h \nu o_h \omega$, for every fuzzy h-quasi-ideal ω, every fuzzy left h-ideal μ and every right h-ideal ν of S.

Proof. (i)$\Rightarrow$(ii)
Suppose (i) holds. Let μ, ν, ω be fuzzy left h-ideal, fuzzy right h-ideal and fuzzy h-bi-ideal respectively. Since S is h-intra-hemiregular and left h-quasi-hemiregular, any element a$\in$S can be expressed as $a + (x_1\alpha_1 a)\beta_1(a\eta a)\delta_1(y_1\gamma_1 a) + z = (x_2\alpha_2 a)\beta_2(a\eta a)\delta_2(y_2\gamma_2 a) + z....(1)$ for $z, x_i, y_i \in$ S and $\eta, \alpha_i, \beta_i, \gamma_i, \delta_i \in \Gamma$, i=1,2. Therefore
$(\mu o_h \nu o_h \omega)(a)$
$$= \sup_i [\min\{\min\{\mu(a_i), \mu(c_i), (\nu o_h \omega)(b_i), (\nu o_h \omega)(d_i)\}\}]$$

$$a + \sum_{i=1}^n a_i \gamma_i b_i + z = \sum_{i=1}^n c_i \delta_i d_i + z$$

$$= \min\{\mu(x_1\alpha_1 a\beta_1 a), \mu(x_2\alpha_2 a\beta_2 a), (\nu o_h \omega)(a\delta_1 y_1\gamma_1 a)(\nu o_h \omega)(a\delta_2 y_2\gamma_2 a))\}$$
$$a+(x_1\alpha_1 a)\beta_1(a\eta a)\delta_1(y_1\gamma_1 a)+z=(x_2\alpha_2 a)\beta_2(a\eta a)\delta_i(y_2\gamma_2 a)+z$$

$\geq \min\{\mu(a), \mu(a), (\nu o_h \omega)(a\delta_1 y_1 \gamma_1 a)(\nu o_h \omega)(a\delta_2 y_2 \gamma_2 a))\}...(2)$

Now, from (1) we have

$a\delta_1 y_1 \gamma_1 a + a\delta_1 y_1 \gamma_1 x_1 \alpha_1 a\beta_1 a\eta a\delta_1 y_1 \gamma_1 a + a\delta_1 y_1 \gamma_1 z = a\delta_1 y_1 \gamma_1 x_2 \alpha_2 a\beta_2 a\eta a\delta_2 y_2 \gamma_2 a + a\delta_1 y_1 \gamma_1 z$ and

$a\delta_2 y_2 \gamma_2 a + a\delta_2 y_2 \gamma_2 x_1 \alpha_1 a\beta_1 a\eta a\delta_1 y_1 \gamma_1 a + a\delta_2 y_2 \gamma_2 z = a\delta_2 y_2 \gamma_2 x_2 \alpha_2 a\beta_2 a\eta a\delta_2 y_2 \gamma_2 a + a\delta_2 y_2 \gamma_2 z$.

So, from (2), we have

$(\mu o_h \nu o_h \omega)$

$\geq \min\{\mu(a), \mu(a), \nu(a\delta_1 y_1 \gamma_1 x_1 \alpha_1 a\beta_1 a), \nu(a\delta_1 y_1 \gamma_1 x_2 \alpha_2 a\beta_2 a), \omega(a\delta_1 y_1 \gamma_1 a),$
$\qquad \omega(a\delta_2 y_2 \gamma_2 a), \nu(a\delta_2 y_2 \gamma_2 x_1 \alpha_1 a\beta_1 a), \nu(a\delta_2 y_2 \gamma_2 x_2 \alpha_2 a\beta_2 a)\}$

$\geq \min\{\mu(a), \nu(a), \omega(a)\} = (\mu \cap \nu \cap \omega)(a)$.

(ii)$\Rightarrow$(iii), follows from Lemma 3.10.

(iii)$\Rightarrow$(i)

Let μ Be any fuzzy left h-ideal of S. Then $\mu = \mu \cap \chi_S \cap \mu \subseteq \mu o_h \chi_S o_h \mu \subseteq \mu o_h \mu \subseteq \mu o_h \chi_S \subseteq \mu$. Therefore from Theorem 6.2, S is left h-quasi-hemiregular.

Now suppose ν be a fuzzy right h-ideal of S. Then $\mu \cap \nu = \mu \cap \nu \cap \chi_S \subseteq \mu o_h \nu o_h \chi_S \subseteq \mu o_h \nu$. Hence from Theorem 5.4, we deduce that S is h-intra-hemiregular. □

7 Fuzzy h-Duo Γ-hemiring

A Γ-hemiring S is called h-duo if every one-sided h-ideal of it is a h-ideal of S. A Γ-hemiring S is called fuzzy h-duo if every one-sided fuzzy h-ideal of S is a fuzzy h-ideal of S. A Γ-hemiring S is called h-hemiregular fuzzy h-duo if it is both h-hemiregular and fuzzy h-duo.

For simplicity in what follows we call h-duo Γ-hemiring a duo Γ-hemiring.

Lemma 7.1. *Let S be a Γ-hemiring. Then the following conditions are equivalent.*
(i) S is h-hemiregular duo Γ-hemiring.
(ii) $L \cap R = \overline{L\Gamma R}$ for every left h-ideal L and every right h-ideal R.

Proof. (i)$\Rightarrow$(ii) Assume that (i) holds. Let L and R be any left h-ideal and any right h-ideal of S respectively. Then since S is duo Γ-hemiring both L and R are h-ideals of S. Moreover since S is h-hemiregular, it follows from Lemma 4.3 that $L \cap R = \overline{L\Gamma R}$.

Conversely, assume that (ii) holds. Let L and R be any left h-ideal and any right h-ideal of S respectively. Since S itself is an h-ideal of S, we have $L = L \cap S = \overline{L\Gamma S} \supseteq L\Gamma S$ and $R = S \cap R = \overline{S\Gamma R} \supseteq S\Gamma R$. Hence both L and R are h-ideals of S and so S is duo. Now for every right h-ideal R and left h-ideal L we have $R \cap L = \overline{R\Gamma L}$ whence S is h-hemiregular. □

Lemma 7.2. *Let S be a Γ-hemiring. Then S is h-hemiregular duo Γ-hemiring if and only if $A \cap Q = \overline{A\Gamma Q\Gamma A}$ for every h-ideal A and every quasi-ideal Q of S.*

Proof. Suppose S be a Γ-hemiring. Assume that S is h-hemiregular duo Γ-hemiring. Let A and Q be any left h-ideal and any h-quasi-ideal of S, respectively. Then $\overline{A\Gamma Q\Gamma A} \subseteq \overline{A} = A$ and $\overline{A\Gamma Q\Gamma A} \subseteq \overline{A\Gamma Q \cap Q\Gamma A} \subseteq \overline{A\Gamma Q \cap Q\Gamma A} \subseteq \overline{S\Gamma Q \cap Q\Gamma S} \subseteq Q$. Hence $\overline{A\Gamma Q\Gamma A} \subseteq A \cap Q$. Now let $x \in A \cap Q$. Since S is h-hemiregular, there exist a,b$\in$S and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$. Then we have $x\alpha a\beta x + x\alpha a\beta x\alpha a\beta x + z\alpha a\beta x = x\gamma b\delta x\alpha a\beta x + z\alpha a\beta x$, $x\gamma b\delta x + x\alpha a\beta x\gamma b\delta x + z\gamma b\delta x = x\gamma b\delta x\gamma b\delta x + z\gamma b\delta x$ and $x + x\alpha a\beta x + z + x\alpha a\beta x\alpha a\beta x + z\alpha a\beta x + x\alpha a\beta x\gamma b\delta x + z\gamma b\delta x = x\alpha a\beta x\alpha a\beta x + z\alpha a\beta x + x\alpha a\beta x\gamma b\delta x + z\gamma b\delta x + x\gamma b\delta x + z$. So, $x + x\gamma b\delta x\alpha a\beta x + x\alpha a\beta x\gamma b\delta x + z' = x\alpha a\beta x\alpha a\beta x + x\gamma b\delta x\gamma b\delta x + z'$ where $z' = z + z\alpha a\beta x + z\gamma b\delta x$. Since A is an h-ideal of S, we have $x\alpha a, x\gamma b, a\beta x, b\delta x \in A$ and in consequence $(x\gamma b\delta x\alpha a\beta x + x\alpha a\beta x\gamma b\delta x), (x\alpha a\beta x\alpha a\beta x + x\gamma b\delta x\gamma b\delta x) \in A\Gamma Q\Gamma A$ and so $x \in \overline{A\Gamma Q\Gamma A}$. Hence $\overline{A\Gamma Q\Gamma A} = A \cap Q$.
Conversely, suppose the condition holds. Let L and R be any left and right h-ideal of S, respectively. Then both L and R are h-quasi ideals of S. Since S itself is an h-ideal of S, we have $\overline{S\Gamma L\Gamma S} = S \cap L = L$ and $\overline{S\Gamma R\Gamma S} = S \cap R = R$. Now, $L\Gamma S = \overline{S\Gamma L\Gamma S\Gamma S} \subseteq \overline{S\Gamma L\Gamma S} = L$ and $S\Gamma R = \overline{S\Gamma S\Gamma R\Gamma S} \subseteq \overline{S\Gamma R\Gamma S} = R$. Hence both L and R are h-ideals of S, i.e. S is duo. Moreover by the assumption, we have $R \cap L = \overline{R\Gamma L\Gamma R} \subseteq \overline{R\Gamma L} \subseteq \overline{R} \cap \overline{L} = R \cap L$. Therefore S is h-hemiregular. $\qquad\square$

Lemma 7.3. *Let S be h-hemiregular Γ-hemiring. Then S is duo if and only if it is fuzzy duo.*

Proof. Assume that S is duo. Let μ be any fuzzy left h-ideal of S. Then since the set $S\Gamma x$ is a left h-ideal of S, it is an h-ideal by the assumption. Since S is h-hemiregular, we have $x\gamma y \in \overline{x\Gamma S\Gamma x\Gamma S} \subseteq \overline{x\Gamma S\Gamma x\Gamma S} \subseteq \overline{S\Gamma x}$ this implies that there exist a,b,z$\in$S and $\alpha, \beta \in \Gamma$ such that $x\gamma y + a\alpha x + z = b\beta x + z$. Since μ is a fuzzy left h-ideal of S, we have $\mu(x\gamma y) \geq \min\{\mu(a\alpha x), \mu(b\beta x)\} \geq \mu(x)$ and so μ is a fuzzy right h-ideal of S. It can be seen in a similar way that any fuzzy right h-ideal of S is a fuzzy h-ideal of S. Thus S is fuzzy duo.
Conversely, assume that S is fuzzy duo. Let L be any left h-ideal of S, then the characteristic function χ_L of L is a fuzzy left h-ideal of S. By the assumption χ_L is a fuzzy h-ideal of S and so L is a h-ideal of S. Thus S is duo. This completes the proof. $\qquad\square$

Lemma 7.4. *Let S be a h-hemiregular fuzzy duo Γ-hemiring. Then every fuzzy h-bi-ideal of S is a h-ideal of S.*

Proof. Let μ be any fuzzy h-bi-ideal of S and x,y$\in$S. Then since the set $S\Gamma x$ is a left h-ideal of S, it is an h-ideal of S. Now since S is h-hemiregular,

$x\gamma y \in \overline{x\Gamma S\Gamma x}\Gamma y \subseteq \overline{x\Gamma S\Gamma x}\Gamma S \subseteq \overline{x\Gamma S\Gamma x\Gamma S} \subseteq \overline{x\Gamma\overline{S\Gamma x}\Gamma S} \subseteq \overline{x\Gamma\overline{\overline{S\Gamma x}}} = \overline{x\Gamma S\Gamma x}$
this implies $x\gamma y + x\alpha a\beta x + z = x\eta b\delta x + z$ for a,b,z∈S and $\alpha, \beta, \eta, \delta \in \Gamma$. Since μ is fuzzy h-bi-ideal of S, we have $\mu(x\gamma y) \geq \min\{\mu(x\alpha a\beta x), \mu(x\eta b\delta x)\} \geq \mu(x)$ and so μ is a fuzzy right h-ideal of S. In a similar way it can be easily seen that μ is a fuzzy left h-ideal of S and so μ is a fuzzy h-ideal of S. $\qquad\square$

By a routine verification we can get the following result.

Theorem 7.5. *Let S be a Γ-hemiring. Then the following conditions are equivalent.*
(1) S is h-hemiregular duo Γ-hemiring.
(2) S is h-hemiregular fuzzy duo Γ-hemiring.
(3) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy h-bi-ideal μ and ν of S.
(4) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy h-bi-ideal μ and every fuzzy h-quasi-ideal ν of S.
(5) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy h-bi-ideal μ and every fuzzy right h-ideal ν of S.
(6) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy h-quasi-ideal μ and every fuzzy h-bi-ideal ν of S.
(7) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy h-quasi-ideal μ and ν of S.
(8) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy h-quasi-ideal μ and every fuzzy right h-ideal ν of S.
(9) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy left h-ideal μ and every fuzzy h-bi-ideal ν of S.
(10) $\mu \cap \nu = \mu o_h \nu$ for every fuzzy left h-ideal μ and every fuzzy right h-ideal ν of S.

Theorem 7.6. *Let S be a Γ-hemiring. Then the following conditions are equivalent.*
(i) S is h-hemiregular duo Γ-hemiring.
(ii) $\mu \cap \nu = \mu o_h \nu o_h \mu$ for every fuzzy h-ideal μ and every fuzzy h-bi-ideal ν of S.
(iii) $\mu \cap \nu = \mu o_h \nu o_h \mu$ for every fuzzy h-ideal μ and every fuzzy h-quasi-ideal ν of S.

Proof. (i)⇒(ii) Assume that (i) holds. Let μ be any fuzzy h-ideal and ν be any fuzzy h-bi-ideal of S. Then we have $\mu o_h \nu o_h \mu \subseteq (\mu o_h \chi_S) o_h \chi_S \subseteq \mu o_h \chi_S \subseteq \mu$. Also, $\mu o_h \nu o_h \mu \subseteq \chi_S o_h \nu o_h \chi_S \subseteq \nu$. Hence $\mu o_h \nu o_h \mu \subseteq \mu \cap \nu$.
Now let x be any element of S. Since S is h-hemiregular, there exist a,b∈S and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $x + x\alpha a\beta x + z = x\gamma b\delta x + z$. Then we have $x\alpha a\beta x + x\alpha a\beta a\alpha a\beta x + z\alpha a\beta x = x\gamma b\delta x\alpha a\beta x + z\alpha a\beta x$, $x\gamma b\delta x + x\alpha a\beta x\gamma b\delta x + z\gamma b\delta x = x\gamma b\delta x\gamma b\delta x + z\gamma b\delta x$ and $x + x\alpha a\beta x + z + x\alpha a\beta x\alpha a\beta x + z\alpha a\beta x + x\alpha a\beta x\gamma b\delta x + z\gamma b\delta x = x\alpha a\beta x\alpha a\beta x + z\alpha a\beta x + x\alpha a\beta x\gamma b\delta x + z\gamma b\delta x + x\gamma b\delta x + z$.

So, $x + x\gamma b\delta x\alpha a\beta x + x\alpha a\beta x\gamma b\delta x + z' = x\alpha a\beta x\alpha a\beta x + x\gamma b\delta x\gamma b\delta x + z'$ where $z' = z + z\alpha a\beta x + z\gamma b\delta x$. Hence we have

$(\mu o_h\nu o_h\mu)(x)$

$= \sup_{x+a\gamma b+z=c\delta d+z}[\min\{(\mu o_h\nu)(a), (\mu o_h\nu)(c), \mu(b), \mu(d)\}]$

$\geq \min\{(\mu o_h\nu)(x), \mu(a\beta x\alpha a\beta x + b\delta x\gamma b\delta x), \mu(b\delta x\alpha a\beta x + a\beta x\gamma b\delta x)\}$

$= \min\{\sup_{x+a\gamma b+z=c\delta d+z}[\min\{\mu(a), \mu(c), \nu(b), \nu(d)\}], \mu(b\delta x\alpha a\beta x + a\beta x\gamma b\delta x), \mu(a\delta x\alpha a\beta x + b\beta x\gamma b\delta x)\}$

$\geq \min\{\mu(x\alpha a), \mu(x\gamma b), \nu(x), \mu(b\delta x\alpha a\beta x + a\beta x\gamma b\delta x), \mu(a\delta x\alpha a\beta x + b\beta x\gamma b\delta x)\}$

$\geq \min\{\mu(x), \nu(x)\} = (\mu \cap \nu)(x)$ and so $(\mu \cap \nu) \subseteq \mu o_h\nu o_h\mu$.

Hence $(\mu \cap \nu) = \mu o_h\nu o_h\mu$.

(ii)$\Rightarrow$(iii) This is straightforward from Lemma 3.10.

(iii)$\Rightarrow$(i) Assume that (iii) holds. Let A and Q be any h-ideal and h-quasi-ideal of S, respectively. Then the characteristic function χ_A, χ_Q of A and Q respectively is a fuzzy h-ideal and fuzzy h-quasi-ideal of S. Now by the assumption we have $\chi_{A\cap Q} = \chi_A \cap \chi_Q = \chi_A o_h\chi_Q o_h\chi_A = \chi_{\overline{A\Gamma Q\Gamma A}}$ and so $A \cap Q = \overline{A\Gamma Q\Gamma A}$. Hence from Lemma 7.2 we have S is h-hemiregular duo Γ-hemiring. $\square$

8   Conclusion

Various relationships between fuzzy h-ideals of a $\Gamma-$hemiring and those of its operator hemirings[2] can be established which will be effective in extending the study of $\Gamma-$hemirings in terms of fuzzy subsets.

References

[1] T.K.Dutta, B.K. Biswas, *Ph.D. dissertation*, University of Calcutta, India

[2] T.K. Dutta, S.K. Sardar, On Operator Semiring of a $\Gamma-$semiring, *Southest Asian Bull of Mathematics, Springer-Vergal*, 26(2002), 203-213

[3] J.S.Golan, Semirings and their applications, *Kluwer Academic Publishers*,(1999)..

[4] M.Henriksen, Ideals in semirings with commutative addition, *Am.Math.Soc.Notices*, 6(1958), 321

[5] X.Huang, H.Li and Y.Yin, The h-hemiregular fuzzy duo hemirings, *Int. J. Fuzzy Systems*, 9(2)(2007), 105-109

[6] K.Iizuka, On the Jacobson radical of semiring, *Tohoku Math.J.*, 11(2)(1959), 409-421

[7] Y.B.Jun, M.A.Öztürk, S.Z.Song, On Fuzzy h-ideals in hemiring, *Information sciences*, 162(2004), 211-226.

[8] M.Kando, W.A.Dudek, On transfer principle in fuzzy theory, *Mathware Soft Comput.*, 12(2005), 41-55.

[9] D.R.La Torre, On h-ideals and k-ideals in hemirings, *Publ. Math. Debrecen*, 12(1965), 219-226.

[10] X.Ma and J.Zahn, Fuzzy h-ideals in h-hemiregular and h-semisimple Γ-hemirings, *Neural Comput and Applic*, 19(2010), 477-485

[11] S.K.Sardar and D.Mandal, On fuzzy h-ideal in Γ-hemiring, *Int.J.Pure.Appl.Math*, 56(3)(2009), 439-450

[12] M.M.K.Rao, Γ−semirings-1, *Southeast Asian Bull. of Math.*, 19(1995), 49-54

[13] A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.*, 35(1971), 512-517

[14] Y.Yin and H.Li, The characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings, *Information Sciences*, 178(2008), 3451-3464

[15] L.A.Zadeh, Fuzzy Sets, *Information and Control*, 8(1965), 338-353.

[16] J.Zhan and W.A.Dudek, Fuzzy h-ideals of hemirings, *Information sciences*, 177(2007), 876-886.