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Some Growth Properties of Entire Functions Represented by Vector Valued Dirichlet Series in Two Variables

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Abstract

In the present paper, we study the entire functions represented by vector valued Dirichlet series of several complex variables. The characterizations of their order and type have been obtained. For the sake of simplicity, we have considered the functions of two variables only.

Keywords: *Entire function, Dirichlet Series, Order, Type*.

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1 Introduction

Consider

$$f(s_1,s_2)=\sum_{m,n=1}^{\infty} a_{mn}e^{(s_1\lambda_m+s_2\mu_n)},\ (s_j=\sigma_j+it_j\ ,\ j=1,2) \tag{1}$$

where $a_{m,n}$'s belong to the Banach space $(E,\|\,.\,\|);0<\lambda_1<...<\lambda_m\to\infty$ as $\mathrm{m}\to\infty$, $0<\mu_1<...<\mu_n\to\infty$ as $\mathrm{n}\to\infty$,

and

$$\lim_{m+n\to\infty}\sup\frac{\log(m+n)}{\lambda_m+\mu_n}=D<+\infty\ . \tag{2}$$

Such a series is called a vector valued Dirichlet series in two complex variables. The concepts of order and type of an entire function (also for analytic function) represented by vector valued Dirichlet series of one complex variable were first introduced in 1983 by O.P.Juneja and B.L.Srivastava.They also obtained the coefficient characterizations of order and type. In this paper we have extended the results to the entire functions represented by vector valued Dirichlet series of several complex variables. For simplicity, we consider here functions of two variables only, though these results can easily be extended to functions of several complex variables.

Let $f(s_1,s_2)$ defined by (1) represent an entire function. We define

$$M(\sigma_1,\sigma_2)=\sup\{\|\ f(\sigma_1+it_1,\sigma_2+it_2)\|;\ -\infty<t_j<\infty,\ j=1,2\}$$

to be the maximum modulus of $f(s_1,s_2)$.Then $M(\sigma_1,\sigma_2)\to\infty$ as $\sigma_1,\sigma_2\to\infty$. We define the order $\rho\,(0\le\rho\le\infty)$ of $f(s_1,s_2)$ as follows.

Class A: An entire function $f(s_1,s_2)$ defined by vector valued Dirichlet series of finite order belongs to class A, if there exist positive constants K_1,γ_1,K_2 and γ_2 such that

(i) For any fixed value of $\sigma_2>0$, there exists a number $\sigma^{(1)}=\sigma^{(1)}(K_1,\gamma_1,\sigma_2)$ such that

$$M(\sigma_1,\sigma_2)<\exp\{K_1\exp(\sigma_1\gamma_1)\}\quad\text{for }\ \sigma_1\ge\sigma^{(1)}.$$

(ii) For any fixed value of $\sigma_1>0$, there exists a number $\sigma^{(2)}=\sigma^{(2)}(K_2,\gamma_2,\sigma_1)$ such that

$$M(\sigma_1,\sigma_2)<\exp\{K_2\exp(\sigma_2\gamma_2)\}\qquad\text{for }\sigma_2\ge\sigma^{(2)}.$$

Therefore, there exists a number $\sigma=\sigma(K_1,K_2,\gamma_1,\gamma_2)$ such that

$$M(\sigma_1,\sigma_2)<\exp\{K_1\exp(\sigma_1\gamma_1)+K_2\exp(\sigma_2\gamma_2)\}\quad\text{for }\ \sigma_1,\sigma_2\ge\sigma.$$

Definition 1.1 An entire function $f(s_1,s_2)$ defined by the vector valued Dirichlet series (1.1) has finite orders ρ_1 and ρ_2 with respect to s_1 and s_2 respectively if

(i) For any arbitrarily small $\varepsilon>0$, and any $\sigma_2>0$, there exists a number $\sigma^{(1)} = \sigma^{(1)}(\varepsilon,\sigma_2)$ such that

$$M(\sigma_1,\sigma_2)<\exp[\exp\{\sigma_1(\rho_1+\varepsilon)\}] \quad \text{for } \sigma_1\ge\sigma^{(1)}.$$

In addition, there exists at least one value of σ_2, say $\sigma_2{}^0(\varepsilon)$ and correspondingly arbitrary large values of $\sigma_1:\{\sigma_{1i}\}$ such that

$$M(\sigma_{1i},\sigma_2{}^0(\varepsilon))>\exp\exp\{\sigma_{1i}(\rho_1-\varepsilon)\}.$$

Hence, (i) is equivalent to

$$\limsup_{\sigma_2\to\infty}\left\{\limsup_{\sigma_1\to\infty}\frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_1}\right\}=\rho_1 \quad .$$

(ii) For any arbitrarily small $\varepsilon>0$, and any $\sigma_1>0$, there exists a number $\sigma^{(2)}=\sigma^{(2)}(\varepsilon,\sigma_1)$
such that

$$M(\sigma_1,\sigma_2)<\exp[\exp\{\sigma_2(\rho_2+\varepsilon)\}] \quad \text{for } \sigma_2\ge\sigma^{(2)}.$$

In addition, there exists at least one value of σ_1, say $\sigma_1{}^0(\varepsilon)$ and correspondingly arbitrarily large values of $\sigma_2:\{\sigma_{2j}\}$ such that

$$M(\sigma_1{}^0(\varepsilon),\sigma_{2j})>\exp\exp\{\sigma_{2j}(\rho_2-\varepsilon)\}.$$

So, (ii) is equivalent to

$$\limsup_{\sigma_1\to\infty}\left\{\limsup_{\sigma_2\to\infty}\frac{\log\log M(\sigma_1,\sigma_2)}{\sigma_2}\right\}=\rho_2 \quad .$$

Definition 1.2 An entire function $f(s_1,s_2)$ defined by vector valued Dirichlet series has a finite order (ρ_1,ρ_2) if

(i) $f(s_1,s_2)\in A$

(ii) $f(s_1,s_2)$ has finite orders ρ_1 and ρ_2 with respect to s_1 and s_2 respectively as above.

(iii) For $\varepsilon>0$, there exists a number $\sigma=\sigma(\varepsilon)$ such that

$$M(\sigma_1,\sigma_2)<\exp\left\{\exp\sigma_1(\rho_1+\varepsilon)+\exp\sigma_2(\rho_2+\varepsilon)\right\} \quad \text{for } \sigma_1,\sigma_2\ge\sigma.$$

Thus, we define the order of vector valued Dirichlet series as

$$\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log\log M(\sigma_1,\sigma_2)}{\log(e^{\sigma_1}+e^{\sigma_2})}=\rho \quad .$$

Similarly, if $0<\rho<\infty$, then the type $T(0\le T\le\infty)$ of $f(s_1,s_2)$ is defined as:

$$\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{\log M(\sigma_1,\sigma_2)}{(e^{\rho\sigma_1}+e^{\rho\sigma_2})}=T \quad .$$

2. Basic Results.

We now prove

Theorem 2.1 The necessary and sufficient condition for the series (1) satisfying the condition (2) to be entire is that

$$\lim_{m+n\to\infty}\sup\frac{\log(\|a_{m,n}\|)}{\lambda_m+\mu_n}=-\infty \ . \tag{3}$$

For proving our result, we need the following result:

Lemma 2.2. The following conditions are equivalent:

(i) $\lim\limits_{m+n\to\infty}\sup\dfrac{\log(m+n)}{\lambda_m+\mu_n}=D<+\infty$

(ii) $\lim\limits_{m\to\infty}\sup\dfrac{\log m}{\lambda_m}=D_1<+\infty$, $\lim\limits_{n\to\infty}\sup\dfrac{\log n}{\mu_n}=D_2<+\infty$,

(iii)There exists $\alpha, 0<\alpha<\infty$, such that the series $\sum\limits_{m,n=0}^{\infty}\exp[-\alpha(\lambda_m+\mu_n)]$ converges.

For the proof of this Lemma we refer to [2].

Proof of Theorem 2.1 We suppose that (1) defines an entire function.Then it converges absolutely for all (s_1,s_2) .Now take the points with coordinates (p,p) with $p>0$ .Then it follows that $\sum\limits_{m,n=0}^{\infty}\|a_{m,n}\|\exp(\lambda_m+\mu_n)p<\infty$

Therefore $\|a_{m,n}\|\exp(\lambda_m+\mu_n)p\le M(p,p)$

and $\lim\limits_{m+n\to\infty}\sup\dfrac{\log\|a_{m,n}\|}{\lambda_m+\mu_n}<-p$.

Since $p>0$ is arbitrary, the necessity part of the theorem is proved.

Conversely, let the given condition (3) be satisfied. It suffices to prove that (1) converges absolutely for all (s_1, s_2).Let us consider $(s_1, s_2) \in C^2$, $\sigma > 0$ such that $\operatorname{Re} s_1 < \sigma, \operatorname{Re} s_2 < \sigma.$ Then by (3) and for some $\delta > 0$,

$$\frac{\log \| a_{m,n} \|}{\lambda_m + \mu_n} < -\sigma - \delta \quad \text{for } m + n \geq N_0(\delta)$$

and so
$$\| a_{m,n} \| \exp\{(\lambda_m + \mu_n)\sigma\} < \exp(-\delta(\lambda_m + \mu_n)) \ . \tag{4}$$

But the double series $\sum_{m,n=0}^{\infty} \exp[-\delta(\lambda_m + \mu_n)]$ being convergent in view of Lemma 2.2, it follows from (4) that the series (1) converges absolutely for all (s_1, s_2), given by $\operatorname{Re} s_1 < \sigma$, $\operatorname{Re} s_2 < \sigma.$ Now, when the series (1) converges at (s_1, s_2) it also converges at $(s_1^{'}, s_2^{'})$ where $\operatorname{Re} s_1^{'} < \operatorname{Re} s_1$, $\operatorname{Re} s_2^{'} < \operatorname{Re} s_2$, hence the result follows.
Next we prove

Theorem 2.3 If

$$f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{mn} e^{(s_1 \lambda_m + s_2 \mu_n)}$$

is an entire function of order $(\rho_1, \rho_2), (0 < \rho_1, \rho_2 < \infty)$, then

$$\lim_{\sigma_1, \sigma_2 \to \infty} \sup \left\{ \frac{\log \log M(\sigma_1, \sigma_2)}{\sigma_1 \rho_1 + \sigma_2 \rho_2} \right\} = 1 \quad .$$

Proof. The proof of this theorem follows on the lines of the proof of Theorem1 in [2].

3 Main Results

Theorem 3.1 If $f(s_1, s_2)$ is an entire function of order $\rho \, (0 < \rho < \infty)$, then

$$\rho = \lim_{m,n \to \infty} \sup \frac{\log(\lambda_m^{\lambda_m} \mu_n^{\mu_n})}{\log \| a_{mn} \|^{-1}} \quad .$$

Proof. Let

$$\mu = \lim_{m,n\to\infty} \sup \frac{\log(\lambda_m^{\lambda_m}\mu_n^{\mu_n})}{\log \| a_{mn} \|^{-1}}$$

First we show that $\rho \geq \mu$. Let us assume that $\mu > 0$, for otherwise the result is trivially true.Then for a given $\varepsilon > 0$, we have two sequences $\{\lambda_{M_p}\}$ and $\{\mu_{N_q}\}$ with $M_p \to \infty$ as $p \to \infty$ and $N_q \to \infty$ as $q \to \infty$ such that

$$\log \| a_{mn} \| > -(\mu-\varepsilon)^{-1}(\lambda_m \log \lambda_m + \mu_n \log \mu_n) \qquad \text{for } m = M_p \text{ and } n = N_q.$$

Since the inequality:

$M(\sigma_1,\sigma_2) \geq \| a_{mn} \| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)$ holds for all σ_1,σ_2 and m,n ,it follows that for all σ_1, and σ_2 and $m = M_p$ and $n = N_q$

$$\log M(\sigma_1,\sigma_2) > \lambda_m\{\sigma_1 - (\mu-\varepsilon)^{-1}\log\lambda_m\} + \mu_n\{\sigma_2 - (\mu-\varepsilon)^{-1}\log\mu_n\}.$$

Taking

$$\sigma_{1,p} = (\mu-\varepsilon)^{-1}\log(e\lambda_{M_p}) \text{ and } \sigma_{2,q} = (\mu-\varepsilon)^{-1}\log(e\mu_{N_q})$$

in the above we find that

$$\log M(\sigma_1,\sigma_2) > \frac{\exp(\sigma_1(\mu-\varepsilon)) + \exp(\sigma_2(\mu-\varepsilon))}{e(\mu-\varepsilon)}$$

for $\sigma_1 = \sigma_{1,p}$ and $\sigma_2 = \sigma_{2,q}$. Proceeding to limits as $\sigma_1,\sigma_2 \to \infty$, we get $\rho \geq \mu$.

Further, to prove the converse part, we suppose that $\mu < \infty$, for otherwise the result is obviously true, so that

$$\| a_{mn} \| < \lambda_m^{-\lambda_m/(\mu+\varepsilon)} . \mu_n^{-\mu_n/(\mu+\varepsilon)} , \qquad m > m_0 \text{ and } n > n_0. \qquad (5)$$

Now,

$$M(\sigma_1,\sigma_2) \leq \left(\sum_{m=1}^{m_0}\sum_{n=1}^{n_0} + \sum_{m=m_0+1}^{\infty}\sum_{n=1}^{n_0} + \sum_{n=n_0+1}^{\infty}\sum_{m=1}^{m_0} + \sum_{m=m_0+1}^{\infty}\sum_{n=n_0+1}^{\infty} \right) \| a_{mn} \| \exp(\sigma_1\lambda_m + \sigma_2\mu_n)$$

$$= \sum_1 + \sum_2 + \sum_3 + \sum_4 . \qquad (6)$$

Now we estimate the four parts of (6). Clearly

$$\sum_1 = O(\exp(\sigma_1\lambda_{m_0} + \sigma_2\mu_{n_0})).$$

In view of (5), we get

$$\sum_4 \le \sum_{m>m_0}\sum_{n>n_0}\exp\{(\sigma_1\lambda_m+\sigma_2\mu_n)+\log(\lambda_m{}^{-\lambda_m/(\mu+\varepsilon)}.\mu_n{}^{-\mu_n/(\mu+\varepsilon)})\}$$
$$\le \sum_{m>m_0}\sum_{n>n_0}\exp\{\sigma_1\lambda_m+\sigma_2\mu_n-(\mu+\varepsilon)^{-1}(\lambda_m\log\lambda_m+\mu_n\log\mu_n)\}$$
$$\le \max[\exp(\sigma_1\lambda_m+\sigma_2\mu_n-(\mu+2\varepsilon)^{-1}(\lambda_m\log\lambda_m+\mu_n\log\mu_n)].$$
$$\sum_{m>m_0}\sum_{n>n_0}\exp\left(-\frac{\lambda_m\log\lambda_m+\mu_n\log\mu_n}{\varepsilon^{-1}(\mu+\varepsilon)(\mu+2\varepsilon)}\right).$$

The series on the right hand side of the above inequality is convergent and the maximum of the expression

$$\exp(\sigma_1\lambda_m+\sigma_2\mu_n-(\mu+2\varepsilon)^{-1}(\lambda_m\log\lambda_m+\mu_n\log\mu_n)$$

is attained at $\lambda_m=e^{-1}\exp(\sigma_1(\mu+2\varepsilon))$ and $\mu_n=e^{-1}\exp(\sigma_2(\mu+2\varepsilon))$, we find that

$$\sum_4 \;\le\; A\exp\{e^{-1}(\mu+2\varepsilon)^{-1}(e^{\sigma_1(\mu+2\varepsilon)}+e^{\sigma_2(\mu+2\varepsilon)})\}$$

where A is an absolute constant. Further, to estimate $\sum_2$, μ being finite, we have for all values of m and n

$$\frac{\log(\lambda_m^{\lambda_m}\mu_n^{\mu_n})}{\log\|a_{mn}\|^{-1}}\le\varsigma,$$

Therefore,

$$\sum_2\le\sum_{m>m_0}\sum_{n=1}^{n_0}\exp(\sigma_1\lambda_m+\sigma_2\mu_n-\varsigma^{-1}\lambda_m\log\lambda_m-\varsigma^{-1}\mu_n\log\mu_n)$$
$$=O(\exp(\sigma_2\mu_{n_0}))\sum_{m>m_0}\exp(\sigma_1\lambda_m-\varsigma^{-1}\lambda_m\log\lambda_m)$$
$$\le O(\exp(\sigma_2\mu_{n_0})\max[\exp(\sigma_1\lambda_m-(\varsigma+\varepsilon)^{-1}\lambda_m\log\lambda_m).\sum_{m>m_0}(-\varepsilon\varsigma^{-1}(\varsigma+\varepsilon)^{-1}\lambda_m\log\lambda_m)]$$

$$=O(\exp(\sigma_2\mu_{n_0})\max[\exp(\sigma_1\lambda_m-(\varsigma+\varepsilon)^{-1}\lambda_m\log\lambda_m)].$$

But the $\max.\exp(\sigma_1\lambda_m-(\varsigma+\varepsilon)^{-1}\lambda_m\log\lambda_m)$ is attained at

$$\lambda_m=e^{-1}\exp(\sigma_1(\varsigma+\varepsilon)).$$

Hence,

$$\sum_2 \;\le\; O(\exp(\sigma_2\mu_{n_0})\exp\{e^{-1}(\varsigma+\varepsilon)^{-1}\exp(\sigma_1(\varsigma+\varepsilon))\}.$$

Similarly

$$\sum_3 \;\le\; O(\exp(\sigma_1\lambda_{m_0})\exp\{e^{-1}(\varsigma+\varepsilon)^{-1}\exp(\sigma_2(\varsigma+\varepsilon))\}.$$

Substituting these estimates of $\sum_i, (1 \le i \le 4)$, in (6), we have

$$\log\log M(\sigma_1, \sigma_2) \le \log\{\exp(\sigma_1(\mu + 2\varepsilon) + \exp(\sigma_2(\mu + 2\varepsilon)\} + O(1),$$

which gives $\rho \le \mu$ on proceeding to limits as $\sigma_1, \sigma_2 \to \infty$. Hence the theorem follows.

Theorem 3.2. Let $f(s_1, s_2)$ be an entire function of order $\rho(0 < \rho < \infty)$ and type $T(0 \le T \le \infty)$. Then

$$\lim_{m,n\to\infty} \sup(\lambda_m^{\lambda_m} \mu_n^{\mu_n} . \| a_{mn} \|^{\rho})^{1/(\lambda_m + \mu_n)} = e\rho T \quad .$$

Proof. We shall only sketch the proof, since it follows on the lines of proof of Theorem1.

For a given $\varepsilon > 0$ we get sequences $m = M_p$ and $n = N_q$, such that

$$\log M(\sigma_1, \sigma_2) > \lambda_m \left(\sigma_1 + \rho^{-1} \log \frac{\alpha - \varepsilon}{\lambda_m} \right) + \mu_n \left(\sigma_2 + \rho^{-1} \log \frac{\alpha - \varepsilon}{\mu_n} \right) \qquad (7)$$

where α is assumed to be positive and is given by

$$\alpha = \lim_{m,n\to\infty} \sup(\lambda_m^{\lambda_m} \mu_n^{\mu_n} . \| a_{mn} \|^{\rho})^{1/(\lambda_m + \mu_n)} \; .$$

Choosing in (7) the sequences of the values of σ_1 and σ_2 given by

$$\sigma_1 = \rho^{-1} \log\left(\frac{e\lambda_{m_p}}{\alpha - \varepsilon} \right) \; , \; \sigma_2 = \rho^{-1} \log\left(\frac{e\mu_{n_q}}{\alpha - \varepsilon} \right)$$

it follows that $e\rho T \ge \alpha$. This assertion is trivially true for the case when $\alpha = 0$.

The converse part follows by using the following estimations of $\sum_i$'s in (6):

$$\sum_1 \le O(\exp(\sigma_1 \lambda_{m_0} + \sigma_2 \mu_{n_0})),$$

$$\sum_2 \le O(\exp(\sigma_1 \lambda_{m_0}) \exp\left(\frac{\varsigma + \varepsilon}{e\rho} . e^{\rho\sigma_2} \right),$$

$$\sum_3 \le O(\exp(\sigma_2 \mu_{n_0}) \exp\left(\frac{\varsigma + \varepsilon}{e\rho} . e^{\rho\sigma_1} \right),$$

$$\sum_4 \le A_1 \exp\left(\frac{\alpha + 2\varepsilon}{e\rho} . (e^{\rho\sigma_1} + e^{\rho\sigma_2}) \right).$$

where A_1 is an absolute constant. Hence

$$M(\sigma_1, \sigma_2) \le \exp(e^{\rho\sigma_1} + e^{\rho\sigma_2}) + O(1),$$

$$\text{or} \quad \log M(\sigma_1, \sigma_2) \le (e^{\rho\sigma_1} + e^{\rho\sigma_2}) + o(1)$$

which gives $e\rho T \le \alpha$ on proceeding to limits as $\sigma_1, \sigma_2 \to \infty$. This proves the result.

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