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THE EQUIVARIANT J-HOMOMORPHISM

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(*communicated by Gunnar Carlsson*)

Abstract

May's J-theory diagram is generalized to an equivariant setting. To do this, equivariant orientation theory for equivariant periodic ring spectra (such as KO_G) is developed, and classifying spaces are constructed for this theory, thus extending the work of Waner. Moreover, *Spin* bundles of dimension divisible by 8 are shown to have canonical KO_G-orientations, thus generalizing work of Atiyah, Bott, and Shapiro. Fiberwise completions for equivariant spherical fibrations are constructed, also on the level of classifying spaces. When G is an odd order p-group, this allows for a classifying space formulation of the equivariant Adams conjecture. It is also shown that the classifying space for stable fibrations with fibers being sphere representations completed at p is a delooping of the 1-component of $Q_G(S^0)\hat{_p}$. The "Adams-May square," relating generalized characteristic classes and Adams operations, is constructed and shown to be a pull-back after completing at p and restricting to G-connected covers. As a corollary, the canonical map from the p-completion of J_G^k to the G-connected cover of $Q_G(S^0)\hat{_p}$ is shown to split after restricting to G-connected covers.

Contents

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1. Introduction

Adams' celebrated series of $J(X)$ papers ([**1**],[**2**],[**3**],[**4**]) laid the foundation for much of modern stable homotopy theory. These papers aimed at a firm computational understanding of the group $J(X)$ of fiber homotopy equivalence classes of virtual real vector bundles over a finite CW complex X. This understanding led Adams to some of the earliest and most successful computations in the stable homotopy groups of spheres.

At the outset of the series, Adams states and proves special cases of the conjecture that now bears his name, later proven completely in ([**10**],[**23**],[**26**]). Together with a study of certain generalized characteristic classes of vector bundles, this conjecture allowed him to construct computable upper and lower bounds for the groups $J(X)$. These bounds, he was able to show, coincide, thus capturing $J(X)$.

At the heart of this work, Adams constructs a commutative diagram, which we will call the Adams square.

$$\begin{tikzcd}
\Sigma_k \tilde{K}SO(X) \arrow[r, "\Sigma k^{e(k)}(\psi^k - 1)"] \arrow[d, "\Sigma\theta^k"'] & \tilde{K}SO(X) \arrow[d, "\Pi\rho^l"] \\
\tilde{K}SO(X)_\otimes \arrow[r, "\Pi\psi^l/1"] & \Pi_l \tilde{K}SO(X)_\otimes[\frac{1}{l}].
\end{tikzcd}$$

The diagram is obtained by taking a sum of individual diagrams for pairs (k, l). Here, $\tilde{K}SO(X)$ and $\tilde{K}SO(X)_\otimes$ are the groups of oriented virtual bundles on X, of virtual dimension 0 and 1 respectively; the tensor symbol indicates that the group structure is given by tensoring bundles rather than summing. The operations ψ^k and ψ^l are the kth and lth Adams operations. The map ρ^l is a certain generalized characteristic class, the "Adams-Bott cannibalistic class." The map θ^k, which is constructed by character-theoretic arguments, makes the diagram commute. Both maps ρ^k and θ^k are exponential. As it turns out, the diagram is a weak pullback, which is central to Adams' demonstration that his bounds for $J(X)$ coincide.

Adams carried out his work handicapped by the absence of a theory $Sph(X)$ of spherical fibrations, with a map from $KO(X)$ having image $J(X)$, as well as a theory $Sph(X; KO)$ of spherical fibrations with a KO-Thom class. Yet even in writing his papers he foresaw the utility such theories would have ([**3**, §7]). When these theories became available through the work of May [**17**] and others, many of Adams' results could be put on a more conceptual and geometric level, with character-theoretic constructions banished by simple fiber sequence arguments giving maps between the classifying spaces of the relevant theories. This project was carried out by May in [**18**]. In particular, May constructs the "J-theory" diagram:

$$\begin{tikzcd}
J^k \arrow[r, "\pi"] \arrow[d, "\alpha^k"'] & BO \arrow[r, "\psi^k - 1"] \arrow[d, "\gamma^k"] & BSpin \arrow[d, equal] & \\
SF \arrow[r, "\tau"] \arrow[d, "\epsilon^k"'] & SF/Spin \arrow[r, "q"] \arrow[d, "f"] & BSpin \arrow[r, "Bj"] \arrow[d, "\rho^k"] & BSF \\
J^k_\otimes \arrow[r, "\pi"] & BO_\otimes \arrow[r, "\psi^k/1"] & BSpin_\otimes. &
\end{tikzcd}$$

Here, rows are fiber sequences and all spaces are localized away from k. The spaces $BO, BSpin$, and BSF classify real vector bundles, Spin bundles, and spherical fibrations over a finite CW-complex. The space $SF \simeq \Omega BSF$ is the monoid of degree 1 maps of spheres; its homotopy groups comprise the positive stems of the stable homotopy groups of spheres. The map Bj represents the forgetful functor from Spin bundles to spherical fibrations. The map $\psi^k - 1$ is determined by the kth Adams operation and the additive Hopf space structure on BO, giving a map $BO \to BO$, which, it's not hard to see, lifts to $BSpin$. The map $\psi^k/1$ is similarly defined, but here the Hopf space structure is multiplicative. The Adams conjecture gives a null-homotopy of $Bj \circ (\psi^k - 1)$, thus determining γ^k. The construction of ρ^k, which represents the cannibalistic class, and f, whose composite with γ^k represents θ^k, requires the introduction of a new space $B(SF; KO[\frac{1}{k}])$ which classifies spherical fibrations with a $KO[\frac{1}{k}]$-Thom class, or orientation. Finally, the maps ϵ^k and α^k are induced maps of fibers.

Within this diagram, the square involving $BO, BSpin, BO_\otimes$, and $BSpin_\otimes$ is of particular importance; we will refer to it as the Adams-May square. May proves in [**18**] that with k suitably chosen, the Adams-May square becomes a pull-back diagram in the homotopy category after localizing at a prime p. On the one hand, this implies Adams' result, that the Adams square is itself a weak pull-back. But it is an especially remarkable result in its own right since pull-backs so rarely exist in homotopy categories. Moreover, one can deduce from the result that the vertical composite $\epsilon^k \circ \alpha^k$ in the J-theory diagram gives a splitting of the p-localization of SF, in which the homotopy groups of the factor J^k_p carry the image of the classical J-homomorphism.

After May's work was completed, Waner constructed a representable theory of equivariant spherical fibrations ([**29**]), and tom Dieck and McClure obtained results generalizing the nonequivariant Adams conjecture ([**22, 27**]), though the obvious equivariant generalization of the Adams conjecture fails. With these results in hand, we revisit the work of Adams and May in this paper to give some equivariant generalizations of their results. Ideally, one might hope to do the following:

1) construct an equivariant J-theory diagram for a compact Lie group G.

2) show that, with k suitably chosen, the associated equivariant Adams-May square becomes a pull-back in the homotopy category at a prime p.

3) obtain a splitting of an equivariant generalization of SF, at each prime p.

As it stands, the above program is not possible. In fact, the equivariant Adams-May square can never be a pull-back diagram when G is not a p-group by [**22**, 5.3]. However, for the sake of generality, we will begin by setting up some foundations for later work with G being a compact Lie group. We will later restrict to finite groups whose order is a power of an odd prime p. Moreover, we must also restrict to the "G-connected cover" of the Adams-May square (see Definition 2.18), and our results only apply to the completion of the Adams-May square at this prime p.

The primary challenges involved in our work are as follows:

1) In order to define cannibalistic classes, we need an equivariant theory of KO-oriented spherical fibrations. Thus, we must consider equivariant orientation theory. In the nonequivariant setting, an E-orientation of a fibration ξ with fiber S^n (where E is a cohomology theory) is determined by a class $\mu \in E^n(T\xi)$, where $T\xi$ is the Thom space of the fibration ξ (obtained by collapsing out the section at infinity). If ξ is an equivariant spherical fibration, then the fiber over a point x is S^{V_x} for some G_x-representation V_x. But V_x may not be isomorphic to V_y if there is no G_x-fixed path from x to y. Thus, it doesn't make sense to try to find a class in $E^V_G(T\xi)$, since there is no suitable choice for V.

2) The equivariant Adams conjecture is not what one might hope. To use this, we need a good theory of equivariant fiberwise completions of spherical fibrations, and some classifying-space level constructions.

3) The proof that the maps $\rho^k, \sigma^k, \psi^k - 1$ and $\psi^k/1$ form a pull-back square in the homotopy category relies on techniques that do not apply equivariantly.

Our main results are as follows:

1) In 4.2, we define orientations for *periodic* cohomology theories, a concept which we will make precise below (4.5). With the right definitions, it is not difficult to show that equivariant *Spin* bundles have KO_G-orientations (5.4).

2) In Section 7, we define a space $B_G(\hat{\mathbb{S}}_p)$ classifying equivariant fibrations whose fibers are sphere representations completed at a prime p, and we define a map on the classifying space level constructing fiberwise completions. It will follow from the equivariant Adams conjecture that $Bj \circ (\psi^k - 1)$ is null-homotopic, when G is a p-group, p odd.

3) With G a p-group, p odd, we construct the equivariant Adams-May square:

$$\begin{tikzcd}
B_G(O)^{\wedge}_p \arrow[r, "\psi^k - 1"] \arrow[d, "\sigma^k"'] & B_G(O)^{\wedge}_p \arrow[d, "\rho^k"] \\
B_G(O)_{\otimes}{}^{\wedge}_p \arrow[r, "\psi^k/1"] & B_G(O)_{\otimes}{}^{\wedge}_p .
\end{tikzcd}$$

We show in Corollary 10.16 that this square is a pull-back diagram in the homotopy category, but only after passing to "G-connected covers." Part of the argument for this involves showing that ρ^k and σ^k are homotopic, but we were only able to do this after passing to G-connected covers, hence the restriction on the result. In the nonequivariant case, there is no loss of generality, since the domains of ρ^k and σ^k are both connected to begin with. In the equivariant case, it is not obvious that σ^k and ρ^k will have the same effect on components of fixed-point spaces.

4) We also construct an equivariant J-theory diagram:

$$\begin{tikzcd} J^k_G \arrow[r, "\pi"] \arrow[d, "\tau^k"] & B_G(O)\hat{_p} \arrow[r, "\psi^k - 1"] \arrow[d, "\sigma^k"] & B_G(O)\hat{_p} \arrow[d, "\rho^k"] \\ J^k_{G\otimes} \arrow[r, "\pi"] & B_G(O)_{\otimes}\hat{_p} \arrow[r, "\psi^k/1"] & B_G(O)_{\otimes}\hat{_p}. \end{tikzcd}$$

The square on the right is the Adams-May square, and the rows are fiber squences. The map τ^k is actually a composite $\epsilon^k \circ \alpha^k$, where the source of ϵ^k (target of α^k) is the component of the identity in $\Omega^\infty \hat{S}_p$, the zero space in the completion of the equivariant sphere spectrum. In Theorem 10.17 we show that τ^k induces a weak equivalence on G-connected covers, thus yielding a splitting of the G-connected cover of $\Omega^\infty \hat{S}_p$. On the level of homotopy groups, the factor J^k_G sits inside $\Omega^\infty \hat{S}_p$ as the image of the equivariant J-homomorphism.

The organization of this paper is as follows. We begin in Sections 2 and 3 with equivariant classifying spaces of bundles and fibrations. We review some of the constructions in [**29**], studying these by examining fixed points, which we can compare with classical nonequivariant classifying spaces (2.8, 3.10). We also show how to stabilize classifying spaces for equivariant Π-bundles, where Π is a structure group such as O, SO, or *Spin* (2.15).

Our primary interest is spherical fibrations, to which we specialize in Section 4. We generalize the theory of Thom classes to the equivariant setting, and we define periodic ring G-spectra in 4.5, a class of ring spectra for which we can define and classify stable equivariant oriented spherical fibrations (4.2 and 4.7). We also show how to construct characteristic classes associated to maps of periodic-ring spectra (4.12)

In Section 5, we consider equivariant KO-theory. We show that the equivariant KO spectrum is a periodic ring spectrum (5.1), and we show that equivariant *Spin*-bundles have canonical KO-Thom classes (5.4), as in the nonequivariant case. We also investigate the relationship between equivariant K-theory and KO-theory, proving a compatibility result relating their periodic structures (5.7).

In the next two sections, we turn to fiberwise completions of spherical fibrations. We give a general construction in Section 6 for a map of classifying spaces associated to a "map of fibers" satisfying appropriate conditions. Fiberwise completion is an example, as shown in Section 7. This material is inspired by an analogous nonequivariant construction in [**20**], though our techniques differ substantially from those used there.

With these foundations set, we turn in Section 9 to defining the maps between the classifying spaces in the equivariant J-theory diagram. We give both a geometric interpretation and a classifying-space construction of the Adams-Bott characteristic class ρ^k and its complex counterpart ρ_c^k. We exploit the construction of fiberwise completions and the equivariant Adams conjecture to obtain an equivariant generalization of the map γ^k in the J-theory diagram above. We use this to construct the map σ^k, which makes the Adams-May square commute.

We come to the heart of the matter in Section 10, where we analyze the equivariant Adams-May square. Unfortunately, the map σ^k has no apparent geometric significance, so it is difficult to get any computational hold of it. In the nonequivariant case, one can show that σ^k and ρ^k are homotopic. We generalize this fact equivariantly, but our proof only shows that σ^k and ρ^k are homotopic when restricted to the G-connected covers of their domains. We believe that, by choosing γ^k appropriately, the resulting σ^k would be homotopic to ρ^k without restricting to G-connected covers, but we have no proof of this. We then show that, after passing to G-connected covers, the equivariant Adams-May square becomes a pull-back in the homotopy category. Again, if we could choose σ^k to be actually homotopic to ρ^k, we could eliminate the need to pass to G-connected covers. Finally, we use this result to show that the G-connected cover of $\Omega^\infty \hat{S}_p$ over the identity splits up to homotopy, with one factor being J_G^k, the fiber of $\psi^k - 1$. Here, $\hat{S}_p$ is the p-completion of the equivariant sphere spectrum. This generalizes the nonequivariant result, since the "connected cover" (i.e. component of the identity) in $\Omega^\infty \hat{S}_p$ is the p-completion of SF.

In the appendix, we prove several technical results. We show how to put weak equivariant Hopf space structures on classifying spaces of stable bundles, fibrations, or fibrations with Thom classes. We also identify the space $\Omega B_G(\hat{\mathbb{S}}_p)$ as the homotopy units in $\Omega^\infty \hat{S}_p$. Lastly, we show that, as in the nonequivariant case, the map $\psi^k - 1 : B_G(O) \to B_G(O)$ lifts to $B_G(Spin)$.

2. Classifying spaces for (G, A)-bundles

In this section, let A be a compact Lie group. A principal (G, A)-bundle is a G-map $p : E \to B$ which is a principal A-bundle such that the action of each $g \in G$ is a map of A-bundles. That is, the actions of G and A commute. In 2.6, we define $B_G(A)$, which, it is shown in [**29**], classifies (G, A)-bundles over G-CW-complexes. We study this space by identifying its fixed point subspaces in Proposition 2.8. As one application of this proposition, we show (2.10) that when A is abelian, and $\xi : P \to B$ is a (G, A)-bundle such that ξ_x has trivial G_x-action for each $x \in B$, then ξ is obtained by pulling back a bundle over X/G. This technical result will be used in Section 10. We also use 2.8 to define maps of classifying spaces representing Whitney sums and stabilization of bundles. Using this, we can construct spaces classifying stable bundles, like $B_G(O)$ and $B_G(Spin)$, appearing in the equivariant J-theory diagram. Finally, we study the "G-connected covers" (2.18) of some of our classifying spaces. Results about these more easily studied G-connected covers will often suffice to answer questions about the classifying spaces we're interested in.

Definition 2.1. Let $\Lambda_H(A)$ be the set of all continuous homomorphisms $\lambda : H \to A$, where H is a closed subgroup of G. Let $\Lambda(A)$ be the disjoint union of the sets $\Lambda_H(A)$

over all closed subgroups $H \leqslant G$.

Definition 2.2. Let $\mathcal{C}_G(A)$ be the topological category with discrete object space $\Lambda(A)$. If $\rho \in \Lambda_H(A)$ and $\sigma \in \Lambda_K(A)$, a morphism $\theta : \rho \to \sigma$ is a map of principal (G, A)-bundles:

$$\begin{tikzcd} G \times_H A_\rho \arrow[r, "\bar{\theta}"] \arrow[d] & G \times_K A_\sigma \arrow[d] \\ G/H \arrow[r, "\tilde{\theta}"] & G/K. \end{tikzcd}$$

Here, A_ρ and A_σ denote A with left H-action and left K-action induced by ρ and σ. The map $\tilde{\theta}$ is a G-map and $\bar{\theta}$ is a $G \times A$-map. The morphism set is topologized as the subspace of all maps from $G \times_H A_\rho$ to $G \times_K A_\sigma$.

Remark 2.3. If $\theta : \rho \to \sigma$ is a map with $\tilde{\theta}(eH) = gK$, then $\bar{\theta}$ determines and is determined by an H-map $A_\rho \to g^*A_\sigma$, which we also designate $\bar{\theta}$, where g^*A_σ denotes A with H-action given by $h \cdot a = \sigma(g^{-1}hg)a$. Thus, θ is determined by a pair $(g, \bar{\theta})$. Now let $k\bar{\theta}$ denote the composite of $\bar{\theta}$ with the map $g^*A_\sigma \to (gk)^*A_\sigma$ given by $\sigma(k^{-1})$. Then the pairs $(gk, \bar{\theta})$ and $(g, k\bar{\theta})$ determine the same map θ. In fact, we could identify the space of maps $\theta : \rho \to \sigma$ as the space of equivalence classes of pairs $(g, \bar{\theta})$, where $(gk, \bar{\theta})$ is equivalent to $(g, k\bar{\theta})$. Composition of equivalence classes is then given by $[g_2, \bar{\theta}_2] \circ [g_1, \bar{\theta}_1] = [g_1g_2, g_1^*\bar{\theta}_2 \circ \bar{\theta}_1]$.

In the following definition and hereafter, $G\mathcal{U}$ is the category of unbased G-spaces.

Definition 2.4. Let $\mathcal{O} : \mathcal{C}_G(A) \to G\mathcal{U}$ be the functor taking an object ρ in $\Lambda_H(A)$ to the orbit G/H, and taking θ to $\tilde{\theta}$.

Remark 2.5. In the following definition, we will use the categorical bar construction. Given a topological category $\mathcal{C}$, together with functors $\mathcal{O} : \mathcal{C} \to G\mathcal{U}$ and $F : \mathcal{C}^{op} \to \mathcal{U}$, one can define a bar construction $B(F, \mathcal{C}, \mathcal{O})$, using the ideas in [**17**, §12]. It is also possible to describe this as the classifying space of a single category. Given $\mathcal{C}$, $\mathcal{O}$ and F as above, there is an associated topological G-category $\mathcal{C}^F_\mathcal{O}$ (or $\mathcal{C}_\mathcal{O}$ when F is trivial) with object space consisting of all triples (f, c, x), (or (c, x)) where c is an object in $\mathcal{C}$, $f \in F(c)$ and $x \in \mathcal{O}(c)$. The group acts on the third coordinate of these objects. A morphism from (f, c, x) to (f', c', x') is a map $\theta : c \to c'$ in $\mathcal{C}$ such that $\mathcal{O}(\theta)(x) = x'$ and $F(\theta)(f') = f$. The group acts trivially on morphism spaces. The classifying space of $\mathcal{C}^F_\mathcal{O}$ is canonically isomorphic to $B(F, \mathcal{C}, \mathcal{O})$.

Definition 2.6. Let $B_G(A)$ be $B(*, \mathcal{C}_G(A), \mathcal{O})$.

We study $B_G(A)$ by passing to fixed points. Letting $\mathcal{O}^H$ denote the composite of $\mathcal{O}$ and the H-fixed point functor, it's easy to check that

$$B(*, \mathcal{C}_G(A), \mathcal{O})^H \cong B(*, \mathcal{C}_G(A), \mathcal{O}^H).$$

Proposition 2.8 below gives a splitting of this space into a disjoint union of simpler classifying spaces. We first need the following definition:

Definition 2.7. Given $H \leqslant G$ and a map $\rho : H \to A$, let A^ρ be the centralizer of ρ, i.e.

$$\{a \in A | \rho(h)a = a\rho(h) \text{ for all } h \in H\}.$$

Now, we can view a topological monoid like A^ρ, or a disjoint union of such monoids, as a category with one object, or a disjoint union of such categories. The following is Proposition 1.3 of [**12**].

Proposition 2.8. *For each $H \leqslant G$, there is an inclusion of categories*

$$i : \coprod_{\rho \in R^+(H,A)} A^\rho \to \mathcal{C}_G(A)_{\mathcal{O}^H}$$

where $R^+(H,A)$ is a set of representatives for the conjugacy classes of homomorphisms $\rho : H \to A$. The functor i has a right adjoint, and therefore induces a homotopy equivalence

$$Bi : \coprod_{\rho \in R^+(H,A)} BA^\rho \to B_G(A)^H$$

The functor i sends the unique object of A^ρ to the object (ρ, eH) and sends $a \in A^\rho$ to the map $(1, \beta_a) : \rho \to \rho$, where $\beta_a(g,p) = (g, ap)$.

As an application of the above proposition, suppose A is abelian and ξ is a (G,A)-bundle over a G-space X such that all the fibers ξ_x have trivial G_x-action. We can then show that ξ is the pull-back of an A-bundle over X/G. This will require the following lemma.

Lemma 2.9. *Suppose A is abelian. Then there is a fiber sequence*

$$BA \xrightarrow{i} B_G(A) \xrightarrow{\pi} L$$

where BA has trivial G-action and each L^H is homotopic to a discrete set, so that in particular, ΩL is G-contractible. The map i classifies the universal bundle on BA with trivial G-action.

Proof. To define L, we construct an $\mathcal{O}_G$-space L' and use the Elmendorf construction C ([**11**]), which takes $\mathcal{O}_G$-spaces to G-spaces. Let $L'(G/H)$ be the discrete set of all conjugacy classes of homomorphisms $H \to A$. Any G-map $f : G/H \to G/K$ is determined by $f(eH) = gK$ for some g satisfying $g^{-1}Hg \subseteq K$. We define $L'(f) : L'(G/K) \to L'(G/H)$ by $L'(f)(\alpha)(h) = \alpha(g^{-1}hg)$. This is independent of the choice of g, since A is abelian. Let $L = CL'$. The trivial map $G \to A$ gives L a G-fixed basepoint.

Now, we define π. Let $\Phi(B_G(A))$ be the $\mathcal{O}_G$-space associated to the G-space $B_G(A)$. By 2.8, $\pi_0(\Phi(B_G(A))(G/H))$ is the set of conjugacy classes of homomorphisms $H \to A$. Thus, there is a functor of $\mathcal{O}_G$-spaces $\Phi(B_G(A)) \to L'$, given on the orbit G/H by discretization. This functor determines a map $\pi : B_G(A) \simeq C\Phi B_G(A) \to L$.

The map $\pi \circ i : BA \to L$ is adjoint to the composite $\Phi(BA) \to \Phi(B_G(A)) \to L'$, which is easily seen to be the trivial map. Thus, $\pi \circ i$ is null-homotopic. This determines a map from BA to the fiber of π. For any $H \leqslant G$, the sequence $BA \to B_G(A)^H \to L^H$ is a split fiber sequence. Now, by the five lemma and the Whitehead theorem, BA is the fiber of π. Moreover, since ΩL^H is contractible for each $H \leqslant G$, ΩL is itself G-contractible. □

Corollary 2.10. *Suppose A is abelian. If ξ is a (G, A)-bundle on a G-CW complex X such that for each $H \leqslant G$ and each $x \in X^H$, the fiber ξ_x has trivial H-action, then ξ is the pull-back of a nonequivariant A-bundle on X/G.*

Proof. Suppose that ξ is classified by a G-map $f : X \to B_G(A)$. Then it follows from the hypothesis on ξ that the composite

$$\Phi X \xrightarrow{\Phi(f)} \Phi B_G(A) \longrightarrow L'$$

is trivial, so that the map $X \simeq C\Phi X \to CL' = L$ is null-homotopic. Therefore, by Lemma 2.9, there is a unique lift $\tilde{f} : X \to BA$. Since the target has trivial G-action, this determines and is determined by a nonequivariant map $X/G \to BA$. □

As another application of Proposition 2.8, we can construct maps classifying products of bundles. Nonequivariantly, if $\xi : P \to B$ and $\xi' : P' \to B'$ are principal A_1 and A_2-bundles, then $\xi \times \xi' : P \times P' \to B \times B'$ is a principal $A_1 \times A_2$-bundle. Taking such products is represented on the classifying space level by the isomorphism $BA_1 \times BA_2 \cong B(A_1 \times A_2)$. In contrast with the nonequivariant case, the natural projection $B_G(A_1 \times A_2) \to B_G(A_1) \times B_G(A_2)$ is not an isomorphism, but we have the following lemma, which follows immediately from 2.8.

Lemma 2.11. *The projection $B_G(A_1 \times A_2) \to B_G(A_1) \times B_G(A_2)$ is a weak G-equivalence.*

Remark 2.12. By "weak G-equivalence," we mean, as usual, an equivariant map that induces a weak equivalence on all fixed point sets.

Since the source and target in Lemma 2.11 are of the homotopy type of G-CW complexes, we can invert the equivalence and obtain a map that classifies taking the product of (G, A_1) and (G, A_2)-bundles. We can use this fact to construct maps of classifying spaces representing Whitney sums of vector bundles. To treat the general case, let $\mathcal{I}$ be the category of finite dimensional complex inner product spaces and isometries. We will sometimes think of a space V in $\mathcal{I}$ as a real inner product space by forgetting the complex structure. As in [**18**], a group-valued $\mathcal{I}$-functor is a functor T from $\mathcal{I}$ to the category of topological groups, together with a commutative, associative, and continuous natural transformation $c : T \times T \to T \circ \oplus$ satisfying two conditions:

1) if $x \in TV$ and if $1 \in T\{0\}$ is the basepoint, then $c(x, 1) = x$ in $T(V \oplus \{0\}) = TV$.
2) if $V = V' \oplus V''$, then the map $TV' \to TV$ induced by the inclusion $V' \to V$ is a homeomorphism onto a closed subset.

For example, the functor SU takes V to $SU(V)$, and O takes V to $O(V)$. There is an evident natural transformation $SU \to O$, taking a special unitary transformation to its underlying orthogonal transformation. In general, we let Π denote a group-valued $\mathcal{I}$-functor, with morphisms $SU \to \Pi \to O$ of group-valued $\mathcal{I}$-functors, such that the composite $SU \to O$ is the canonical map. Interesting examples include SU, U, SO, and O. In addition, the inclusion $SU(V) \to SO(V)$ lifts to a map $i : SU(V) \to Spin(V)$ by elementary covering space theory, so that $Spin$ is also an example. Taking the Whitney sum of a $(G, \Pi(V))$-bundle and a $(G, \Pi(W))$-bundle yields a $(G, \Pi(V \oplus W))$-bundle; this operation is classified by the composite obtained by inverting the equivalence below.

$$B_G(\Pi(V)) \times B_G(\Pi(W)) \xleftarrow{\simeq} B_G(\Pi(V) \times \Pi(W)) \xrightarrow{c_*} B_G(\Pi(V \oplus W)).$$

Notation 2.13. We will let $\Lambda_H^\Pi(V)$ and $\Lambda^\Pi(V)$ denote $\Lambda_H(\Pi(V))$ and $\Lambda(\Pi(V))$. If $\lambda \in \Lambda_H^\Pi(V)$, let V_λ denote V with the corresponding H-action. If $\lambda_i \in \Lambda_H^\Pi(V_i)$ for $i = 1, 2$, then the sum c determines an element $\lambda_1 \oplus \lambda_2$ in $\Lambda_H^\Pi(V_1 \oplus V_2)$.

We will now construct classifying spaces $B_G(\Pi)$ for stable (G, Π)-bundles. For example, $B_G(O)$ classifies stable orthogonal G-bundles. Since we will be stabilizing other classifying space constructions in later sections, we consider the following generic construction.

Construction 2.14. Let $\mathcal{V}$ be the groupoid whose objects are finite dimensional complex inner product spaces V equipped with an action $\rho : G \to SU(V)$ of G, and whose morphisms are equivariant isomorphisms. Let $h\mathcal{V}$ be the associated homotopy category, and let $hG\mathcal{T}$ be the homotopy category of based G-spaces having the homotopy type of G-CW complexes. We define an $h\mathcal{V}$ functor to be a functor $A : h\mathcal{V} \to hG\mathcal{T}$ together with a commutative and associative natural transformation $c : A \times A \to A \circ \oplus$.

If $\iota : V \to W$ is an equivariant isometry, then $W \cong V \oplus U$ for some $U \in \mathcal{V}$. Using the basepoint of $A(U)$, the map c, and functoriality of A, we obtain a map

$$A(V) \to A(V) \times A(U) \to A(V \oplus U) \cong A(W),$$

which we call $A(\iota)$. Since c is natural and associative, this construction extends A to a functor on the category of representations $V \in \mathcal{V}$ and equivariant isometries.

We say that a sequence $\mathbb{V} = V_1 \subseteq V_2 \subseteq \cdots$ of inclusions in $\mathcal{V}$ is complete if $\cup V_i$ forms a complete G-universe. A map $\mathbb{V} \to \mathbb{W}$ consists of a sequence of equivariant isometries $V_i \to W_i$ compatible with the inclusions. We let $A(\mathbb{V})$ denote the homotopy colimit of the spaces $A(V_i)$, with maps $A(\iota) : A(V_i) \to A(V_{i+1})$, where ι is the inclusion of V_i in V_{i+1}. A map $i : \mathbb{V} \to \mathbb{W}$ induces a map $i_* : A(\mathbb{V}) \to A(\mathbb{W})$. It is easy to check that i_* is a G-equivalence, since $\mathbb{V}$ and $\mathbb{W}$ are complete. Given any two complete sequences $\mathbb{V}$ and $\mathbb{W}$, there are canonical and coherent equivalences $A(\mathbb{V}) \to A(\mathbb{W})$, since we can always form the sequence $\mathbb{V} \oplus \mathbb{W} = \{V_i \oplus W_i\}$, and we then have equivalences from $A(\mathbb{V})$ and $A(\mathbb{W})$ to $A(\mathbb{V} \oplus \mathbb{W})$.

The Whitney sum maps $B_G(\Pi(V)) \times B_G(\Pi(W)) \to B_G(\Pi(V \oplus W))$ make $B_G(\Pi(-))$ into an $h\mathcal{V}$ functor.

Definition 2.15. Let $B_G(\Pi) = B_G(\Pi(\mathbb{V}))$ for some complete sequence $\mathbb{V}$.

For example, we could take Π to be $Spin, SO$, or O. Then the maps $Spin \to SO \to O$ induce maps $B_G(Spin) \to B_G(SO) \to B_G(O)$, or we could take Π to be SU or U, and the inclusion induces a map $B_G(SU) \to B_G(U)$. We record the following two lemmas for future applications.

Lemma 2.16. *Suppose G is a finite group of odd order. Then for all $H \leqslant G$, the group of components in $B_G(Spin)^H, B_G(SO)^H$, and $B_G(O)^H$ are equal. Moreover, the fiber of $B_G(SO) \to B_G(O)$ is the discrete space $\mathbb{Z}/2$ with trivial G-action, and the fiber of $B_G(Spin) \to B_G(SO)$ is $K(\mathbb{Z}/2, 1)$, again with trivial G-action.*

Proof. By Definition 2.15, $B_G(\Pi)$ is the colimit of spaces $B_G(\Pi(V_i))$, and by Proposition 2.8, for $H \leqslant G$, $B_G(\Pi(V_i))^H$ is the disjoint union of the spaces $B\Pi(V_i)^\rho$ over all $\rho \in R^+(H, \Pi(V_i))$. The basepoint is in the component corresponding to the canonical action $\rho : G \to \Pi(V_i)$.

To show the groups of components are equal, we show that any map $\rho : H \to O(V)$ lift uniquely to $SO(V)$ and $Spin(V)$. Indeed, since H has odd order, the composite of any map $\rho : H \to O(V)$ with the determinant must be trivial, so ρ factors uniquely through $SO(V)$. Now let $\tilde{H}$ be the pull-back in the diagram below:

$$\begin{tikzcd} \tilde{H} \arrow[r] \arrow[d] & Spin(V) \arrow[d] \\ H \arrow[r] & SO(V). \end{tikzcd}$$

Then we have an extension

$$\mathbb{Z}/2 \to \tilde{H} \to H.$$

By Schur-Zassenhaus, such extensions split, and since $\mathbb{Z}/2$ has no nontrivial automorphisms, $\tilde{H} \cong H \times \mathbb{Z}/2$. This implies that ρ lifts to $Spin(V)$, and uniqueness again follows since every map $H \to \mathbb{Z}/2$ is trivial.

The proof of the lemma will be completed by the following claim: if V^ρ is nontrivial, then there are short exact sequences of groups:

$$0 \longrightarrow SO(V)^\rho \longrightarrow O(V)^\rho \xrightarrow{\det} \mathbb{Z}/2 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z}/2 \longrightarrow Spin(V)^\rho \xrightarrow{\pi} SO(V)^\rho \longrightarrow 0.$$

Indeed, the kernel of the determinant is clearly $SO(V)^\rho$, and since any reflection in V^ρ is in $O(V)^\rho$, the determinant is surjective, so the first sequence is exact. The kernel of the projection $Spin(V) \to SO(V)$, generated by an element t, is contained in the center of $Spin(V)$, and hence lies in $Spin(V)^\rho$. So, we are left to show that $Spin(V)^\rho \to SO(V)^\rho$ is surjective. Suppose $s \in SO(V)^\rho$, so that $\rho(h)s = s\rho(h)$ for any $h \in H$. Let $\tilde{s}$ be either of the two elements in $Spin(V)$ mapping to s. We claim that $\tilde{s}$ is in $Spin(V)^\rho$. The commutator $[\rho(h), \tilde{s}]$ must be either the identity or t. If it is t, then

$$\rho(h)\tilde{s} = \tilde{s}\rho(h)t.$$

Let l be the order of h in H. Then we have

$$\tilde{s} = \rho(h)^l \tilde{s} = \tilde{s}\rho(h)^l t^l = \tilde{s}t^l$$

whence $t^l = 1$, so that l must be even, but H has odd order. □

Lemma 2.17. *There is a fiber sequence*

$$B_G(SU) \to B_G(U) \to B_G(S^1),$$

where the first map is induced by the inclusion and the second map by the determinant.

Proof. To construct the sequence, we take a colimit over $V_i \in \mathbb{V}$ of the sequences

$$B_G(SU(V_i)) \to B_G(U(V_i)) \to B_G(S^1).$$

Passing to fixed points and arguing as in 2.16, we see that this is a fiber sequence whenever V_i has a trivial summand. □

Finally, we need to study the "G-connected cover" of some of our classifying spaces – these are the equivariant homotopy theorist's best substitute for looking at the basepoint component of a space.

Definition 2.18. A G-connected cover of a based G-space X is a map $\iota : X_0 \to X$ such that for each $H \leqslant G$, $\iota^H : X_0^H \to X^H$ is, up to homotopy, the inclusion of the basepoint component. Sometimes, we just denote the G-connected cover by X_0, with the inclusion ι implied.

The G-connected cover of a based G-space X exists and is unique up to equivariant homotopy equivalence. It can be constructed abstractly using the Elmendorf construction [**11**]. In our case, the G-connected covers of our classifying spaces can often be constructed as colimits of fairly simple classifying spaces, and thus lend themselves to easier analysis. Moreover, whenever we are interested in the loops on a space, we can replace that space with its G-connected cover.

Suppose given a distinguished map $\rho : G \to A$, endowing $B_G(A)$ with a G-fixed basepoint. Let $\mathcal{C}_G(A)_0$ be the full subcategory of $\mathcal{C}_G(A)$ whose objects are the restrictions of ρ to a subgroup $H \leqslant G$. The bar construction yields a subspace $B_G(A)_0$ of $B_G(A)$.

Lemma 2.19. *The inclusion $B_G(A)_0 \to B_G(A)$ is a G-connected cover.*

Proof. We have a commutative diagram

$$\begin{tikzcd} A^{\rho|H} \arrow[r, "i_0"] \arrow[d] & (\mathcal{C}_G(A)_0)_{\mathcal{O}^H} \arrow[d] \\ \coprod_{\sigma \in R^+(H,A)} A^\sigma \arrow[r, "i"] & (\mathcal{C}_G(A))_{\mathcal{O}^H}, \end{tikzcd}$$

where i_0 is defined as i was. Again, i_0 has a right adjoint. Passing to classifying spaces yields the lemma. □

To get a simpler model, note that the G-action $g \cdot a = \rho(g)a\rho(g)^{-1}$ on A induces a G-action on BA, with $(BA)^H = B(A^{\rho|H})$. The following lemma then shows that, with this G-action, BA is another model for the G-connected cover of $B_G(A)$.

Lemma 2.20. *There is a weak G-equivalence $BF : B_G(A)_0 \to BA$, natural in A.*

Proof. By Remark 2.5, we may view $B_G(A)_0$ as the classifying space of $(\mathcal{C}_G(A)_0)_{\mathcal{O}}$, and we may view BA as the classifying space of a G-category A with one object. A morphism $\theta : (\rho|H, x) \to (\rho|K, y)$ in $(\mathcal{C}_G(A)_0)_{\mathcal{O}}$ is a map of (G, A)-bundles:

$$\begin{tikzcd} G \times_H A_\rho \arrow[r, "\bar{\theta}"] \arrow[d] & G \times_K A_\rho \arrow[d] \\ G/H \arrow[r, "\tilde{\theta}"] & G/K, \end{tikzcd}$$

such that $\tilde{\theta}(x) = y$. Suppose $\bar{\theta}(e, e) = (g, a)$. Since $\bar{\theta}$ is a $G \times A$-map, $\rho(g)a$ commutes with elements $\rho(h)$ for $h \in H$. Thus, $\rho(x)\rho(g)a\rho(x)^{-1}$ is a well-defined element of A, depending only on the coset of $x \in G/H$. We let $F(\theta)$ be this map. This defines a G-functor $F : (\mathcal{C}_G(A)_0)_{\mathcal{O}} \to A$, which then induces a G-map $BF : B_G(A)_0 \to BA$. But the functor

$$i_0 : A^{\rho|H} \to (\mathcal{C}_G(A)_0)_{\mathcal{O}^H} = ((\mathcal{C}_G(A)_0)_{\mathcal{O}})^H$$

is left inverse to F^H, and Bi_0 is a homotopy equivalence, so BF induces a weak G-equivalence, as needed. $\square$

3. Classifying Spaces for Equivariant Fibrations

Given a nonequivariant, based space F, equivariant F-fibrations are defined and classified in [**29**]. One constructs a category $G\mathcal{F}(F)$ of admissible fibers, whose objects are sectioned G-maps $p : P \to Q$, where Q is a G-orbit G/H and P is weakly G-equivalent to $G \times_H F_\lambda$ for some H-action λ on F. A $G\mathcal{F}(F)$-space is then a sectioned map $\xi : P \to B$ which restricts over each G-orbit in B to an object in $G\mathcal{F}(F)$. One defines $G\mathcal{F}(F)$-quasifibrations and $G\mathcal{F}(F)$-fibrations as $G\mathcal{F}(F)$-spaces which are G-quasifibrations ([**29**, 1.2.1]) or which satisfy the covering homotopy property ([**29**, 1.2.2]). In this section, we review and generalize the classifying space constructions in [**29**]. In particular, we classify equivariant fibrations with "Y-structures," in preparation for our discussion of orientations. As in Section 2, we study fixed point subspaces and G-connected covers of our classifying spaces.

The treatment of [**29**] is limited by the need to choose a single fiber F supporting all the actions of subgroups of G. We will need to consider collections of equivariant based spaces as candidates for fibers. Thus, we let a *set of admissible fibers*, or just a *set of fibers*, be a set of spaces $\{F_\lambda\}$ with left actions of closed subgroups H_λ of G; such a set is denoted by the symbol $\mathbb{F}$. In this paper, whenever F_λ is a fiber in $\mathbb{F}$, and $g^{-1}Kg \leqslant H_\lambda$, we implicitly include the K-space g^*F_λ in $\mathbb{F}$; this denotes F_λ with K-action given by $k \cdot f = (g^{-1}kg)f$. We now give our amended version of Definition 1.1.1 in [**29**].

Definition 3.1. An *equivariant category of fibers* with distinguished set of fibers $\mathbb{F}$ is a category $G\mathcal{F}(\mathbb{F})$, in which an object is a sectioned G-map $p : P \to Q$ over a G-orbit Q, and a morphism is a section-preserving pair of G-maps $(\tilde{\theta}, \bar{\theta})$ yielding a commutative square:

$$\begin{tikzcd} P \arrow[r, "\bar{\theta}"] \arrow[d, "p"'] & P' \arrow[d, "p'"] \\ Q \arrow[r, "\tilde{\theta}"] & Q'. \end{tikzcd}$$

In addition, $G\mathcal{F}(\mathbb{F})$ is required to satisfy the following properties:

1) For each $F_\lambda \in \mathbb{F}$, the maps $p_\lambda : G \times_{H_\lambda} F_\lambda \to G/H_\lambda$ are in $G\mathcal{F}(\mathbb{F})$.

2) For each morphism $(\tilde{\theta}, \bar{\theta}) : p \to p'$ in $G\mathcal{F}(\mathbb{F})$, $\bar{\theta}$ restricts to a weak G_x-equivalence: $p^{-1}(x) \to {p'}^{-1}(\theta(x))$ for each $x \in Q$.

3) If $p : P \to Q$ is an object in $G\mathcal{F}(\mathbb{F})$, and $\tilde{\theta} : G/K \to Q$ is a G-map, then there is a morphism $(\tilde{\theta}, \bar{\theta}) : p_\mu \to p$ for some $F_\mu \in \mathbb{F}$.

4) Observing that $G\mathcal{F}(\mathbb{F})$ is a topological category, the natural projections

$$\pi_{\lambda\mu} : G\mathcal{F}(\mathbb{F})(p_\lambda, p_\mu) \to \tilde{G}\mathcal{U}(G/H_\lambda, G/H_\mu)$$

(where $\tilde{G}\mathcal{U}(G/H_\lambda, G/H_\mu) \subset G\mathcal{U}(G/H_\lambda, G/H_\mu)$ denotes the image of $\pi_{\lambda\mu}$) are quasi-fibrations such that if $\phi \in G\mathcal{F}(\mathbb{F})(p_\mu, p_\nu)$, then, in the commutative diagram

$$\begin{tikzcd} G\mathcal{F}(\mathbb{F})(p_\lambda, p_\mu) \arrow[r, "\bar{\phi}_*"] \arrow[d, "\pi_{\lambda\mu}"'] & G\mathcal{F}(\mathbb{F})(p_\lambda, p_\nu) \arrow[d, "\pi_{\lambda\nu}"] \\ G\mathcal{U}(G/H_\lambda, G/H_\mu) \arrow[r, "\tilde{\phi}_*"] & G\mathcal{U}(G/H_\lambda, G/H_\nu) \end{tikzcd}$$

induced by ϕ, $\bar{\phi}_*$ restricts to a weak equivalence on each fiber.

(5) If $p : P \to Q$ is in $G\mathcal{F}(\mathbb{F})$, then so is $p \times 1_t : P \times \{t\} \to Q \times \{t\}$ for each $t \in I$. If $\Theta : p \to q$ is in $G\mathcal{F}(\mathbb{F})$, then so is $\Theta \times 1_t : p \times 1_t \to q \times 1_t$ for each $t \in I$.

Remark 3.2. As in Remark 2.3, if $\theta : p_\lambda \to p_\mu$ is a map with $\tilde{\theta}(eH_\lambda) = gH_\mu$, then $\bar{\theta}$ is uniquely determined by an H_λ-map $F_\lambda \to g^* F_\mu$, and when $\tilde{\theta}$ is understood, we sometimes refer to this H-map as $\bar{\theta}$. Again, we identify the space of maps $\theta : p_\lambda \to p_\mu$ as a space of equivalence classes $[g, \bar{\theta}]$.

Notation 3.3. If $\xi : P \to B$ is a $G\mathcal{F}(\mathbb{F})$-space, we let $T\xi$ denote P/B, the quotient of P by the section. We call this the *Thom space* of ξ.

In all our examples, the objects in $G\mathcal{F}(\mathbb{F})$ are the sectioned G-maps $p : P \to Q$ such that there is a fiberwise equivariant weak equivalence $\theta : p_\lambda \to p$ for some G-homeomorphism $G/H_\lambda \to Q$ and some $F_\lambda \in \mathbb{F}$. Likewise, morphisms are the pairs $(\tilde{\theta}, \bar{\theta})$ satisfying condition (2) above. Then, conditions (1), (2), and (5) above are immediate; conditions (3) and (4) follow in all our examples as in [**29**, 1.1.3].

Equivariantly or not, classifying space constructions for fibrations yield universal quasifibrations which need to be replaced by fibrations. For this purpose, there is a functorial Γ-construction ([**17**, §3]), which can be applied equivariantly. Given $\xi : P \to B$, ΓP is the set of pairs (λ, p) where λ is a Moore path in B beginning at $\xi(p)$ for a point $p \in P$, and $\Gamma\xi$ takes (λ, p) to the endpoint of λ. There are then "constant path" maps $\eta : P \to \Gamma P$ over B and "path addition" maps $\mu : \Gamma(\Gamma P) \to \Gamma P$. We can characterize $G\mathcal{F}(\mathbb{F})$-fibrations as those $G\mathcal{F}(\mathbb{F})$-spaces with a $G\mathcal{F}(\mathbb{F})$-lifting function $\xi : \Gamma P \to P$ left inverse to η (cf. [**29**, 1.2.4]). There is an analogous Γ'-construction when fibers are based ([**17**, §5]).

For these constructions to be useful, we need the category of fibers $G\mathcal{F}(\mathbb{F})$ to satisfy the Γ'-completeness condition below. All our categories of fibers will satisfy these conditions (cf. [**29**, 1.2.7]).

Definition 3.4. ([**29**, 1.2.6]) A category of fibers $G\mathcal{F}(\mathbb{F})$ is Γ'-complete in $G\mathcal{T}$ if the following statements are true for $G\mathcal{F}(\mathbb{F})$ quasifiberings $p : E \to B$ with B and E in $G\mathcal{T}$:
(i) $\Gamma' p$ is a $G\mathcal{F}(\mathbb{F})$ fibration with $G\mathcal{F}(\mathbb{F})$-lifting function ξ;
(ii) $\eta : E \to \Gamma' E$ is a $G\mathcal{F}(\mathbb{F})$ map over B;
(iii) Γ' takes $G\mathcal{F}(\mathbb{F})$ maps between quasifiberings to $G\mathcal{F}(\mathbb{F})$ maps.

As with bundles, $G\mathcal{F}(\mathbb{F})$-fibrations are classifed by introducing a category whose objects are fibrations over orbits.

Definitions 3.5. ([**29**, 2.2.2]) Given an equivariant category of fibers $G\mathcal{F}(\mathbb{F})$, let $\mathcal{A}(\mathbb{F})$ denote the full subcategory of $G\mathcal{F}(\mathbb{F})$ consisting only of those objects of the form p_λ (3.1(i)). We will often write λ for the object p_λ. The object set of $\mathcal{A}(\mathbb{F})$ is discrete, and morphisms are topologized as subspaces of the usual function spaces. Let $\mathcal{O} : \mathcal{A}(\mathbb{F}) \to G\mathcal{U}$ be the functor taking λ to the G-orbit G/H_λ and taking a map $\theta : \lambda \to \mu$ to $\tilde{\theta}$. Let $B_G(\mathbb{F}) = B(*, \mathcal{A}(\mathbb{F}), \mathcal{O})$.

With the obvious adjustments to accomodate our modifications in Definition 3.1, the proof of [**29**, 2.3.6] goes through to show that the space $B_G(\mathbb{F})$ classifies $G\mathcal{F}(\mathbb{F})$-fibrations over G-CW-complexes.

Remark 3.6. When the spaces F_λ are not compact, the space $B_G(\mathbb{F})$ may not be of the homotopy type of a G-CW-complex, but since we are only interested in classifying $G\mathcal{F}(\mathbb{F})$-fibrations over G-CW-complexes, we can and will take a CW-approximation as the classifying space for $G\mathcal{F}(\mathbb{F})$-fibrations. We perform such replacements throughout implicitly.

We also need to generalize the definition of Y-structures in [**17**, §10] to an equivariant context. Let $\mathbb{F}$ be a set of fibers, and let Z be a G-space. A Z-orientation functor Y on $\mathbb{F}$ is a contravariant functor $Y : \mathcal{A}(\mathbb{F}) \to \mathcal{U}$ such that for each $F_\lambda \in \mathbb{F}$, $Y(\lambda)$ is a subspace consisting of some of the components in $F_{H_\lambda}(F_\lambda, Z) = F_G(G_+ \wedge_{H_\lambda} F_\lambda, Z)$, and given a map $\theta : \mu \to \lambda$, the map $Y(\theta)$ is induced by restriction along $T\theta^*$.

Definitions 3.7. A Y-structure on a $G\mathcal{F}(\mathbb{F})$-space $\xi : P \to B$ is a G-map $\mu : T\xi \to Z$ such that for any map $\theta : p_\lambda \to \xi$ of $G\mathcal{F}(\mathbb{F})$-spaces, the composite

$$\mu \circ T\theta : G_+ \wedge_{H_\lambda} F_\lambda \to Z$$

is in $Y(\lambda)$. A morphism between $G\mathcal{F}(\mathbb{F})$-spaces ξ and ξ' with Y-structures μ and μ' is a map $\theta : \xi \to \xi'$ with $\mu' \circ T\theta \sim \mu$.

To classify $G\mathcal{F}(\mathbb{F})$-fibrations with Y-structure, we will need (Y, Z) to be admissible, as defined below. This definition is adapted from [**17**, 10.2].

Definition 3.8. Suppose $G\mathcal{F}(\mathbb{F})$ is Γ'-complete in $G\mathcal{T}$. A pair (Y, Z) is admissible if the following statements hold for $G\mathcal{F}(\mathbb{F})$-quasifibrations $p : P \to B$ with Y-structure μ.

1) $\Gamma' p : \Gamma' P \to B$ admits a Y-structure $\Gamma'\mu : T\Gamma' P \to Z$.

2) $\eta : P \to \Gamma' P$ defines a $G\mathcal{F}(\mathbb{F})$-map $P \to \Gamma' P$ over B such that $\eta^*(\Gamma'\mu)$ is homotopic to μ.

3) Γ' takes $G\mathcal{F}(\mathbb{F})$-maps $(p, \mu) \to (q, \nu)$ to $G\mathcal{F}(\mathbb{F})$-maps $(\Gamma' p, \Gamma'\mu) \to (\Gamma' q, \Gamma'\nu)$.

We can classify Y structures on $G\mathcal{F}(\mathbb{F})$-fibrations. Let $B_G(\mathbb{F}; Y)$ denote the G-space $B(Y, \mathcal{A}(\mathbb{F}), \mathcal{O})$. The proof of the following proposition is easily adapted from [**17**] and [**29**].

Proposition 3.9. *Given a Z-orientation functor Y on $\mathbb{F}$, the space $B_G(\mathbb{F}; Y)$ classifies $G\mathcal{F}(\mathbb{F})$ fibrations with Y-structure.*

As with bundles, we can analyze the fixed-point spaces of these classifying spaces. Just as we viewed a topological monoid A^ρ as a subcategory of $\mathcal{C}_G(A)_{\mathcal{O}^H}$ in Proposition 2.8, now, when $F_\lambda \in \mathbb{F}$ and $H = H_\lambda$, we let $\mathcal{A}(F_\lambda)^Y \subset \mathcal{A}(\mathbb{F})^Y_{\mathcal{O}^H}$ be the subcategory consisting of objects (y, λ, eH), with morphisms of the form $(1, \bar{\theta})$. We write $\mathcal{A}(F_\lambda)$ for $\mathcal{A}(F_\lambda)^*$. The proof of the following analogue of Proposition 2.8 is an easy generalization of the argument in [**12**, 1.3].

Proposition 3.10. *For each $H \leqslant G$, there is an inclusion of categories*

$$i : \coprod_{\lambda \in R_h^+(H,\Lambda)} \mathcal{A}(F_\lambda)^Y \to \mathcal{A}(\mathbb{F})^Y_{\mathcal{O}^H}$$

where $R_h^+(H, \Lambda)$ is a set of representatives of equivalence classes of elements λ such that $H_\lambda = H$, with $\rho \sim \sigma$ if $F_\rho \simeq_H F_\sigma$. The functor i has a right adjoint, and therefore induces a homotopy equivalence

$$Bi : \coprod_{\lambda \in R_h^+(H,\Lambda)} B\mathcal{A}(F_\lambda)^Y \to B_G(\mathbb{F}; Y)^H.$$

We use 3.10 to identify the fiber of the quasifibration $q : B_G(\mathbb{F}; Y) \to B_G(\mathbb{F})$ which represents forgetting the Y-structure. We assume that $\mathbb{F}$ comes equipped with a distinguished fiber F_ρ with $H_\rho = G$, giving $B_G(\mathbb{F})$ a G-fixed basepoint.

Since $G \times_H F_\rho \cong G/H \times F_\rho$, any G-map $G/H \to G/K$ induces a map $j(f) : \rho|H \to \rho|K$ in $\mathcal{A}(\mathbb{F})$. This yields a functor $j : \mathcal{O}_G \to \mathcal{A}(\mathbb{F})$. If $Y : \mathcal{A}(\mathbb{F}) \to \mathcal{U}$ is a Z-orientation functor, let $j^*Y : \mathcal{O}_G \to \mathcal{U}$ be the composite $Y \circ j$, so $j^*Y(G/H)$ is a subspace consisting of some of the components in $F_H(F_\rho, Z)$. The Elmendorf Construction ([**11**]) yields a G-space $C(j^*Y) = B(j^*Y, \mathcal{O}_G, \mathcal{O})$, where $\mathcal{O} : \mathcal{O}_G \to G\mathcal{U}$ is the inclusion. Now, j yields a map:

$$j_* : C(j^*Y) = B(j^*Y, \mathcal{O}_G, \mathcal{O}) \to B(Y, \mathcal{A}(\mathbb{F}), \mathcal{O}) = B_G(\mathbb{F}; Y)$$

Proposition 3.11. *The map j_* is the homotopy fiber of q.*

Proof. It suffices by the Whitehead theorem to consider H-fixed points for each $H \leqslant G$. We have a commutative diagram:

$$\begin{tikzcd}
j^*Y(G/H) \arrow[r] \arrow[d] & B(\mathcal{A}(F_{\rho|H})^Y) \arrow[r] \arrow[d, "Bi"] & B(\mathcal{A}(F_{\rho|H})) \arrow[d, "Bi"] \\
B(j^*Y, \mathcal{O}_G, \mathcal{O}^H) \arrow[r, "j_*^H"] & B(Y, \mathcal{A}(\mathbb{F}), \mathcal{O}^H) \arrow[r, "q^H"] & B(*, \mathcal{A}(\mathbb{F}), \mathcal{O}^H).
\end{tikzcd}$$

By 3.10, the right vertical maps is the inclusion of the basepoint component, and the middle vertical map is the inclusion of the components in $B_G(\mathbb{F}, Y)^H$ mapping to the basepoint component in $B_G(\mathbb{F})^H$. By [**11**], the left vertical map is an equivalence. The top sequence is a quasifibration sequence by [**17**, 7.9], so that the inclusion of $j^*Y(G/H)$ is equivalent to the inclusion of the fiber. It follows that $C(j^*Y)^H$ is equivalent to the fiber of q^H. $\square$

Remark 3.12. Let $*$ be the restriction of ρ to the trivial subgroup. The space $Y(*) \subseteq F(F_\rho, Z)$ has a G-action given by conjugation. If we let $\Phi(Y(*))(G/H) = Y(*)^H$, then there is an inclusion $j^*Y \to \Phi(Y(*))$ induced by the inclusion of fixed points. Applying C, this induces an inclusion $Fib(q) \to Y(*)$.

From Proposition 3.10, we also obtain an analogue of Lemma 2.11, allowing us to construct (3.14) maps on classifying spaces representing fiberwise smash products. Suppose given sets of fibers $\mathbb{F}^i$ and Z^i-orientation functors Y^i on $\mathbb{F}^i$ for $i = 1, 2$. Let $\mathcal{A}(\mathbb{F}^1 \times \mathbb{F}^2)$ be the subcategory of $\mathcal{A}(\mathbb{F}^1) \times \mathcal{A}(\mathbb{F}^2)$ whose objects are those pairs (λ, μ) for which $H_\lambda = H_\mu$ and whose morphisms are pairs (θ^1, θ^2) with $\tilde{\theta}^1 = \tilde{\theta}^2$. Let $\mathcal{O}(\lambda, \mu)$ be $G/H_\lambda = G/H_\mu$, and let $(Y^1 \times Y^2)(\lambda, \mu) = Y^1(\lambda) \times Y^2(\mu)$; these define covariant and contravariant functors from $\mathcal{A}(\mathbb{F}^1 \times \mathbb{F}^2)$ to $G\mathcal{U}$ and $\mathcal{U}$. Let $B_G(\mathbb{F}^1 \times \mathbb{F}^2; Y^1 \times Y^2)$ denote $B(Y^1 \times Y^2, \mathcal{A}(\mathbb{F}^1 \times \mathbb{F}^2), \mathcal{O})$. The following lemma and its proof (using 3.10) are analogous to Lemma 2.11 and its proof.

Lemma 3.13. *The projection*

$$B_G(\mathbb{F}^1 \times \mathbb{F}^2; Y^1 \times Y^2) \to B_G(\mathbb{F}^1; Y^1) \times B_G(\mathbb{F}^2; Y^2)$$

is a weak G-equivalence.

Construction 3.14. Suppose given a set of fibers $\mathbb{F}$ such that for each pair $(F^1_\lambda, F^2_\mu) \in \mathbb{F}^1 \times \mathbb{F}^2$, with $H_\lambda = H = H_\mu$, the H-space $F^1_\lambda \wedge F^2_\mu$ is a fiber in $\mathbb{F}$. Denote the corresponding object in $\mathcal{A}(\mathbb{F})$ by $\lambda \wedge \mu$. Suppose also given a G-space Z and a G-map $\psi : Z^1 \wedge Z^2 \to Z$, along with a Z-orientation functor Y on $\mathbb{F}$, such that the composite

$$F_H(F^1_\lambda, Z^1) \times F_H(F^2_\mu, Z^2) \xrightarrow{\wedge} F_H(F^1_\lambda \wedge F^2_\mu, Z^1 \wedge Z^2) \xrightarrow{\psi_*} F_H(F^1_\lambda \wedge F^2_\mu, Z)$$

restricts to a map $Y^1(\lambda) \times Y^2(\mu) \to Y(\lambda \wedge \mu)$. Suppose $\xi^1 : P^1 \to B^1$ and $\xi^2 : P^2 \to B^2$ are $G\mathcal{F}(\mathbb{F}^1)$ and $G\mathcal{F}(\mathbb{F}^2)$-fibrations with Y^1 and Y^2-structures $\mu : T\xi^1 \to Z^1$ and $\mu^2 : T\xi^2 \to Z^2$, and with sections $\sigma^1 : B^1 \to P^1$ and $\sigma^2 : B^2 \to P^2$. Let $\xi^1 \wedge \xi^2$ represent the $G\mathcal{F}(\mathbb{F})$-fibration over $B^1 \times B^2$ whose total space is the quotient $P^1 \times P^2/\sim$, where we identify $(\sigma^1(b), p)$ to $(\sigma^1(b), \sigma^2\xi^2(p'))$ and $(p, \sigma^2(b))$ to $(\sigma^1\xi^1(p), \sigma^2(b))$. This has a section induced by $\sigma^1 \times \sigma^2$. Moreover, $T(\xi^1 \wedge \xi^2) \cong T\xi^1 \wedge T\xi^2$, so that we have a Y-orientation $\mu^1 \wedge \mu^2$ on $\xi^1 \wedge \xi^2$ given by the composite

$$T\xi^1 \wedge T\xi^2 \xrightarrow{\mu^1\wedge\mu^2} Z^1 \wedge Z^2 \xrightarrow{\psi} Z\ .$$

We call the pair $(\xi^1 \wedge \xi^2, \mu^1 \wedge \mu^2)$ the *fiberwise smash product* of (ξ^1, μ^1) and (ξ^2, μ^2). This is classified by the composite obtained by inverting the equivalence below:

$$B_G(\mathbb{F}^1; Y^1) \times B_G(\mathbb{F}^2; Y^2) \xleftarrow{\simeq} B_G(\mathbb{F}^1 \times \mathbb{F}^2; Y^1 \times Y^2) \xrightarrow{\psi_*} B_G(\mathbb{F}; Y).$$

Finally, we consider G-connected covers. A distinguished object $F_\rho \in \mathbb{F}$, with $H_\rho = G$, endows $B_G(\mathbb{F})$ with a G-fixed base point. Let $\mathcal{A}(\mathbb{F})_0$ denote the full subcategory of $\mathcal{A}(\mathbb{F})$ having those objects of the form $\rho|H$ for some $H \leqslant G$. The bar construction yields a subspace $B_G(\mathbb{F})_0$ of $B_G(\mathbb{F})$. As in Lemma 2.19, we have

Lemma 3.15. *The inclusion $B_G(\mathbb{F})_0 \to B_G(\mathbb{F})$ is a G-connected cover.*

Given a homomorphism $\rho : G \to A$, A can be thought of as the group of A-linear self-maps of A, with G acting by conjugation. In this spirit, let $\tilde{\mathcal{A}}(F_\rho)$ be the monoid under composition of the space of based self-maps of F_ρ which are nonequivariant equivalences, with G acting through conjugation. We then have a classifying space $B(\tilde{\mathcal{A}}(F_\rho))$ with a G-action.

As in Lemma 2.20, there is a map $B_G(\mathbb{F})_0 \to B(\tilde{\mathcal{A}}(F_\rho))$. To see this, regard $\tilde{\mathcal{A}}(F_\rho)$ as a category with one object, and let $F : (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}} \to \tilde{\mathcal{A}}(F_\rho)$ be the functor given on the morphism

$$\theta = (\tilde{\theta}, \bar{\theta}) : (\rho|H, xH) \to (\rho|K, yK)$$

by the formula

$$\bar{\theta}([x, x^{-1}f]) = [y, y^{-1}F(\theta)f].$$

It is not difficult to verify that F is a well-defined G-functor, and therefore induces a G-map

$$BF : B_G(\mathcal{A}(\mathbb{F}))_0 \to B(\tilde{\mathcal{A}}(F_\rho)).$$

Unlike in the bundle case, BF is not always a weak equivalence, but we do have the following analogue of 2.20.

Lemma 3.16. *Suppose $\tilde{\mathcal{A}}(F_\rho)$ is a grouplike G-space; that is, $\pi_0^H(\tilde{\mathcal{A}}(F_\rho))$ is a group for each $H \leqslant G$. Then $BF : B_G(\mathcal{A}(\mathbb{F}))_0 \to B(\tilde{\mathcal{A}}(F_\rho))$ is a weak G-equivalence.*

Proof. By hypothesis, any H-equivariant map $f : F_\rho \to F_\rho$ in $\tilde{\mathcal{A}}(F_\rho)$ is an H-equivalence. We may therefore define a functor

$$L^H : \tilde{\mathcal{A}}(F_\rho)^H \to (\mathcal{A}(\mathbb{F})_0)_{\mathcal{O}^H}$$

taking the unique object to $(\rho|H, eH)$ and taking $\alpha : F_\rho \to F_\rho$ to $(\mathrm{id}, \mathrm{id} \times_H \alpha)$. The functor L^H has a right adjoint R^H, which sends a map

$$(\tilde{\theta}, \bar{\theta}) : (\rho|H, xH) \to (\rho|K, yK)$$

to the H-map

$$F_\rho \xrightarrow{\nu} G \times_H F_\rho \xrightarrow{\bar{\theta}} G \times_K F_\rho \xrightarrow{\mu} F_\rho \ .$$

Here, ν is the H-map taking f to $(x, x^{-1}f)$, and μ is the action map of F_ρ. Therefore, BR^H is an equivalence. But clearly $Q^H \circ R^H$ is the identity, so that Q^H_* is an equivalence, whence Q_* is a weak G-equivalence. □

4. Spherical fibrations and Thom Classes

In this section, we define E-orientations (or Thom classes) of equivariant spherical fibrations, where E is a commutative, unital ring G-spectrum. When E is an equivariant periodic ring spectrum, as defined in 4.5, we can define and classify stable E-oriented spherical fibrations with a space $B_G(\mathbb{S}; E)$. We will show in Section 5 that the equivariant KO spectrum furnishes an example of an equivariant periodic ring spectrum, and that equivariant *Spin*-bundles have KO-orientations. This generalizes a classical nonequivariant result, and is critical for defining the cannibalistic class ρ^k, as well as the map σ^k, in the J-theory diagram. We also show in this section (4.12) how to construct a self-map of $B_G(\mathbb{S}; E)$ associated to a collection of self-maps of $\Omega^\infty E$ satisfying certain properties. We will need this in Section 9 to define cannibalistic classes on the classifying space level.

Suppose that Π is a group-valued $\mathcal{I}$-functor, and V is an object in $\mathcal{I}$. Given $\lambda \in \Lambda^\Pi_H(V)$, we denote the one-point compactification of the H-representation V_λ by S^{V_λ}, and we denote the associated set of fibers by $\mathbb{S}^V$.

Remark 4.1. We can associate to a principal $(G, \Pi(V))$-bundle $\xi : P \to B$ a $G\mathcal{F}(\mathbb{S}^V)$-fibration $S(\xi) : S(P) \to B$ by letting $S(P) = P \times_{\Pi(V)} S^V$. This is represented by a map $Bj : B_G(\Pi(V)) \to B_G(\mathbb{S}^V)$ of classifying spaces.

Nonequivariantly, if ξ a spherical fibration with fiber S^n, then an E-orientation of ξ is an element $\mu \in \tilde{E}^n(T\xi)$ restricting to a generator of the free E_*-module $\tilde{E}^*(T\xi_x)$ for each fiber ξ_x of ξ. There is a classifying space $B(F; E)$ for E-oriented spherical fibrations ([**18**, III.2]).

Equivariantly, if $\xi : P \to B$ is a $G\mathcal{F}(\mathbb{S}^V)$-fibration, then for each $x \in B^H$, the fiber $T\xi_x$ of $T\xi$ over x is equivalent to S^{V_λ} for some H-representation V_λ. Thus, $\tilde{E}^*_H(T\xi_x)$ is a free E^H_*-module with a generator γ in $\tilde{E}^{V_\lambda}_H(T\xi_x)$. Since V_λ and H depend on x, it is too restrictive to define an E-orientation for ξ to be a class in $\tilde{E}^{V_\lambda}_H(T\xi)$. We will adopt the following definition:

Definition 4.2. An E-orientation of a $G\mathcal{F}(\mathbb{S}^V)$-fibration ξ is an element $\mu \in \tilde{E}^0_G(T\xi)$, which, for each $x \in B^H$, restricts to a generator in $\tilde{E}^*_H(T\xi_x)$. Such a generator will be a product of a generator $\gamma \in \tilde{E}^*_H(T\xi_x)$ and a unit in E^*_H. If $\xi : P \to B$ is a principal $(G, \Pi(V))$-bundle, then an E-orientation of ξ is an E-orientation of the associated $G\mathcal{F}(\mathbb{S}^V)$-fibration $S(\xi)$.

Remark 4.3. Note that if for some $x \in B^H$, $T\xi_x$ is equivalent to S^{V_λ}, then an E-orientation for ξ cannot exist unless $E^{-V_\lambda}_H$ contains a unit.

If μ is an E-orientation of ξ, then an easy Mayer-Vietoris argument shows that $\tilde{E}^*_G(T\xi)$ is a free $E_G(B)$-module generated by μ.

In order to speak of stable E-oriented $G\mathcal{F}(\mathbb{S})$-fibrations, we need a way to induce an E-orientation of $\Sigma^V\xi : P \times S^V \to B$ from an E-orientation of $\xi : P \to B$, where V comes equipped with a map $\rho : G \to \Pi(V)$. This can be done when E is a *periodic ring G-spectrum*, which we will define in 4.5. Equivariant KO-theory, with its Bott classes, is the motivating example.

In the following definition, let $\mathcal{I}'$ be an additive subcollection of $\mathcal{I}$ (e.g. those spaces of dimension divisible by 8), let Π be a group-valued $\mathcal{I}$-functor (e.g. $Spin$), and let $E = E_G$ be a commutative unital ring G-spectrum, (e.g. KO_G).

Definition 4.4. A *collection of periodicity classes* on E is a collection b of elements

$$b^{V_\lambda} \in \tilde{E}^0_G(G_+ \wedge_{H_\lambda} S^{V_\lambda}) \cong \tilde{E}^0_{H_\lambda}(S^{V_\lambda}),$$

for each V in $\mathcal{I}'$ and $\lambda \in \Lambda^\Pi(V)$. When λ is understood, we write b^V for b^{V_λ}. These classes must satisfy the following two conditions.

1) The class b^{V_λ} generates $\tilde{E}^0_{H_\lambda}(S^{V_\lambda})$ as a free $E^{H_\lambda}_*$-module.

2) If $V, W \in \mathcal{I}', H \leqslant G, \lambda \in \Lambda^\Pi_H(V), \mu \in \Lambda^\Pi_H(W)$, then the multiplication

$$\tilde{E}_H(S^{V_\lambda}) \otimes \tilde{E}_H(S^{W_\mu}) \to \tilde{E}_H(S^{V_\lambda} \wedge S^{W_\mu}) \cong \tilde{E}_H(S^{V_\lambda \oplus W_\mu})$$

takes $b^{V_\lambda} \otimes b^{W_\mu}$ to $b^{V_\lambda \oplus W_\mu}$.

We now define periodic ring G-spectra (see [**18**, II.3.11] for a nonequivariant analogue).

Definition 4.5. A (*unital*) *periodic ring G-spectrum* (E, b) is a (unital) commutative ring G-spectrum E, equipped with a collection of periodicity maps b.

Now let $Z = \Omega^\infty E$, where E is a periodic ring G-spectrum. We construct a Z-orientation functor $F^V E$ so that an E-orientation on a $G\mathcal{F}(\mathbb{S}^V)$-fibration can be interpreted as an $F^V E$-structure, and thus classified. First, notice that

$$\pi_0(F_{H_\lambda}(S^{V_\lambda}, \Omega^\infty E)) \cong \tilde{E}^0_{H_\lambda}(S^{V_\lambda}).$$

By condition (1) of 4.4, this is a free $E^{H_\lambda}_0$-module.

Construction 4.6. Let $F^V E(\lambda)$ be the subspace of $F_{H_\lambda}(S^{V_\lambda}, \Omega^\infty E)$ consisting of all components of generators in $\tilde{E}^0_{H_\lambda}(S^{V_\lambda})$. A map $\theta : \lambda \to \mu$ in $\mathcal{A}(\mathbb{S}^V)$ induces a map

$$T\theta^* : F_G(G_+ \wedge_{H_\mu} S^{V_\mu}, \Omega^\infty E) \to F_G(G_+ \wedge_{H_\lambda} S^{V_\lambda}, \Omega^\infty E).$$

Since $\bar{\theta}$ is a G-equivalence, this map restricts to a map $F^V E(\mu) \to F^V E(\lambda)$.

The proof that $(F^V E, \Omega^\infty E)$ is admissible is formally identical to the proof in [**17**, 10.6]. The following now follows from Proposition 3.9.

Proposition 4.7. *The space* $B_G(\mathbb{S}^V; E) := B_G(\mathbb{S}^V; F^V E)$ *classifies* E*-oriented* $G\mathcal{F}(\mathbb{S}^V)$*-fibrations.*

Let $\mu : \Omega^\infty E \wedge \Omega^\infty E \to \Omega^\infty E$ be the multiplication map. Suppose $\lambda \in \Lambda_H^\Pi(V), \mu \in \Lambda_H^\Pi(W)$. By condition (2) of 4.4, the map

$$F_H(S^{V_\lambda}, \Omega^\infty E) \times F_H(S^{W_\mu}, \Omega^\infty E) \xrightarrow{\mu_* \circ \wedge} F_H(S^{V_\lambda} \wedge S^{W_\mu}, \Omega^\infty E)$$

restricts to a map

$$F^V E(\lambda) \times F^W E(\mu) \to F^{V \oplus W}(\lambda \wedge \mu).$$

Construction 3.14 yields maps representing fiberwise smash products

$$B_G(\mathbb{S}^V; E) \times B_G(\mathbb{S}^W; E) \to B_G(\mathbb{S}^{V \oplus W}; E).$$

As with bundles, we can construct classifying spaces for equivariant stable E-oriented spherical fibrations. Suppose V is a G-representation in $\mathcal{V}$ (see 2.14 above). The action map $\rho : G \to \Pi(V)$ endows $\mathcal{A}(\mathbb{S}^V)$ with a distinguished object. Moreover, a choice of G-map $S^V \to \Omega^\infty E$ representing the class of b^V determines a distinguished point in $F^V E(\rho)$. We thus obtain a G-fixed basepoint for $B_G(\mathbb{S}^V; E)$. The fiberwise smash product maps above make $B_G(\mathbb{S}^{(-)}; E)$ an $h\mathcal{V}$-functor.

Definition 4.8. Let $B_G(\mathbb{S}; E) = B_G(\mathbb{S}^{\mathbb{V}}; E)$ for some complete sequence $\mathbb{V}$.

We now make a few remarks on the fiber of the map which represents forgetting the orientation.

Remark 4.9. We can identify the fiber of the map $q : B_G(\mathbb{S}; E) \to B_G(\mathbb{S})$ using 3.11. If $V \in \mathcal{V}$, then $j^* F^V E$ is the functor $\mathcal{O}_G \to \mathcal{U}$ taking G/H to the components in $F_H(S^V, \Omega^\infty E)^H$ representing units in $\tilde{E}_H(S^V)$. Thus, the fiber of q is the colimit of the spaces $C(j^* F^{V_i} E)$ for any complete sequence V_i. The map $j^* F^{V_i} E(G/H) \to j^* F^{V_{i+1}} E(G/H)$ associated to an inclusion $V_i \subseteq V_{i+1}$ is induced by taking H-fixed points of the G-map

$$F(S^{V_i}, \Omega^\infty E) \to F(S^{V_i} \wedge S^{V_{i+1} - V_i}, \Omega^\infty E) \cong F(S^{V_{i+1}}, \Omega^\infty E),$$

where the first map is induced from the periodicity class of $V_{i+1} - V_i$ and the multiplication μ. This is an equivalence by periodicity (4.4). Therefore, the fiber of q is $C(j^* F^{V_i} E)$ for any V_i. By Remark 3.12, since $F^V E(*)$ consists of the homotopy units of $\Omega^\infty E$, which we denote $\Omega^\infty E^\times$, there is a canonical inclusion $\iota : Fib(q) \to \Omega^\infty E^\times$.

Remark 4.10. The inclusion $\tau : Fib(q) \to B_G(\mathbb{S}; E)$ has the following geometric interpretation. If X is a G-space and $\alpha \in E_G(X)$ restricts to a unit in $E_H(*)$ for every H-fixed point $* \in X$, then we may represent α by a map $\tilde{\alpha} : X \to Fib(q)$. If $b^V \in \tilde{E}_G(S^V)$ is the periodicity class for $V \in \mathcal{V}$, then we have an orientation $b^V \alpha \in \tilde{E}_G(\Sigma^V X_+)$, of the trivial fibration over X with fiber S^V. The map $\tau \circ \tilde{\alpha} : X \to B_G(\mathbb{S}; E)$ represents the trivial fibration with orientation $b^V \alpha$.

We now show how a collection of self-maps Ψ of $\Omega^\infty E$ satisfying certain conditions induces a map $c(\Psi)$ from $B_G(\mathbb{S}; E)$ to $\Omega^\infty E^\times$. The map $c(\Psi)$ is called the cannibalistic class associated to Ψ. Let $\mathcal{V}'$ be the full subcategory of $\mathcal{V}$ consisting of representations whose underlying inner product spaces are in $\mathcal{I}'$.

Definition 4.11. A collection Ψ of self-maps ψ_V of $\Omega^\infty E$, one for each $V \in \mathcal{V}'$, is called *exponential* if the induced map $\psi_{V*} : \tilde{E}_G(S^V) \to \tilde{E}_G(S^V)$ takes b^V to b^V, and the following diagram commutes up to homotopy, where V and W are in $\mathcal{V}'$.

$$\begin{tikzcd}
\Omega^\infty E \wedge \Omega^\infty E \arrow[r, "\mu"] \arrow[d, "\psi_V \wedge \psi_W"'] & \Omega^\infty E \arrow[d, "\psi_{V\oplus W}"] \\
\Omega^\infty E \wedge \Omega^\infty E \arrow[r, "\mu"] & \Omega^\infty E.
\end{tikzcd}$$

Construction 4.12. Using the above diagram, it is easy to check that for any $V \in \mathcal{V}'$, ψ_V induces a natural transformation $F^V E \to F^V E$, and hence a self-map ψ_{V*} of $B_G(\mathbb{S}^V; E)$. The diagram and the condition that ψ_{V*} preserve the periodicity class together imply that these maps stabilize to give a self-map Ψ_* of $B_G(\mathbb{S}; E)$.

The space $B_G(\mathbb{S}; E)$ has the structure of a weak grouplike G-Hopf space (see Section 11), and therefore has a weak homotopy inverse map by 11.5, so that we may apply $\Psi_* - 1$ to $B_G(\mathbb{S}; E)$. Since Ψ_* only sees the orientation of a spherical fibration, $q \circ (\Psi_* - 1)$ is null-homotopic when restricted to finite skeleta of $B_G(\mathbb{S}; E)$. We may choose null-homotopies and construct a map $c(\Psi) : B_G(\mathbb{S}; E) \to Fib(q)$ restricting on finite skeleta to a lift of $\Psi_* - 1$. Recall from 4.9 that $Fib(q)$ is a subspace of $\Omega^\infty E^\times$. We refer to the composite $B_G(\mathbb{S}; E) \to Fib(q) \subseteq \Omega^\infty E^\times$ as $c(\Psi)$ as well, relying on context to resolve the ambiguity.

Remark 4.13. Using Remark 4.10, we obtain the following geometric interpretation for $c(\Psi)$. Suppose $f : X \to B_G(\mathbb{S}; E)$ classifies a stable E-oriented $G\mathcal{F}(\mathbb{S})$-fibration, represented by a $G\mathcal{F}(\mathbb{S}^W)$-fibration $\xi : P \to X$ and a class $\mu \in \tilde{E}_G(T\xi)$. The map $c(\Psi) \circ f : X \to Fib(q) \subseteq \Omega^\infty E^\times$ classifies an element $\alpha \in E_G(X)$ (restricting to a unit in $E_H(*)$ for each $* \in X^H$) such that for some $V \in \mathcal{V}'$, $b^V \psi_W(\mu) = (b^V \alpha)\mu$ in $\tilde{E}_G(\Sigma^V T\xi)$. Since multiplication by b^V is an equivalence, we have $\psi_W(\mu) = \alpha\mu$ in $\tilde{E}_G(T\xi)$. That is, $\alpha = \psi_W(\mu)/\mu$. (This is independent of W, since by exponentiality $\psi_{V\oplus W}(b^V\mu)/b^V\mu = \psi_V(b^V)\psi_W(\mu)/b^V\mu = \psi_W(\mu)/\mu$.) It follows from 4.10 that the composite $c(\psi) \circ \tau : Fib(q) \to Fib(q)$ classifies $\psi_{\{0\}}/1$, when we regard $Fib(q)$ as a subspace of $\Omega^\infty E^\times$.

5. Equivariant real and complex K-theory

In this section, we show that the equivariant ring spectrum KO is periodic, and that stable $(G, Spin)$-bundles have KO-orientations. As observed in the last section, these constructions are needed to define cannibalistic classes. We then investigate some relationships between KO-theory and complex K-theory, which will be computationally useful, since complex K-theory is so much more well-behaved than KO-theory. Throughout, let $\mathcal{I}'$ be the collection of complex inner product spaces V with complex dimension divisible by 4. More details on Clifford algebras can be found in [**7**].

To any (real) inner product space V, there is an associated Clifford algebra C_V, multiplicatively generated by a unit 1 and an orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ of

V, subject to the relations

$$e_i^2 = -1, e_i e_j = -e_j e_i \; i \neq j.$$

The Clifford algebra has a $\mathbb{Z}/2$-grading, with subspaces C_V^0 and C_V^1 additively generated by even and odd products of generators respectively. The group $Spin(V)$ is a subgroup of the units of C_V^0, giving C_V a $Spin(V)$-action (conjugation) compatible with the algebra structure. Moreover, any C_V-module inherits a $Spin(V)$-action compatible with the module structure.

If $\{e_1, e_2, \cdots, e_{8n}\}$ is an orthonormal basis of $V \in \mathcal{I}'$, there is a central element $\omega = e_1 e_2 \cdots e_{8n}$ in C_V^0, depending only on the orientation of the basis. Thus, to each $V \in \mathcal{I}'$ is associated a canonical choice of $\omega \in C_V^0$, since the underlying real inner product space of V has a canonical orientation. One can show that there are two irreducible graded C_V-modules; ω acts by the scalar $+1$ on one of these and -1 on the other. Let $\lambda_V = \lambda_V^0 \oplus \lambda_V^1$ be the module on which ω acts as $+1$.

A graded C_V-module gives rise to a $Spin(V)$-bundle over S^V. In brief, let $B(V)$ and $S(V)$ be the unit ball and unit sphere in V. Then we have a map of $Spin(V)$-equivariant bundles over $B(V)$, restricting to an isomorphism over $S(V)$:

$$B(V) \times \lambda_V^1 \to B(V) \times \lambda_V^0$$

$$(v, \lambda) \rightarrowtail (v, v \cdot \lambda).$$

By [**7**, §11], such data determines a class

$$b^V \in KO_{Spin(V)}(B(V), S(V)) \cong \tilde{K}O_{Spin(V)}(S^V).$$

Given a homomorphism $\lambda : H \to Spin(V)$, the restriction of b^V under λ is the Bott class $b^{V_\lambda} \in \tilde{K}O_H(S^{V_\lambda})$.

Proposition 5.1. *The maps b^{V_λ} form a collection of periodicity maps for KO.*

Proof. The first condition of Definition 4.4 follows by equivariant Bott periodicity (see [**6**, 6.1]). For condition (ii), suppose V and W are in $\mathcal{I}'$. Let

$$c^* : \tilde{K}O_{Spin(V \oplus W)}(S^{V \oplus W}) \to \tilde{K}O_{Spin(V) \times Spin(W)}(S^{V \oplus W})$$

be induced by $c : Spin(V) \times Spin(W) \to Spin(V \oplus W)$. It suffices to show that $c^*(b^{V \oplus W}) = b^V \cdot b^W$. The element $b^{V \oplus W}$ is obtained from the $C_{V \oplus W}$ module $\lambda_{V \oplus W}$, while $b^V \cdot b^W$ is obtained from the graded tensor product $\lambda_V \tilde{\otimes} \lambda_W$, which may be viewed as a $C_{V \oplus W}$ module using the canonical isomorphism of algebras $C_{V \oplus W} \to C_V \tilde{\otimes} C_W$ (cf. [**7**, §6, 10, 11]). By a dimension argument, $\lambda_V \tilde{\otimes} \lambda_W$ is irreducible, and since ω acts on it by the scalar $+1$, it is isomorphic as a $C_{V \oplus W}$ module to $\lambda_{V \oplus W}$. $\square$

We next show that *Spin* bundles have canonical KO-orientations. This is most easily seen as a consequence of equivariant Bott periodicity. Let $V \in \mathcal{I}'$. Associated to the projection $\pi : G \times Spin(V) \to Spin(V)$ is a Bott class b^{V_π} in $\tilde{K}O_{G \times Spin(V)}(S^{V_\pi})$. If $\xi : P \to B$ is a principal $(G, Spin(V))$-bundle, then by Bott periodicity [**6**, 6.1], multiplication by $[b^{V_\pi}]$ induces an isomorphism

$$KO_{G \times Spin(V)}(P) \to \tilde{K}O_{G \times Spin(V)}(P_+ \wedge S^{V_\pi}).$$

But P has a free $Spin(V)$-action with quotient B, and $P_+ \wedge S^{V_\pi}$ has a $Spin(V)$-action, free away from the basepoint, with quotient $T\xi$. So, the above map induces an isomorphism

$$\Phi^\xi : KO_G(B) \to \tilde{K}O_G(T\xi),$$

natural in $(G, Spin(V))$-bundles.

Definition 5.2. For any $(G, Spin(V))$-bundle $\xi : P \to B$, let $\mu^\xi = \Phi^\xi(1)$.

To show that μ^ξ is an orientation of ξ, we will need the following lemma.

Lemma 5.3. $\Phi^{p_\lambda}(1) = [b^{V_\lambda}]$.

Proof. Let $\lambda' : H \to G \times Spin(V)$ be given by $\lambda'(h) = (h, \lambda(h))$. The element $[1] \otimes [b^{V_\pi}]$ is mapped under the two composites of the commutative diagram below to the cited elements. We write $\Pi(V)$ for $Spin(V)$ in the following diagram.

$$\begin{tikzcd}
KO_{G\times\Pi(V)}(G \times_H \Pi(V)_\lambda) \otimes \tilde{K}O_{G\times\Pi(V)}(S^{V_\pi}) \arrow[r, "(\lambda')^*"] \arrow[d, "\otimes"] & KO_H(*) \otimes \tilde{K}O_H(S^{V_\lambda}) \arrow[d, "\otimes"] \\
\tilde{K}O_{G\times\Pi(V)}((G \times_H \Pi(V)_\lambda)_+ \wedge S^{V_\pi}) \arrow[r, "(\lambda')^*"] & \tilde{K}O_H(S^{V_\lambda}).
\end{tikzcd}$$

□

Proposition 5.4. *μ^ξ is a KO-orientation of ξ.*

Proof. Any $x \in B^H$ determines an orbit inclusion $x : G/H \to B$, and ξ restricts to p_λ over G/H, for some $\lambda \in \Lambda_H^{Spin}(V)$. By naturality, μ^ξ restricts to μ^{p_λ}, which is b^{V_λ} by Lemma 5.3. The result follows by Definitions 4.2 and 4.4(1). □

Equivariant complex K-theory is also periodic. As in the nonequivariant setting, it is often simplest to perform computations in K-theory, and then deduce results in KO-theory. To this end, we will need a formal comparison between real and complex Bott classes.

First, let us recall the construction of the complex Bott classes. Given a complex inner product space V of complex dimension n, we have a $U(V)$-equivariant sequence of bundles over $B(V)$, acyclic over $S(V)$:

$$B(V) \times \Lambda^0(V) \xrightarrow{d^0} B(V) \times \Lambda^1(V) \xrightarrow{d_1} \cdots \xrightarrow{d^{n-1}} B(V) \times \Lambda^n(V).$$

Here $\Lambda^i(V)$ is the ith exterior power of V, and $d^i(v, w) = (v, v \wedge w)$. Such a sequence determines a class $b_c^V \in \tilde{K}_{U(V)}(S^V)$ whose restriction under $\lambda : H \to U(V)$ is the complex Bott class $b_c^{V_\lambda}$, a generator of the free $R(H)$-module $\tilde{K}_H(S^{V_\lambda})$ (see [**6**, §4]).

Just as real Bott classes yield KO-orientations for $Spin$ bundles, so too complex Bott classes endow unitary bundles with K-orientations. If V is a complex inner product space, then there is a complex Bott class $b_c^{V_\pi}$ associated to the projection $\pi : G \times U(V) \to U(V)$. If $\zeta : P \to B$ is a $(G, U(V))$-bundle, then, as above, multiplication by $b_c^{V_\pi}$ induces an isomorphism $\Phi_c^\zeta : K_G(B) \to \tilde{K}_G(T\zeta)$. We define μ_c^ζ to be $\Phi_c^\zeta(1)$. Then μ_c^ζ is a K-orientation of ζ.

Now suppose that $V \in \mathcal{I}'$, meaning the complex dimension of V is divisible by 4. We can associate to any $\lambda : G \to SU(V)$ a complex Bott class $b_c^{V_\lambda}$ and a real Bott class b^{V_λ}, where we use λ to denote both compositions $G \to SU(V) \subseteq U(V)$ and $G \to SU(V) \subseteq Spin(V)$. We aim to prove that the complexification of b^{V_λ} is $b_c^{V_\lambda}$. We will need the following construction.

Construction 5.5. Within the group of units of the complex Clifford algebra $C_V \otimes \mathbb{C}$ sits a subgroup $Spin^c(V)$, isomorphic to $Spin(V) \times_{\mathbb{Z}/2} U(1)$ (see [**7**]). Let $h : Spin(V) \to Spin^c(V)$ be the inclusion, and let $l : U(V) \to SO(V) \times U(1)$ be the product of the standard inclusion and the determinant. Let

$$\pi : Spin^c(V) \cong Spin(V) \times_{\mathbb{Z}/2} U(1) \to SO(V) \times U(1)$$

be the $\mathbb{Z}/2$-covering given by $\pi(f, z) = (\sigma(f), z^2)$, where σ is the projection of $Spin(V)$ on $SO(V)$. By elementary covering space theory, there is a map $\tilde{l} : U(V) \to Spin^c(V)$ lifting l, and the following diagram commutes:

$$\begin{tikzcd}
SU(V) \arrow[r, "i"] \arrow[d] & Spin(V) \arrow[d, "h"] \\
U(V) \arrow[r, "\tilde{l}"] & Spin^c(V).
\end{tikzcd}$$

Let c denote complexification. We now have a commutative diagram

$$\begin{tikzcd}
\tilde{K}O_{Spin^c(V)}(S^V) \arrow[rr, "\tilde{l}^*"] \arrow[dr, "c"] \arrow[dd, "h^*"'] & & \tilde{K}O_{U(V)}(S^V) \arrow[dr, "c"] \arrow[dd] & \\
& \tilde{K}_{Spin^c(V)}(S^V) \arrow[rr, "\tilde{l}^*"] \arrow[dd, "h^*"'] & & \tilde{K}_{U(V)}(S^V) \arrow[dd] \\
\tilde{K}O_{Spin(V)}(S^V) \arrow[rr, "i^*"] \arrow[dr, "c"] & & \tilde{K}O_{SU(V)}(S^V) \arrow[dr, "c"] & \\
& \tilde{K}_{Spin(V)}(S^V) \arrow[rr, "i^*"] & & \tilde{K}_{SU(V)}(S^V).
\end{tikzcd}$$

Lemma 5.6. *There exists an element $\beta \in \tilde{K}_{Spin^c(V)}(S^V)$ satisfying $h^*(\beta) = c(b^V)$ and $\tilde{l}^*(\beta) = b_c^V$.*

Proof. Let $\lambda_V^c = \lambda_V \otimes \mathbb{C}$. This can be considered as a $C_V \otimes \mathbb{C}$-module or as the complexification of a C_V-module. As such, it has a $Spin(V)$ and a $Spin^c(V)$-action, compatible under the inclusion h. Consider the following map of trivial bundles over $B(V)$, restricting to an isomorphism over $S(V)$:

$$B(V) \times (\lambda_V^c)^1 \to B(V) \times (\lambda_V^c)^0$$

$$(v, \lambda) \rightarrowtail (v, v \cdot \lambda).$$

This map can be viewed as a map of $Spin^c(V)$-equivariant complex bundles, thus determining a class β in $\tilde{K}_{Spin^c(V)}(S^V)$. It can also be viewed as the complexification of the map of $Spin(V)$-bundles determining the class b^V, and thus determines the class $c(b^V)$ in $\tilde{K}_{Spin(V)}(S^V)$. Therefore, $c(b^V) = h^*(\beta)$.

Next, there is an alternative construction for the complex Bott class b_c^V, presented in [**7**, §11]. Let $\Lambda^*(V) = \oplus_{j=0}^n \Lambda^j(V)$ be the exterior algebra of V, with $\mathbb{Z}/2$ grading given by $\Lambda(V)^0 = \oplus\Lambda^{2j}(V)$ and $\Lambda(V)^1 = \oplus\Lambda^{2j+1}(V)$ and with inner product induced by that on V. For $v \in V$, let $d_v : \Lambda^*(V) \to \Lambda^*(V)$ be given by $d_v(w) = v \wedge w$, and let δ_v be its adjoint. The maps

$$V \otimes_{\mathbb{R}} \Lambda^*(V) \to \Lambda^*(V)$$

$$v \otimes w \rightarrowtail (d_v - \delta_v)(w)$$

make $\Lambda^*(V)$ into a graded module over $C_V \otimes \mathbb{C}$ (see [**7**, 5.10]). Now consider the following map of trivial complex bundles over $B(V)$, restricting to an isomorphism over $S(V)$,

$$B(V) \times \Lambda(V)^1 \to B(V) \times \Lambda(V)^0$$

$$(v, w) \rightarrowtail (v, (d_v - \delta_v)(w)).$$

On the one hand, we may view this as a map of $Spin^c(V)$-bundles. A dimension count shows that $\Lambda^*(V)$ is irreducible, and it is easy to check that the element ω acts by the scalar $+1$ ([**7**, 5.11]), so that $\Lambda^*(V)$ is isomorphic to λ_V^c as a graded $(C_V \otimes \mathbb{C})$-module. Thus, the map above determines the element β. On the other hand, by restricting the $Spin^c(V)$-action along $\tilde{l}$ to give a $U(V)$-action (which coincides with the usual $U(V)$-action, by [**7**, §5]), we have a map of $U(V)$-equivariant bundles. This map determines the class $b_c^V \in \tilde{K}_{U(V)}(S^V)$ (see [**7**, §9, 11.6]). Therefore, $\tilde{l}^*(\beta) = b_c^V$, as needed. □

From this and an easy diagram chase, we have the following proposition.

Proposition 5.7. *The complexification of b^{V_λ} is $b_c^{V_\lambda}$.*

Corollary 5.8. *Suppose V has complex dimension divisible by 4, and ζ is a $(G, SU(V))$-bundle with underlying $(G, Spin(V))$-bundle $r\zeta$. Then $c\mu^{r\zeta} = \mu_c^\zeta$.*

6. Change of Fiber

In Section 7, we will define a map $\hat{k}_p : B_G(\mathbb{S}, E) \to B_G(\hat{\mathbb{S}}_p, \hat{E}_p)$, which will, by appeal to the classification theorem, give a construction for fiberwise completion of E-oriented equivariant spherical fibrations. Some of the ideas behind this construction are inspired by [**20**], in which fiberwise completions of nonequivariant fibrations are constructed. Moreover, we will need these fiberwise completions in Section 9, since the equivariant Adams conjecture, in its most natural interpretation, involves the fiberwise completion of equivariant spherical fibrations. But to prepare lay the groundwork for this construction, we work more generally in this section, and construct maps of classifying spaces associated to general "change of fiber," of which completion is an example.

The general idea is as follows. Suppose that for each fiber F^1_λ in a set of fibers $\mathbb{F}^1$, we have a fiber F^2_λ in another set of fibers $\mathbb{F}^2$, and an H_λ-map $l_\lambda : F^1_\lambda \to F^2_\lambda$. Suppose also that the restriction map $l^*_\lambda : F(F^2_\lambda, F^2_\mu) \to F(F^1_\lambda, F^2_\mu)$ is a weak H_λ-equivalence, so that, in particular, any H_λ-map $F^1_\lambda \to F^1_\mu$ gives rise to an H_λ-map $F^2_\lambda \to F^2_\mu$. Then, we can define a category $\mathcal{B}(\mathbb{F}^{12})$ together with functors $\Phi^i : \mathcal{B}(\mathbb{F}^{12}) \to \mathcal{A}(\mathbb{F}^i)$ $(i = 1, 2)$, so that Φ^1 is a weak equivalence on all

morphism spaces and a bijection on objects. Taking classifying spaces and inverting the weak equivalence will then yield a map $B\mathcal{A}(\mathbb{F}^1) \to B\mathcal{A}(\mathbb{F}^2)$.

The objects in $\mathcal{B}(\mathbb{F}^{12})$ will be *cofibration cubes* of based G-spaces. We proceed therefore to a discussion of these, listing some of their useful properties.

Given a finite set S, let $\mathcal{P}(S)$ be the poset of subsets of S. An S-cube of based spaces is a functor $\mathcal{X} : \mathcal{P}(S) \to \mathcal{T}$. If $S = \{1, 2, \cdots, n\}$, then an S-cube is sometimes called an n-cube. If Z is a based space, $\mathcal{X} \wedge Z$ is the S-cube obtained by smashing each space $\mathcal{X}(U)$ with Z. A map of S-cubes is a natural transformation. Two maps $f, g : \mathcal{X} \to \mathcal{Y}$ are homotopic if there is a map $h : \mathcal{X} \wedge I_+ \to \mathcal{Y}$ restricting to f and g on the endpoints of I. A map $f : \mathcal{X} \to \mathcal{Y}$ is a homotopy equivalence if there exists a homotopy inverse $g : \mathcal{Y} \to \mathcal{X}$, and a level equivalence if $f_U : \mathcal{X}(U) \to \mathcal{Y}(U)$ is a homotopy equivalence for each $U \subseteq S$.

A map $f : \mathcal{X} \to \mathcal{Y}$ of S-cubes can be thought of as an $S \cup \{t\}$-cube $(\mathcal{X} \to \mathcal{Y})$. Namely, if $t \in U$, let $(\mathcal{X} \to \mathcal{Y})(U) = \mathcal{Y}(U - \{t\})$; otherwise, let $(\mathcal{X} \to \mathcal{Y})(U) = \mathcal{X}(U)$. The maps f_U then determine the structure maps of $\mathcal{X} \to \mathcal{Y}$.

The set of all maps $f : \mathcal{X} \to \mathcal{Y}$, denoted $F_S(\mathcal{X}, \mathcal{Y})$, is contained in the product over all $U \subseteq S$ of the spaces $F(\mathcal{X}(U), \mathcal{Y}(U))$, and inherits the subspace topology.

For $U \subset T \subset S$, denote the $T - U$-cube $\{V \to \mathcal{X}(V \cup U) : V \subset T - U\}$ by $\partial^T_U \mathcal{X}$. This is a *face* of $\mathcal{X}$. Denote $\partial^T_\emptyset \mathcal{X}$ by $\partial^T \mathcal{X}$ and $\partial^S_U \mathcal{X}$ by $\partial_U \mathcal{X}$.

We say that $\mathcal{X}$ is a *cofibration cube* if for every $T \subset S$, the map

$$\operatorname{colim}\{\mathcal{X}(U) : U \subsetneq T\} \to \mathcal{X}(T)$$

is a cofibration. A map $f : \mathcal{X} \to \mathcal{Y}$ of cofibration S-cubes is a *cofibration* if $(\mathcal{X} \to \mathcal{Y})$ is a cofibration cube. It's easy to check by induction that a cofibration satisfies the obvious version of the homotopy lifting property. Lemma 6.1 below follows from the standard adjunction, and Lemma 6.2 by induction on the size of the cubes.

Lemma 6.1. *If $f : \mathcal{X} \to \mathcal{Y}$ is a cofibration, then for any S-cube $\mathcal{Z}$, the restriction map $f^* : F_S(\mathcal{Y}, \mathcal{Z}) \to F_S(\mathcal{X}, \mathcal{Z})$ is a fibration.*

Lemma 6.2. *A cofibration that is a level equivalence is a homotopy equivalence.*

Now suppose S is ordered. A cofibration S-cube $\mathcal{X}$ is *monotonic* if each map $\mathcal{X}(T) \to \mathcal{X}(U)$ induced by an inclusion $T \subseteq U$ with $max(T) = max(U)$ is a homotopy equivalence; here $max(T)$ is the largest element of $T \subseteq S$. We say $\mathcal{X}$ is *strongly monotonic* if *all* maps $\mathcal{X}(T) \to \mathcal{X}(U)$ induced by inclusions $T \subseteq U$ are homotopy equivalences. We observe that if $s = max(S)$ and $\mathcal{X}$ is monotonic, then $\partial_{\{s\}} X$ is strongly monotonic.

Lemma 6.3. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are cofibration S-cubes, with $\mathcal{Y}$ strongly monotonic. The map $F_S(\mathcal{X},\mathcal{Y}) \to F(\mathcal{X}(S),\mathcal{Y}(S))$ $(f \to f_S)$ is a homotopy equivalence.*

Proof. Let $T = S - \{s\}$ for some $s \in S$. Let $i_{\mathcal{X}} : \partial^T\mathcal{X} \to \partial_{\{s\}}\mathcal{X}$ be the evident cofibration of T-cubes, and similarly define $i_{\mathcal{Y}}$. Observe that the following is a pull-back diagram:

$$\begin{tikzcd} F_S(\mathcal{X},\mathcal{Y}) \arrow[r, "\rho"] \arrow[d] & F_T(\partial_{\{s\}}\mathcal{X},\partial_{\{s\}}\mathcal{Y}) \arrow[d, "i_{\mathcal{X}}^*"] \\ F_T(\partial^T\mathcal{X},\partial^T\mathcal{Y}) \arrow[r, "i_{\mathcal{Y}*}"'] & F_T(\partial^T\mathcal{X},\partial_{\{s\}}\mathcal{Y}). \end{tikzcd}$$

By 6.1, $i_{\mathcal{X}}^*$ is a fibration. By 6.2, $i_{\mathcal{Y}*}$ is a homotopy equivalence. So, the restriction ρ is an equivalence, and the lemma follows by induction on the size of S. □

Proposition 6.4. *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are cofibration S-cubes, with $\mathcal{Y}$ monotonic. Let $s = max(S)$, and let $T = S - \{s\}$. If the map*

$$F(\mathcal{X}(S),\mathcal{Y}(S)) \to F(\mathcal{X}(T),\mathcal{Y}(S))$$

induced by $\mathcal{X}(T) \to \mathcal{X}(S)$ is a weak equivalence, then the restriction map

$$F_S(\mathcal{X},\mathcal{Y}) \to F_T(\partial^T\mathcal{X},\partial^T\mathcal{Y})$$

is a fibration and a weak equivalence.

Proof. Recycling notation from 6.3, consider the following commutative diagram:

$$\begin{tikzcd} F_S(\mathcal{X},\mathcal{Y}) \arrow[r] \arrow[d] & F_T(\partial_{\{s\}}\mathcal{X},\partial_{\{s\}}\mathcal{Y}) \arrow[r] \arrow[d, "i_{\mathcal{X}}^*"] & F(\mathcal{X}(S),\mathcal{Y}(S)) \arrow[d] \\ F_T(\partial^T\mathcal{X},\partial^T\mathcal{Y}) \arrow[r, "i_{\mathcal{Y}*}"'] & F_T(\partial^T\mathcal{X},\partial_{\{s\}}\mathcal{Y}) \arrow[r] & F(\mathcal{X}(T),\mathcal{Y}(S)). \end{tikzcd}$$

The left square is a pull-back square. The map $i_{\mathcal{X}}^*$ is a fibration by 6.1. Using the hypothesis and Lemma 6.3, we see that $i_{\mathcal{X}}^*$ is also a weak equivalence, which yields the proposition. □

Remark 6.5. Let an S-cube of based G-spaces be a functor $\mathcal{P}(S) \to G\mathcal{T}$ for some group G. The equivariant analogue of Proposition 6.4 follows by a similar proof.

We next turn to defining maps of fibers and orientation functors, and constructing the associated maps of classifying spaces. We will give maps of the triples used in defining the classifying spaces, and then apply the bar construction.

Definition 6.6. If $\mathbb{F}^1$ and $\mathbb{F}^2$ are sets of fibers indexed on Λ^1 and Λ^2, a *map of fibers* $l : \mathbb{F}^1 \to \mathbb{F}^2$ is a map of sets $\lambda \to \lambda'$ (where $H_\lambda = H_{\lambda'}$ for each $\lambda \in \Lambda^1$), along with a set of equivariant maps, $l_\lambda : F^1_\lambda \to F^2_{\lambda'}$. We will often suppress the prime in the notation. If Y^i is a Z^i orientation functor on $\mathbb{F}^i$ $(i = 1, 2)$, then a map of orientation functors $l_Z : Y^1 \to Y^2$ is a map $l_Z : Z^1 \to Z^2$.

For each $\lambda \in \Lambda^1$, let F_λ^{12} be the reduced mapping cylinder of l_λ, let $F_\lambda^\emptyset$ be a point, and let $\alpha_\lambda : F_\lambda^1 \to F_\lambda^{12}$ and $\beta_\lambda : F_\lambda^2 \to F_\lambda^{12}$ be the evident cofibrations. Then we have monotonic cofibration 2-cubes $\mathcal{F}_\lambda^{12}$ of H_λ-spaces

$$\begin{tikzcd} F_\lambda^\emptyset \arrow[r] \arrow[d] & F_\lambda^1 \arrow[d, "\alpha_\lambda"] \\ F_\lambda^2 \arrow[r, "\beta_\lambda"'] & F_\lambda^{12}. \end{tikzcd}$$

When $g^{-1}Kg \leqslant H_\lambda$, we write $g^*\mathcal{F}_\lambda^{12}$ for the cube of K-spaces obtained by applying g^* to each space in the cube. Likewise, let Z^{12} be the reduced mapping cylinder of l_Z, let $Z^\emptyset$ be a point, and let $\alpha_Z : Z^1 \to Z^{12}$ and $\beta_Z : Z^2 \to Z^{12}$ be the evident cofibrations. Then we have a monotonic cofibration 2-cube $\mathcal{Z}^{12}$ of G-spaces as above.

Let $\mathcal{B}(\mathbb{F}^{12})$ be the category whose objects are the cofibration 2-cubes $\mathcal{F}_\lambda^{12}$, and whose maps $\theta : \mathcal{F}_\lambda^{12} \to \mathcal{F}_\mu^{12}$ are equivalence classes $[g, \bar\theta]$, where $g^{-1}H_\lambda g \leqslant H_\mu$ and $\bar\theta : \mathcal{F}_\lambda^{12} \to g^*\mathcal{F}_\mu^{12}$ is a weak equivalence of cofibration 2-cubes of H_λ-spaces, and we identify $[gk, \bar\theta]$ with $[g, k\bar\theta]$ for $k \in H_\mu$ (cf 2.3). Let $Y^{12}(\mathcal{F}_\lambda^{12})$ be the space of all maps of cofibration 2-cubes of H_λ-spaces $\theta : \mathcal{F}_\lambda^{12} \to \mathcal{Z}^{12}$ such that $\theta_{\{1\}} \in Y^1(\lambda)$ and $\theta_{\{2\}} \in Y^2(\lambda)$. This determines a functor $Y^{12} : \mathcal{B}(\mathbb{F}^{12}) \to \mathcal{U}$. Finally, let $\mathcal{O} : \mathcal{B}(\mathbb{F}^{12}) \to G\mathcal{U}$ be the functor taking $\mathcal{F}_\lambda^{12}$ to G/H_λ and taking a map $(g, \bar\theta)$ to left multiplication by g.

For $i = 1, 2$, we have functors $\Phi^i : \mathcal{B}(\mathbb{F}^{12}) \to \mathcal{A}(\mathbb{F}^i)$ taking the object $\mathcal{F}_\lambda^{12}$ to p_λ^i, and taking a map $[g, \bar\theta]$ to $[g, \bar\theta_{\{i\}}]$. We also have natural transformations $\chi^i : Y^{12} \to Y^i \circ \Phi^i$ taking θ to $\theta_{\{i\}}$. Finally, note that $\mathcal{O} \circ \Phi^i = \mathcal{O}$. We want Φ^1 to give a weak equivalence on all morphism spaces, and χ^1 to give a weak equivalence on all objects in $\mathcal{B}(\mathbb{F}^{12})$. For this, we need the following condition.

Definition 6.7. A map of fibers l is *good* if for any pair $\lambda, \mu \in \Lambda^1$ with $H_\lambda = H_\mu$, the restriction map $l_\lambda^* : F(F_\lambda^2, F_\mu^2) \to F(F_\lambda^1, F_\mu^2)$ is a weak H_λ-equivalence. A map of orientation functors $l_Z : Z^1 \to Z^2$ is *good* if, for any $\lambda \in \Lambda^1$, the restriction map $l_\lambda^* : F(F_\lambda^2, Z^2) \to F(F_\lambda^1, Z^2)$ is a weak H_λ-equivalence and the map

$$[l_\lambda^*]^{-1} \circ [l_{Z*}] : [F_\lambda^1, Z^1]_{*H_\lambda} \to [F_\lambda^2, Z^2]_{*H_\lambda}$$

takes the components of $Y^1(\lambda)$ to components in $Y^2(\lambda)$.

Lemma 6.8. *If l and l_Z are good, then Φ^1 is a weak equivalence and fibration on all morphism spaces, and χ^1 is a weak equivalence for all objects in $\mathcal{B}(\mathbb{F}^{12})$.*

Proof. The following homotopy commutative diagram, in which all horizontal arrows are equivalences by monotonicity and the right arrow is an equivalence by hypothesis, implies that we may apply proposition 6.4 to attain the first statement.

$$\begin{tikzcd} F(F_\lambda^{12}, F_\mu^{12}) \arrow[r, "\simeq"] \arrow[d] & F(F_\lambda^2, F_\mu^{12}) & F(F_\lambda^2, F_\mu^2) \arrow[l, "\simeq"'] \arrow[d, "\simeq"] \\ F(F_\lambda^1, F_\mu^{12}) & & F(F_\lambda^1, F_\mu^2). \arrow[ll, "\simeq"] \end{tikzcd}$$

By the same argument, the space of maps from $\mathcal{F}^{12}_\lambda$ to $\mathcal{Z}^{12}$ is weakly H_λ-equivalent (by restriction) to the space of maps from F^1_λ to Z^1. Since $[l^*_\lambda]^{-1} \circ [l_Z]$ takes components of $Y^1(\lambda)$ to components in $Y^2(\lambda)$, it follows that any map from $\mathcal{F}^{12}_\lambda$ to $\mathcal{Z}^{12}$ restricting to a map in $Y^1(\lambda)$ must itself be in $Y^{12}(\mathcal{F}^{12}_\lambda)$, so that $Y^{12}(\mathcal{F}^{12}_\lambda)$ restricts by a weak H_λ-equivalence to $Y^1(\lambda)$. □

It follows that if $l : \mathbb{F}^1 \to \mathbb{F}^2$ and $l_Z : Y^1 \to Y^2$ are good, then the map

$$(\Phi^1, \chi^1)_* : B(Y^{12}, \mathcal{B}(\mathbb{F}^{12}), \mathcal{O}) \to B(Y^1, \mathcal{A}(\mathbb{F}^1), \mathcal{O})$$

is an equivalence.

Definition 6.9. The *change of fiber map* l_* associated to a pair (l, l_Z) as above is the map obtained by inverting the equivalence in the diagram below:

$$B(Y^1, \mathcal{A}(\mathbb{F}^1), \mathcal{O}) \xleftarrow[\simeq]{(\Phi^1,\chi^1)_*} B(Y^{12}, \mathcal{B}(\mathbb{F}^{12}), \mathcal{O}) \xrightarrow{(\Phi^2,\chi^2)_*} B(Y^2, \mathcal{A}(\mathbb{F}^2), \mathcal{O}).$$

The next proposition shows functoriality of change of fiber.

Proposition 6.10. *Suppose $l : \mathbb{F}^1 \to \mathbb{F}^2, k : \mathbb{F}^2 \to \mathbb{F}^3$ and $k \circ l : \mathbb{F}^1 \to \mathbb{F}^3$ are good maps of fibers, and $l_Z : Y^1 \to Y^2, k_Z : Y^2 \to Y^3$, and $k_Z \circ l_Z : Y^1 \to Y^3$ are good maps of orientiation functors. Then $k_* \circ l_*$ is homotopic to $(k \circ l)_*$.*

Proof. For each $\lambda \in \Lambda^1$, let F^{123}_λ be the reduced mapping cylinder of the map $F^{12}_\lambda \to F^3_\lambda$, given by composing the projection $F^{12}_\lambda \to F^2_\lambda$ with k_λ. The cofibrations from F^1_λ and F^2_λ to F^{12}_λ induce cofibrations from F^{13}_λ and F^{23}_λ to F^{123}_λ. Now the spaces F^U_λ for $U \subseteq \{1, 2, 3\}$ determine a monotonic cofibration 3-cube, $\mathcal{F}^{123}_\lambda$. We define the cofibration cube $\mathcal{Z}^{123}$ similarly. Also, we define the category $\mathcal{C}(\mathbb{F}^{123})$ and the functors $Y^{123} : \mathcal{C}(\mathbb{F}^{123}) \to \mathcal{U}$ in analogy with $\mathcal{B}(\mathbb{F}^{12})$ and Y^{12}.

Restriction of 3-cubes to faces yields functors Φ^{12}, Φ^{13}, and Φ^{23} from $\mathcal{C}(\mathbb{F}^{123})$ to $\mathcal{B}(\mathbb{F}^{12}), \mathcal{B}(\mathbb{F}^{13})$, and $\mathcal{B}(\mathbb{F}^{23})$ as well as natural transformations χ^{12}, χ^{13}, and χ^{23} from Y^{123} to Y^{12}, Y^{13}, and Y^{23}. The argument used in Lemma 6.8 shows that Φ^{12} is a weak equivalence and fibration on all morphism spaces, and χ^{12} is a weak equivalence for all objects in $\mathcal{B}(\mathbb{F}^{12})$. Now the result follows by chasing the following commutative diagram

$$\begin{tikzcd}
 & & B(Y^{13}, \mathcal{B}(\mathbb{F}^{13}), \mathcal{O}) \arrow[ddll, bend right, "\simeq"'] \arrow[ddrr, bend left] & & \\
 & & B(Y^{123}, \mathcal{C}(\mathbb{F}^{123}), \mathcal{O}) \arrow[u, "\simeq"'] \arrow[dl, "\simeq"'] \arrow[dr] & & \\
 & B(Y^{12}, \mathcal{B}(\mathbb{F}^{12}), \mathcal{O}) \arrow[dl, "\simeq"'] \arrow[dr] & & B(Y^{23}, \mathcal{B}(\mathbb{F}^{23}), \mathcal{O}) \arrow[dl, "\simeq"'] \arrow[dr] & \\
B(Y^1, \mathcal{A}(\mathbb{F}^1), \mathcal{O}) & & B(Y^2, \mathcal{A}(\mathbb{F}^2), \mathcal{O}) & & B(Y^3, \mathcal{A}(\mathbb{F}^3), \mathcal{O}).
\end{tikzcd}$$

□

We can use this result to show that change of fiber is homotopy invariant. In the proposition below, if $\mathbb{F}^1$ is a set of fibers, then $\mathbb{F}^1 \wedge I_+$ denotes the set $\{F^1_\lambda \wedge I_+\}$.

For $t \in I$, we have maps of fibers $i_t : \mathbb{F}^1 \to \mathbb{F}^1 \wedge I_+$. Similarly, if Y^1 is a Z^1-orientation functor on $\mathbb{F}^1$, then $(Y^1 \wedge I_+)(\lambda)$ denotes the space of all H_λ-maps $G : F^1_\lambda \wedge I_+ \to Z^1 \wedge I_+$ such that $G \circ i_0$ is homotopic to $i_0 \circ g$ for some H_λ-map $g : F^1_\lambda \to Z^1$ in $Y^1(\lambda)$. This determines a $Z^1 \wedge I_+$-orientation functor on $\mathbb{F}^1 \wedge I_+$, and we have good maps of orientation functors $(i_Z)_t : Y^1 \to Y^1 \wedge I_+$.

Proposition 6.11. *Suppose that $l : \mathbb{F}^1 \wedge I_+ \to \mathbb{F}^3$ is a good map of fibers, and $l_Z : Y^1 \wedge I_+ \to Y^3$ is a good map of orientation functors. Then the change of fiber maps associated to the pairs $(l \circ i_0, l_Z \circ i_{Z0})$ and $(l \circ i_1, l_Z \circ i_{Z1})$ are homotopic.*

Proof. Let $\mathbb{F}^2 = \mathbb{F}^1 \wedge I_+$ and let $Y^2 = Y^1 \wedge I_+$. By functoriality (6.10), it suffices to show that $(i_0)_*$ is homotopic to $(i_1)_*$. To see this, we construct a map

$$i_* : B(Y^1, \mathcal{A}(\mathbb{F}^1), \mathcal{O}) \to B(Y^2, \mathcal{A}(\mathbb{F}^2), \mathcal{O})$$

which we claim is homotopic to $(i_0)_*$ and $(i_1)_*$. Let $\Phi : \mathcal{A}(\mathbb{F}^1) \to \mathcal{A}(\mathbb{F}^2)$ be the functor taking λ to λ' and extending maps by the identity along the interval. Let $\chi : Y^1 \to Y^2 \circ \Phi$ be the natural transformation extending maps by the identity along the interval. Let i_* be induced by the pair (Φ, χ).

We show that the functors $\Phi^2, \Phi \circ \Phi^1 : \mathcal{B}(\mathbb{F}^{12}) \to \mathcal{A}(\mathbb{F}^2)$, which coincide on objects, are homotopic on morphism spaces. A similar argument shows that the natural transformations $\chi^2 : Y^{12} \to Y^2 \circ \Phi^2$ and $\chi \circ \chi^1 : Y^{12} \to Y^2 \circ \Phi \circ \Phi^1$, whose source and target coincide on objects, are homotopic on all such objects. Thus, $i_* \sim (i_1)_*$, and a similar argument shows $i_* \sim (i_0)_*$.

Since $F^2_\lambda = F^1_\lambda \wedge [0,1]_+$, we have $F^{12}_\lambda = F^{12}_\lambda \wedge [0,2]_+$. Let $\mu_t(s) = (2-s)t + s$, let $\nu_t(s) = (s-1)(t-1)+1$, and let $\xi(s) = min(1,s)$. Given a map $\bar{\theta} : \mathcal{F}^{12}_\lambda \to g^*\mathcal{F}^{12}_\mu$, we have maps

$$F^2_\lambda = F^1_\lambda \wedge [0,1]_+ \to g^* F^1_\mu \wedge [0,1]_+ = g^* F^2_\mu$$

given by $(\mathrm{id} \times \xi) \circ \bar{\theta}_{\{1,2\}} \circ (\mathrm{id} \times \mu_t)$ for $0 \leqslant t \leqslant 1$ and $\bar{\theta}_{\{1\}} \times \nu_t$ for $1 \leqslant t \leqslant 2$. This gives the necessary homotopy

$$\Phi^2 \simeq \Phi \circ \Phi^1 : \mathcal{B}(\mathbb{F}^{12})(\lambda, \mu) \to \mathcal{A}(\mathbb{F}^2)(\lambda, \mu).$$

□

7. Fiberwise Completion

We now specialize the results of the previous section to completions. Recall from [**21**] that a nilpotent space X is *p-complete* if $\pi_n(X^H)$ is p-complete for all $H \leqslant G$, $n \geqslant 1$, and that completion is a map γ from X to a p-complete space Y which is initial (in the homotopy category) among all such maps. If $\mathbb{F}$ is a set of fibers such that each F_λ is a nilpotent H_λ-space, we let $\hat{\mathbb{F}}_p$ be the set of fibers $\{\hat{F}_{\lambda p}\}$.

When $\mathbb{F} = \mathbb{S}^V$, the spaces S^{V_λ} are not all nilpotent H_λ-spaces, as they are not all G-connected. We therefore let $\tilde{\Lambda}^\Pi_H(V) \subset \Lambda^\Pi_H(V)$ be the subset of all $\lambda : H \to \Pi(V)$ such that V_λ has a trivial summand, and we let $\tilde{\Lambda}^\Pi(V)$ be the union over all $H \leqslant G$ of the sets $\tilde{\Lambda}^\Pi_H(V)$. Let $\tilde{\mathbb{S}}^V$ denote the corresponding set of fibers. When $\lambda \in \tilde{\Lambda}^\Pi(V)$, all the fixed point spaces of S^{V_λ} are spheres of dimension at least 1 and hence nilpotent. Therefore, we can complete each fiber to obtain a set of fibers $\hat{\mathbb{S}}^V_p$, and hence a classifying space $B_G(\hat{\mathbb{S}}^V_p)$ for $G\mathcal{F}(\hat{\mathbb{S}}^V_p)$-fibrations.

Remark 7.1. The inclusion $B_G(\tilde{\mathbb{S}}^V) \to B_G(\mathbb{S}^V)$ is not in general an equivalence. For example, $\pi_0^G(B_G(\mathbb{S}^V))$ is the set of equivalence classes of elements in $\Lambda_G^\Pi(V)$, where $\lambda \sim \mu$ when $S^{V_\lambda} \simeq_G S^{V_\mu}$, while $\pi_0^G(B_G(\tilde{\mathbb{S}}^V))$ is the corresponding set for elements in $\tilde{\Lambda}_G^\Pi(V)$. This distinction is lost after stabilization.

We can also complete a G-spectrum E, at any ideal I in the Burnside ring $A(G)$ (cf. [**14**]), but we will only be interested in the ideal generated by a prime p. For a finite G-CW-complex X, the $\hat{E}_p$-cohomology of X is the p-completion of the E-cohomology of X; in general we will therefore let $\tilde{E}_G(X)\hat{_p}$ denote the reduced $\hat{E}_p$-cohomology of any space X; this represents an abuse of notation only when X is not finite. Moreover, the G-connected cover of $\Omega^\infty E$, $(\Omega^\infty E)_0$, is p-complete, since all its homotopy groups are complete. Since $[X,Y]_G = [X,Y_0]_G$ for any G-connected space X, it follows easily that the map $k^* : \tilde{E}^*_G(\hat{X}_p) \to \tilde{E}^*_G(X)$ is an equivalence whenever E is p-complete.

Completion at p preserves unital commutative ring spectra. The product structure, for example, is given by

$$\hat{E}_p \wedge \hat{E}_p \xrightarrow{k} (\hat{E}_p \wedge \hat{E}_p)\hat{_p} \xleftarrow{\simeq} (E \wedge E)\hat{_p} \xrightarrow{\mu\hat{_p}} \hat{E}_p \ .$$

If E is periodic, then the image of the elements b^{V_λ} in $\tilde{E}_{H_\lambda}(S^{V_\lambda})\hat{_p}$, denoted $\hat{b}^{V_\lambda}_p$, determine a collection of periodicity classes for $\hat{E}_p$.

Now suppose E is periodic, and let $Z = \Omega^\infty \hat{E}_p$. We can then construct a Z-orientation functor $F^V\hat{E}_p$ on $\hat{\mathbb{S}}^V_p$. As in Construction 4.6, we let $F^V\hat{E}_p(\lambda)$ be the subspace of $F_{H_\lambda}(\hat{S}^{V_\lambda}_p, \Omega^\infty \hat{E}_p)$ consisting of all components of generators in $\tilde{E}^0_{H_\lambda}(\hat{S}^{V_\lambda}_p)\hat{_p}$. An $\hat{E}_p$-orientation of a $G\mathcal{F}(\hat{\mathbb{S}}^V_p)$-fibration can be interpreted as an $F^V\hat{E}_p$-structure. Then $\hat{E}_p$-oriented $G\mathcal{F}(\hat{\mathbb{S}}^V_p)$-fibrations are classified by the space $B_G(\hat{\mathbb{S}}^V_p, F^V\hat{E}_p)$, denoted $B_G(\hat{\mathbb{S}}^V_p, \hat{E}_p)$.

Lemma 7.2. *The completion maps $k_\lambda : S^{V_\lambda} \to \hat{S}^{V_\lambda}_p$ determine a good map of fibers $k : \tilde{\mathbb{S}}^V \to \hat{\mathbb{S}}^V_p$. Similarly, if E is a periodic ring G-spectrum, then the zeroth space of the completion map, $\Omega^\infty E \to \Omega^\infty \hat{E}_p$, determines a good map of orientation functors $k_E : F^V E \to F^V\hat{E}_p$.*

Proof. We show that when $H_\lambda = H_\mu$, the maps

$$k^*_\lambda : F(\hat{S}^{V_\lambda}_p, \hat{S}^{V_\mu}_p) \to F(S^{V_\lambda}, \hat{S}^{V_\mu}_p)$$

yield weak H-equivalences. This follows by definition of completions, which implies that the indicated maps in the diagram below are bijections for any $K \leqslant H$.

$$\begin{tikzcd}
[(S^n \wedge \hat{S}_p^{V_\lambda})\hat{}_p, \hat{S}_p^{V_\mu}]_K \arrow[r, "\cong"] \arrow[d, "\cong"] & [(S^n \wedge S^{V_\lambda})\hat{}_p, \hat{S}_p^{V_\mu}]_K \arrow[d, "\cong"] \\
[S^n \wedge \hat{S}_p^{V_\lambda}, \hat{S}_p^{V_\mu}]_K \arrow[r] & [S^n \wedge S^{V_\lambda}, \hat{S}_p^{V_\mu}]_K.
\end{tikzcd}$$

By a similar argument, $k_E : F^V E \to F^V \hat{E}_p$ is a good map of orientation functors. □

Using Definition 6.9, we have fiberwise completion maps

$$B_G(\tilde{\mathbb{S}}^V; E) \to B_G(\hat{\mathbb{S}}_p^V; \hat{E}_p).$$

We next aim to construct a map for the fiberwise smash product of $\hat{E}_p$-oriented $G\mathcal{F}(\hat{\mathbb{S}}_p^V)$ and $G\mathcal{F}(\hat{\mathbb{S}}_p^W)$-fibrations.

Construction 7.3. Let $\hat{\mathbb{S}}_p^V \wedge \hat{\mathbb{S}}_p^W = \{\hat{S}_p^{V_\lambda} \wedge \hat{S}_p^{W_\mu} : H_\lambda = H_\mu\}$. We denote the corresponding objects in $\mathcal{A}(\hat{\mathbb{S}}_p^V \wedge \hat{\mathbb{S}}_p^W)$ by $\lambda \wedge \mu$. The spaces in this set are not themselves p-complete; their completions are the spaces $\hat{S}_p^{V_\lambda \oplus W_\mu}$. Thus the restriction maps

$$[\hat{S}_p^{V_\lambda \oplus W_\mu}, \Omega^\infty \hat{E}_p]_{H_\lambda} \to [\hat{S}_p^{V_\lambda} \wedge \hat{S}_p^{W_\mu}, \Omega^\infty \hat{E}_p]_{H_\lambda}$$

are bijections. Let $(F^V \hat{E}_p \wedge F^W \hat{E}_p)(\lambda \wedge \mu)$ consist of the components of the image of $F^{V\oplus W}\hat{E}_p(\lambda \oplus \mu)$ under this bijection. Then, $\hat{\mathbb{S}}_p^V \wedge \hat{\mathbb{S}}_p^W \to \hat{\mathbb{S}}_p^{V\oplus W}$ is a good map of fibers, and $F^V \hat{E}_p \wedge F^W \hat{E}_p \to F^{V\oplus W}\hat{E}_p$ is a good map of orientation functors. Definition 6.9 yields

$$k_* : B(F^V \hat{E}_p \wedge F^W \hat{E}_p, \mathcal{A}(\hat{\mathbb{S}}_p^V \wedge \hat{\mathbb{S}}_p^W), \mathcal{O}) \to B(F^{V\oplus W}\hat{E}_p, \mathcal{A}(\hat{\mathbb{S}}_p^{V\oplus W}), \mathcal{O}).$$

Precomposing k_* with the fiberwise smash product map of Construction 3.14 yields

$$B_G(\hat{\mathbb{S}}_p^V, \hat{E}_p) \times B_G(\hat{\mathbb{S}}_p^W, \hat{E}_p) \to B_G(\hat{\mathbb{S}}_p^{V\oplus W}, \hat{E}_p),$$

which we take as a construction on the classifying space level for the fiberwise smash product of $\hat{E}_p$-oriented $G\mathcal{F}(\hat{\mathbb{S}}_p^V)$ and $G\mathcal{F}(\hat{\mathbb{S}}_p^W)$-fibrations.

This construction also allows us to stabilize these fibrations. If $V \in \mathcal{V}$ (2.14), with action map $\rho : G \to SU(V)$, then the category $\mathcal{A}(\hat{\mathbb{S}}_p^V)$ has a distinguished object ρ corresponding to the fiber $\hat{\mathbb{S}}_p^{V_\rho}$. We can choose a distinguished point in $F^W \hat{E}_p(\rho)$ corresponding to the map $\hat{b}_p^{V_\rho}$. These determine a G-fixed basepoint for $B_G(\hat{\mathbb{S}}_p^V; \hat{E}_p)$. These basepoints and the maps above make $B_G(\hat{\mathbb{S}}_p^{(-)}; \hat{E}_p)$ an $h\mathcal{V}$-functor (cf. 2.14).

Definition 7.4. Let $B_G(\hat{\mathbb{S}}_p; \hat{E}_p) = B_G(\hat{\mathbb{S}}_p^{\mathbb{V}}; \hat{E}_p)$ for some complete sequence $\mathcal{V}$.

The next lemma shows that completions are compatible with fiberwise smash products.

Lemma 7.5. *The following diagram commutes up to homotopy.*

$$\begin{tikzcd}
B_G(\mathbb{S}^V;E)\times B_G(\mathbb{S}^W;E) \arrow[r] \arrow[dd] & B_G(\hat{\mathbb{S}}_p^V;\hat{E}_p)\times B_G(\hat{\mathbb{S}}_p^W;\hat{E}_p) \arrow[d] \\
 & B(F^V\hat{E}_p\wedge F^W\hat{E}_p,\mathcal{A}(\hat{\mathbb{S}}_p^V\wedge\hat{\mathbb{S}}_p^W),\mathcal{O}) \arrow[d, "k_*"] \\
B_G(\mathbb{S}^{V\oplus W};E) \arrow[r] & B_G(\hat{\mathbb{S}}_p^{V\oplus W};\hat{E}_p).
\end{tikzcd}$$

Proof. To simplify the argument, we will take all the orientation functors to be trivial. The general argument follows similarly.

Let $l_V:\mathbb{S}^V\to\hat{\mathbb{S}}_p^V$ and $l_W:\mathbb{S}^W\to\hat{\mathbb{S}}_p^W$ be the completion maps. We also have maps of fibers

$$\mathbb{S}^{V\oplus W}\xrightarrow{l}\hat{\mathbb{S}}_p^V\wedge\hat{\mathbb{S}}_p^W\xrightarrow{k}\hat{\mathbb{S}}_p^{V\oplus W}.$$

Note that l is not necessarily a good map of fibers, but k and $k\circ l$ are (being completions). We abbreviate the categories of cofibration 2-cubes by $\mathcal{B}(l_V),\mathcal{B}(l_W)$, etc. From the definitions, the claim quickly reduces to showing that the outer circuit of the following diagram, from top left to bottom, commutes:

$$\begin{tikzcd}
B(*,\mathcal{A}(\mathbb{S}^V\times\mathbb{S}^W),\mathcal{O}) \arrow[d] & B(*,\mathcal{B}(l_V\times l_W),\mathcal{O}) \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & B(*,\mathcal{A}(\hat{\mathbb{S}}_p^V\times\hat{\mathbb{S}}_p^W),\mathcal{O}) \arrow[d] \\
B(*,\mathcal{A}(\mathbb{S}^{V\oplus W}),\mathcal{O}) \arrow[dr, "(k\circ l)_*"'] & B(*,\mathcal{B}(l),\mathcal{O}) \arrow[l, "(\Phi^1)_*"'] \arrow[r, "(\Phi^2)_*"] & B(*,\mathcal{A}(\hat{\mathbb{S}}_p^V\wedge\hat{\mathbb{S}}_p^W),\mathcal{O}) \arrow[dl, "k_*"] \\
 & B(*,\mathcal{A}(\hat{\mathbb{S}}_p^{V\oplus W}),\mathcal{O}). &
\end{tikzcd}$$

Clearly, the top two squares commute. Moreover, while we cannot conclude that $(\Phi^1)_*$ is a weak equivalence, the argument used in Proposition 6.10 still goes through to show that the bottom triangle in the diagram above commutes. Since the top left horizontal arrow is an equivalence, we are done. □

It follows immediately that completion is compatible with stabilization.

Definition 7.6. Let

$$\hat{k}_p:B_G(\mathbb{S};E)\to B_G(\hat{\mathbb{S}}_p;\hat{E}_p)$$

be obtained by taking the colimit of the maps k_* above.

Remark 7.7. Consider again the diagram of Lemma 7.5, taking all orientation functors to be trivial. We could replace $B_G(\hat{\mathbb{S}}_p^V)$ in the upper right with $B_G(\mathbb{S}^V)$, and we could replace $B(*,\mathcal{A}(\hat{\mathbb{S}}_p^V\wedge\hat{\mathbb{S}}_p^W),\mathcal{O})$ with $B(*,\mathcal{A}(\mathbb{S}^V\wedge\hat{\mathbb{S}}_p^W),\mathcal{O})$. The new diagram also commutes; the same proof applies. It follows that the map $B_G(\mathbb{S}^V) \to B_G(\hat{\mathbb{S}}_p)$, representing completion followed by stabilization, factors through the map $B_G(\mathbb{S}^V) \to B_G(\mathbb{S}^V \wedge \hat{\mathbb{S}}_p^W)$ representing fiberwise smash product with $\hat{S}_p^W$.

8. The homotopy units of the sphere

In this brief section, we show how to identify the space $\Omega B_G(\hat{\mathbb{S}}_p)$ with the homotopy units in $\Omega^\infty \hat{S}_p$, when G is a p-group, p odd. This space will appear in section 10, when we analyze the equivariant J-theory diagram.

Recall from section 3 that if $F_\rho \in \mathbb{F}$ is a distinguished fiber, then $\tilde{\mathcal{A}}(F_\rho)$ is the monoid of based self-maps of F_ρ which are nonequivariant homotopy equivalences, with G acting through conjugation. If every H-map $F_\rho \to F_\rho$ which is a nonequivariant equivalence is also an equivariant equivalence, then $\pi_0(\tilde{\mathcal{A}}(F_\rho)^H)$ is a group for all $H \leqslant G$, and by Lemma 3.16, $B(\tilde{A}(F_\rho))$ is equivalent to the G-connected cover of $B_G(\mathcal{A}(\mathbb{F}))$. The following lemma, which easily applies to any subgoup $H \leqslant G$, provides an example.

Lemma 8.1. *If $V \in \mathcal{V}$ is sufficiently large, then every G-map $\hat{S}_p^V \to \hat{S}_p^V$ which is a nonequivariant equivalence is also an equivariant equivalence.*

Proof. When $\dim V^G > 1$, the monoid $[\hat{S}_p^V, \hat{S}_p^V]_G$ is the p-completion of the group $[S^V, S^V]_G$. When V is sufficiently large, $[S^V, S^V]_G$ is the Burnside ring $A(G)$. We must therefore show that every element in $A(G)\hat{_p}$ mapping under the augmentation to a unit in $\hat{\mathbb{Z}}_p$ is itself a unit. Since the augmentation splits, it is enough to show that $1 + I(G)\hat{_p}$ is a group under multiplication.

To see this, we claim that p is in every maximal ideal of $A(G)\hat{_p}$. If p were not in $\mathcal{M}$, then we could write $1 = ap + m$ for some $a \in A(G)\hat{_p}$ and $m \in \mathcal{M} - \{0\}$. But $1 - ap$ is a unit with inverse $1 + ap + (ap)^2 + \cdots$, which is a contradiction.

Now, by [**27**, Ex. 1.9.4], some power of $I(G)$ is contained in $pA(G)$. It follows that if x is in $I(G)\hat{_p}$, then some power of x is divisible by p, and hence in every maximal ideal of $A(G)\hat{_p}$. Since maximal ideals are prime, x itself must be in every maximal ideal, hence in the Jacobson radical. By [**8**, 1.9], $1 + x$ is invertible. $\square$

Together with Lemma 3.16, Lemma 8.1 implies the following corollary.

Corollary 8.2. *For $V \in \mathcal{V}$ sufficiently large, $\Omega B_G(\hat{\mathbb{S}}_p^V)$ is equivalent to the space of self-maps $\hat{S}_p^V$ which are nonequivariant equivalences, with G-acting by conjugation.*

Note that $\Omega B_G(\mathbb{S}^V)$ is *not* equivariantly equivalent to the space of self-maps of S^V of degree ± 1, though this does hold nonequivariantly.

Corollary 8.2 shows that $\Omega B_G(\hat{\mathbb{S}}_p^V)$ is equivalent to $\tilde{\mathcal{A}}(\hat{S}_p^V)$ for V sufficiently large. Smashing with the identity on $\hat{S}_p^W$ yields a map $\tilde{\mathcal{A}}(\hat{S}_p^V) \to \tilde{\mathcal{A}}(\hat{S}_p^V \wedge \hat{S}_p^W)$. We can adapt the change of fiber construction to get a map $\tilde{\mathcal{A}}(\hat{S}_p^V \wedge \hat{S}_p^W) \to \tilde{\mathcal{A}}((S^V \wedge S^W)\hat{_p})$, so that change of fiber is compatible with the equivalence BF of 3.16. Thus, $\Omega B_G(\hat{\mathbb{S}}_p)$ is equivalent to $\operatorname{colim} \tilde{\mathcal{A}}(\hat{\mathbb{S}}_p^{V_i})$ over a cofinal sequence $V_i \in \mathcal{V}$.

Corollary 8.3. $\Omega B_G(\hat{\mathbb{S}}_p) \simeq \Omega^\infty \hat{S}_p^\times$.

Remark 8.4. From the above corollary, we can show that the G-connected cover of $B_G(\hat{\mathbb{S}}_p)$ is itself p-complete. Indeed, the group of components in $(\Omega^\infty \hat{S}_p^\times)^H$ is the group of units in $A(H)\hat{_p}$. As in the proof of Lemma 8.1, the units in $A(H)\hat{_p}$ are those elements mapping to units in $\hat{\mathbb{Z}}_p$. Since the units in $\hat{\mathbb{Z}}_p$ are closed, it follows that the units in $A(H)\hat{_p}$ are closed and hence complete. Moreover, all the higher homotopy groups of $\Omega^\infty \hat{S}_p^\times$ are p-complete. Thus, $\pi_n(B_G(\hat{\mathbb{S}}_p)^H)$ is p-complete for all $H \leqslant G$ and all $n \geqslant 1$.

9. Maps between Classifying Spaces

In this section, we construct the maps between the classifying spaces of the equivariant J-theory diagram. We begin with a geometric description of the Adams-Bott cannibalistic class ρ^k and its complex analogue ρ_c^k. These are obtained by applying Adams operations to Thom classes of *Spin*-bundles and inverting the Thom isomorphism to get virtual bundles. We also show how to describe ρ^k on the classifying space level. After this has been done, we use the equivariant Adams conjecture of [**27**] to obtain an equivariant version of the map γ^k in the J-theory diagram. From here, it becomes straightforward to define σ^k and the remaining maps in the equivariant J-theory diagram. Throughout this section, we take G to be a p-group, p odd, and $k \geqslant 2$ to be an odd integer prime to p.

Suppose ζ is a $(G, U(V))$-bundle over B. Since $\tilde{K}_G(T\zeta)$ is a free $K_G(B)$-module, we may define $\rho_c^k(\zeta)$ to be the element in $K_G(B)$ satisfying $\psi^k(\mu^\zeta) = \rho_c^k(\zeta)\mu^\zeta$. The operation ρ_c^k is natural and is *exponential* in the sense that $\rho_c^k(\zeta \oplus \zeta') = \rho_c^k(\zeta) \cdot \rho_c^k(\zeta')$. Also, $\rho_c^k(\zeta) = 1 + \zeta + \cdots + \zeta^{k-1}$ if ζ is a line bundle. By the splitting principle, these three facts completely determine the behavior of ρ_c^k.

Similarly, if ξ is a $(G, Spin(V))$-bundle over B, where V has real dimension divisible by 8, then we may define $\rho^k(\xi)$ to be the element in $KO_G(B)$ satisfying $\psi^k(\mu^\xi) = \rho^k(\xi)\mu^\xi$. Again, ρ^k is natural and exponential. The following lemma now follows from Corollary 5.8 and the fact that c is a ring homomorphism commuting with the Adams operations.

Lemma 9.1. *If V has complex dimension divisible by 4 and ζ is a $(G, SU(V))$-bundle, then $\rho_c^k(\zeta) = c\rho^k(r\zeta)$.*

We record the following corollary for future use.

Corollary 9.2. *The Adams operation ψ^k commutes with ρ_c^k, and commutes with ρ^k if we invert 2.*

Proof. It's easy to check that ρ_c^k and ψ^k commute on line bundles, so that by the splitting principle, they commute in general. By lemma 9.1, $\rho_c^k(\zeta) = c\rho^k(r\zeta)$, where ζ is a $(G, SU(V))$-bundle with underlying $(G, Spin(V))$-bundle $r\zeta$. Since $rc = 2$ and ψ^k commutes with c, we have

$$\begin{aligned} 2\psi^k\rho^k(2\xi) &= r(c\psi^k\rho^k(2\xi) = r\psi^k(c\rho^k r(c\xi)) \\ &= r(c\rho^k r)\psi^k(c\xi) = rc\rho^k(rc\psi^k\xi) = 2\rho^k\psi^k(2\xi). \end{aligned}$$

The middle equality follows from Lemma 9.1. If 2 is invertible, the result follows. □

We would like to define ρ_c^k and ρ^k on stable (G, SU) and $(G, Spin)$-bundles. To do this, we formally think of a stable bundle as a difference $\zeta - \mathbf{V}$, or $\xi - \mathbf{V}$. Here ζ denotes a $(G, SU(V))$-bundle, ξ denotes a $(G, Spin(V))$-bundle, and $\mathbf{V}$ is the trivial bundle $B \times V \to B$, where $V \in \mathcal{V}$ has complex dimension divisible by 4. Exponentiality forces us to define $\rho_c^k(\zeta - \mathbf{V})$ as $\rho_c^k(\zeta)/\rho_c^k(\mathbf{V})$ and $\rho^k(\xi - \mathbf{V})$ as $\rho^k(\xi)/\rho^k(\mathbf{V})$, but to make sense of this, we need to be sure that $\rho_c^k(\mathbf{V})$ and $\rho^k(\mathbf{V})$ are units.

Let $\theta_c^k(V)$ and $\theta^k(V)$ be the elements in $R(G)$ and $RO(G)$ satisfying

$$\psi^k(b_c^V) = \theta_c^k(V) \cdot b_c^V, \psi^k(b^V) = \theta^k(V) \cdot b^V.$$

Clearly, the orientations $\mu_c^{\mathbf{V}}$ and $\mu^{\mathbf{V}}$ are the products of the respective Bott classes on S^V and the identity, so that $\rho_c^k(\mathbf{V}) = \theta_c^k(V)$ and $\rho^k(\mathbf{V}) = \theta^k(V)$.

By Corollary 2.5 of [**15**], when k is prime to $|G|$, $\theta^k(V)$ becomes a unit after inverting k. We may therefore make the following definition.

Definition 9.3. Suppose $\zeta - \mathbf{V}$ is a stable (G, SU)-bundle over a G-space B. Then let $\rho_c^k(\zeta - \mathbf{V})$ be the element $\rho_c^k(\zeta)/\theta_c^k(V)$ in $K_G(B)[1/k]$.

We wish to extend this to the real case.

Lemma 9.4. *Suppose $V \in \mathcal{V}$ has complex dimension divisible by 4, and k is prime to $|G|$. Then the element $\theta^k(V)$ maps to a unit in $RO(G)[1/2k]$.*

Proof. Let $u_c = \theta_c^k(V)$ and let $u = \theta^k(V)$. Let r, c, and t be the homomorphisms induced by forgetting complex structure, complexification, and complex conjugation. By Proposition 5.7, $c(b^V) = b_c^V$. Since complexification commutes with Adams operations, we then have $cu = u_c$. Let $v \in R(G)[1/k]$ be the inverse of u_c. Then

$$c(u \cdot rv) = c(u) \cdot cr(v) = u_c \cdot (v + tv) = u_c \cdot v + u_c \cdot tv = (u_c \cdot v) + t(u_c \cdot v) = 2.$$

Applying r, we have $2(u \cdot rv) = 4$, so $u \cdot (\frac{rv}{2}) = 1$. $\square$

Example 9.5. If G acts on $\mathbb{C}[G]^{\oplus 4}$ by permuting generators, then the action map $G \to U(\mathbb{C}[G]^{\oplus 4})$ factors through $SU(\mathbb{C}[G]^{\oplus 4})$. Therefore, if $V \in \mathcal{V}$ is isomorphic to $\mathbb{C}[G]^{\oplus 4}$, then $\theta^k(V)$ is a unit in $RO(G)[1/2k]$.

Corollary 9.6. *For any $V \in \mathcal{V}$ of complex dimension divisible by 4, $\theta^k(V)$ is a unit in $RO(G)[1/2k]$.*

Proof. There is a $W \in \mathcal{V}$ such that $V \oplus W$ is isomorphic to $\mathbb{C}[G]^{\oplus 4n}$, for some n. The operation θ^k is exponential, so the corollary follows by Example 9.5. $\square$

We may now make the following definition.

Definition 9.7. Suppose $\xi - \mathbf{V}$ is a stable $(G, Spin)$-bundle over a G-space B. Then let $\rho^k(\xi - \mathbf{V})$ be the element $\rho^k(\xi)/\theta^k(V)$ in $KO_G(B)[1/2k]$.

We may also construct the Adams-Bott cannibalistic class ρ^k on the level of classifying spaces. For $V \in \mathcal{V}$ of complex dimension divisible by 4, let $\psi_V^k : \Omega^\infty KO[1/2k] \to \Omega^\infty KO[1/2k]$ be represented by $\frac{\psi^k}{\theta^k(V)}$. This makes sense by 9.6. Lemma 9.8, together with Construction 4.12, yields a map

$$c(\Psi^k) : B_G(\mathbb{S}; KO[1/2k]) \to Fib(q) \subseteq \Omega^\infty KO[1/2k]^\times.$$

Lemma 9.8. *The maps ψ_V^k form an exponential collection.*

Proof. Clearly $\psi_V^k(b^V) = b^V$. The diagram of Definition 4.11 commutes because ψ^k is multiplicative, multiplication by $\theta^k(V)$, $\theta^k(W)$ and $\theta^k(V \oplus W)$ are all equivalences, and $\psi^k = \theta^k(V) \cdot \psi_V^k$. □

Now let ρ^k be the map $c(\Psi^k) \circ g$, where $g : B_G(Spin) \to B_G(\mathbb{S}; KO[\frac{1}{2k}])$ is induced by the KO-orientation. Since $B_G(Spin)$ is nonequivariantly connected, ρ^k factors through the base-point component, $B_G(O)_\otimes[1/2k] \subseteq \Omega^\infty KO[1/2k]^\times$. This factorization represents the operation ρ^k described above, as one can see by considering the geometric interpretation of $c(\Psi^k)$ (4.13).

We can compose ρ^k with the map $\hat{k}_p : \Omega^\infty KO[1/2k]^\times \to \Omega^\infty KO\hat{_p}$ induced by completion. We then have a unique dotted arrow, which we also label ρ^k, making the following diagram commute up to homotopy.

$$\begin{tikzcd}
B_G(Spin) \arrow[r, "\rho^k"] \arrow[d] & \Omega^\infty KO[1/2k]^\times \arrow[r, "\hat{k}_p"] & \Omega^\infty KO\hat{_p}^\times \\
B_G(O)\hat{_p} \arrow[rr, dotted, "\rho^k"] & & B_G(O)_{\otimes}\hat{_p}. \arrow[u, "\subseteq"']
\end{tikzcd}$$

Remark 9.9. Using a construction similar to 4.12, one obtains a map $c(\psi^k) : B_G(\hat{\mathbb{S}}_p; KO\hat{_p}) \to \Omega^\infty KO\hat{_p}^\times$, and we can choose the maps $c(\psi^k)$ to be compatible under completion. Thus, ρ^k can be described as the composite

$$B_G(Spin) \xrightarrow{g} B_G(\hat{\mathbb{S}}_p; KO\hat{_p}) \xrightarrow{c(\psi^k)} \Omega^\infty KO\hat{_p}^\times ,$$

or its factorization through $B_G(O)_\otimes\hat{_p}$.

Now, we turn to the equivariant Adams conjecture. First, recall the nonequivariant version of the Adams conjecture ([**1**, 1.2]).

Conjecture 9.10. *If k is an integer, X is a finite CW-complex and $y \in K_R(X)$, then there exists a non-negative integer $e = e(k, y)$ such that $k^e(\Psi^k - 1)y$ maps to zero in $J(X)$.*

Equivariantly, the Adams conjecture is significantly more subtle. The following, which is Proposition 11.3.7 of [**27**], must certainly lie at the heart of any attempt to generalize the Adams conjecture.

Proposition 9.11. *Let G be a compact Lie group and let $E \to B$ be an orthogonal G-vector bundle. Then there exist stable G-maps $f : S(E) \to S(\psi^k E)$ if k is odd ($S(E \oplus E) \to S(\psi^k(E \oplus E))$ if k is even) of fiber-degree dividing a power of k.*

That is, for some inner product space V, the stable bundles E and $\psi^k E$ can be represented as $(G, O(V))$-bundles, and there is a map $f : S(E) \to S(\psi^k E)$ between the associated $G\mathcal{F}(\mathbb{S}^V)$-fibrations of fiber-degree dividing a power of k.

Remark 9.12. The reader may be familiar with a version of the equivariant Adams Conjecture proven in [**16**]. In this version, it is shown that $sk^n(\psi^k - 1)(\mu)$ is fiber-homotopy trivial for any $\mu \in K_G(X)$, where s is the order of k mod $|G|$. This is in some sense stronger than 9.11, but it is more restrictive in that it requires G to be cyclic.

In the following corollary, we restore the convention that G is a p-group, p odd, and k is an odd integer prime to p and greater than 1.

Corollary 9.13. *Let $\xi : E \to B$ be an orthogonal G-vector bundle. Let V be a G-representation in $\mathcal{V}$ with $V^G \neq 0$. Let $S(\xi) \wedge \hat{S}^V_p$ and $S(\psi^k\xi) \wedge \hat{S}^V_p$ denote the fiberwise smash products of ξ and $\psi^k\xi$ with the trivial bundle $B \times \hat{S}^V_p$. Then $S(\xi) \wedge \hat{S}^V_p$ and $S(\psi^k\xi) \wedge \hat{S}^V_p$ are fiberwise G-homotopy equivalent.*

Proof. The proposition gives us a map $f : S(\xi) \to S(\psi^k\xi)$. Since G is a p-group, the fiber degree of f^H must be prime to p for all $H \leqslant G$, as observed in [**27**, 11.4.3]. Since $\hat{S}^V_p$ is p-local, and the suspension of any p-local space is p-local, the H-fixed points of the fibers of $S(\xi) \wedge \hat{S}^V_p$ and $S(\psi^k\xi) \wedge \hat{S}^V_p$ are p-local for any $H \leqslant G$. Therefore, the map $f \wedge 1$ restricts on the H-fixed points of each fiber to a homotopy equivalence, so that $f \wedge 1$ restricts on each fiber to an equivariant homotopy equivalence. By the G-Dold Theorem (11.4), this implies that $f \wedge 1$ is a fiberwise G-homotopy equivalence. □

We wish to put this on the level of classifying spaces. The Adams operation ψ^k induces a map $B_G(O) \to B_G(O)$, and since the Hopf space $B_G(O)$ has a homotopy inverse, we have a map $\psi^k - 1 : B_G(O) \to B_G(O)$. We now show that this map lifts to $B_G(Spin)$, as in the nonequivariant setting.

When $|G|$ is odd, the fibers of the maps $B_G(Spin) \to B_G(SO)$ and $B_G(SO) \to B_G(O)$ are $F_1 = K(\mathbb{Z}/2, 1)$ and $F_2 = K(\mathbb{Z}/2, 0)$, by Lemma 2.16. In either case, the map $F_i \to Map(EG, F_i)$ is an equivariant equivalence, since the spectral sequence

$$H^*(G, \pi_*(F_i)) \Longrightarrow \pi_*(F_i^{hG})$$

collapses at the E^2-term, since $|G|$ is odd, so the edge homomorphism is an equivalence. (The same argument works for subgroups of G). This shows that the diagram below is, equivariantly, a fiber square:

$$\begin{tikzcd}
B_G(Spin) \arrow[r] \arrow[d] & B_G(O) \arrow[d] \\
Map(EG, B_G(Spin)) \arrow[r] & Map(EG, B_G(O)).
\end{tikzcd}$$

Now, since the inclusion $BO \to B_G(O)$ is a nonequivariant equivalence, it follows that the map $Map(EG, BO) \to Map(EG, B_G(O))$ is an equivariant equivalence. Since the nonequivariant map $\psi^k - 1 : BO \to BO$ lifts to $BSpin$, it follows that the equivariant map $\psi^k - 1 : Map(EG, B_G(O)) \to Map(EG, B_G(O))$ must lift to $Map(EG, B_G(Spin))$. Precomposing with the inclusion

$B_G(O) \to Map(EG, B_G(Spin))$, we have the left arrow in the diagram below, which then determines the dotted arrow, lifting $\psi^k - 1$.

$$
\begin{tikzcd}
B_G(O) \arrow[dr, dotted] \arrow[drr, bend left, "\psi^k-1"] \arrow[ddr, bend right] & & \\
& B_G(Spin) \arrow[r] \arrow[d] & B_G(O) \arrow[d] \\
& Map(EG, B_G(Spin)) \arrow[r] & Map(EG, B_G(O)).
\end{tikzcd}
$$

We can compose the map $Bj : B_G(Spin) \to B_G(\mathbb{S})$ (cf. 4.1) with the fiberwise completion map $\hat{k}_p$ of §7 to obtain a map $B\hat{j}_p : B_G(Spin) \to B_G(\hat{\mathbb{S}}_p)$. The above corollary, together with Remark 7.7, then implies that the composite

$$B_G(O) \xrightarrow{\psi^k-1} B_G(Spin) \xrightarrow{B\hat{j}_p} B_G(\hat{\mathbb{S}}_p)$$

is equivariantly null-homotopic when restricted to any finite subcomplex of $B_G(O)$. We claim that $B\hat{j}_p \circ (\psi^k - 1)$ is itself null-homotopic. The following lemma is key to showing this.

Lemma 9.14. *Suppose that X is a G-space which can be expressed as a union $\cup X_i$ of finite G-CW complexes, and Y is a G-space whose G-connected cover is p-complete. Then the map of pointed sets $[X, Y]_G \to \lim[X_i, Y]_G$ has trivial fiber.*

Proof. We have a $\lim^1$ exact sequence:

$$\lim{}^1[X_i, \Omega Y]_G \hookrightarrow [X, Y]_G \twoheadrightarrow \lim[X_i, Y]_G.$$

By assumption, $\pi_n^H(\Omega Y)$ is p-complete for all $n \geqslant 0$ and all $H \leqslant G$. By induction up the skeletal filtration of X_i, exactness of p-completion, and the 5-lemma, the group $[X_i, \Omega Y]_G$ is p-complete for each i, and therefore compact. The $\lim^1$-term of an inverse system of compact groups vanishes. □

The G-connected cover of $B_G(\hat{\mathbb{S}}_p)$ is p-complete, since $\Omega B_G(\hat{\mathbb{S}}_p)$ has p-complete homotopy groups by Remark 8.4, so we have the corollary:

Corollary 9.15. *The composite $B\hat{j}_p \circ (\psi^k - 1)$ is null-homotopic, so that there is a lift $\gamma^k : B_G(O) \to Fib(B\hat{j}_p)$ of $\psi^k - 1$.*

We now use the KO-orientation of $Spin$ bundles together with the map γ^k above to define a map σ^k. Let $\hat{g}_p : B_G(Spin) \to B_G(\hat{\mathbb{S}}_p; KO_p^{\wedge})$ be the composite $\hat{k}_p \circ g$, where $\hat{k}_p$ is the fiberwise completion map of Section 7. In the diagram below, τ is the inclusion of fibers, and f is induced by $\hat{g}_p$.

$$\begin{tikzcd} Fib(B\hat{j}_p) \arrow[r, "\tau"] \arrow[d, "f"] & B_G(Spin) \arrow[r, "B\hat{j}_p"] \arrow[d, "\hat{g}_p"] & B_G(\hat{\mathbb{S}}_p) \arrow[d, equal] \\ Fib(q) \arrow[r, "\tau"] & B_G(\hat{\mathbb{S}}_p; KO\hat{_p}) \arrow[r, "q"] & B_G(\hat{\mathbb{S}}_p). \end{tikzcd}$$

By an easy adaption of Remark 4.9, we have a map $\iota : Fib(q) \to \Omega^\infty KO\hat{_p}^\times$. Precomposing $\iota \circ f$ with γ^k, one obtains a map from $B_G(O)$ to $\Omega^\infty KO\hat{_p}^\times$.

Since $B_G(O)$ is nonequivariantly connected and $\Omega^\infty KO\hat{_p}$ is p-complete, we have a unique dotted arrow σ^k making the following diagram homotopy commute.

$$\begin{tikzcd} B_G(O) \arrow[r, "\gamma^k"] \arrow[d] & Fib(B\hat{j}_p) \arrow[r, "\iota\circ f"] & \Omega^\infty KO\hat{_p}^\times \\ B_G(O)\hat{_p} \arrow[rr, dotted, "\sigma^k"] & & B_G(O)_{\otimes}\hat{_p} \arrow[u] \end{tikzcd}$$

Definition 9.16. Let σ^k be the dotted arrow in the diagram above.

10. The Adams-May square

Using the maps constructed in the previous section, we now have the homotopy commuting Adams-May square:

$$\begin{tikzcd} B_G(O)\hat{_p} \arrow[r, "\psi^k-1"] \arrow[d, "\sigma^k"'] & B_G(O)\hat{_p} \arrow[d, "\rho^k"] \\ B_G(O)_{\otimes}\hat{_p} \arrow[r, "\psi^k/1"] & B_G(O)_{\otimes}\hat{_p}. \end{tikzcd}$$

The bottom map is technically induced by the composite $c(\psi^k) \circ \tau$, which classifies $\psi^k/1$ by Remark 4.13.

In this section, we analyze this square by showing that ρ^k and σ^k are homotopic, at least after restricting to G-connected covers. (Since the basepoint of $B_G(O)_\otimes$ represents the trivial virtual bundle of dimension 1, we denote the G-connected cover of $B_G(O)_\otimes$ as $B_G(O)_{\otimes 1}$.) We then show that the G-connected cover of the Adams-May square is a pull-back in the homotopy category. The demonstration of this fact relies heavily on the theory of p-adic γ-rings in [**9**]. Finally, we use our result to obtain a splitting of the G-connected cover of $\Omega^\infty \hat{S}_p$, based at the identity.

We first outline the idea for why ρ^k and σ^k are homotopic on G-connected covers. Since p is odd, 2 is a unit, so that by Corollary 9.2, ρ^k commutes with ψ^k. Thus, if we were to replace σ^k by ρ^k in the Adams-May square, the square would still commute. It would therefore be enough to show that $\psi^k/1$ is an equivalence, but unfortunately, this is not the case. Nevertheless, we *can* show that $\psi^k/1$ becomes an equivalence after inverting p (10.3), so that ρ^k and σ^k become homotopic after mapping to $(B_G(O)_{\otimes 1\hat{p}})\,[1/p]$. We can also show that the map

$$[B_G(O)_{0\hat{p}}, B_G(O)_{\otimes 1\hat{p}}]_G \to [B_G(O)_{0\hat{p}}, (B_G(O)_{\otimes 1\hat{p}})\,[1/p]]_G$$

is an injection (10.6), which proves the claim.

We need two lemmas to prepare for 10.3.

Lemma 10.1. *For any $r \geqslant 1$, the map $(k^r\psi^k) - 1 : R(G) \to R(G)$ is injective.*

Proof. Suppose $k^r\psi^k(x) = x$. For some $l > 0$, $k^l \equiv 1 \mod p$, so we have

$$x = (k^r\psi^k)^l(x) = k^{rl}\psi^{k^l}(x) = k^{rl}x.$$

The first equality follows from iterating $k^r\psi^k(x) = x$, the second since $\psi^m\psi^n = \psi^{mn}$ and the third since ψ^k is periodic on $R(G)$, with period $|G|$ and ψ^1 is the identity. Since $l, r \geqslant 1$ and $k \geqslant 2$, $k^{rl} > 1$. Since $R(G)$ is torsion free, this implies $x = 0$. □

Lemma 10.2. *For $n \geqslant 1$, the map below is injective.*

$$\psi^k/1 : 1 + \tilde{K}_G(S^n) \to 1 + \tilde{K}_G(S^n)$$

Proof. We have $\tilde{K}_G(S^n) \cong \tilde{K}(S^n) \otimes R(G)$. When n is odd, $\tilde{K}(S^n) = 0$, so there is nothing to prove. On the other hand, $\tilde{K}(S^{2r})$ is a free abelian group on a generator κ, with $\kappa^2 = 0$. Thus, the multiplicative structure on $1 + \tilde{K}_G(S^n)$ is the same as the additive. Moreover, $\psi^k(\kappa) = k^r\kappa$. Thus, $\psi^k/1$ acts on $1 + \tilde{K}_G(S^{2r}) \cong R(G)$ by $k^r\psi^k - 1$, so we may appeal to Lemma 10.1. □

Proposition 10.3. *The map $\psi^k/1 : (B_G(O)_{\otimes 1\hat{p}})\,[1/p] \to (B_G(O)_{\otimes 1\hat{p}})\,[1/p]$ is a weak G-equivalence.*

Proof. Let c denote complexification. Since c commutes with Adams operations, the following diagram commutes:

$$\begin{tikzcd}
1 + \tilde{K}O_G(S^n) \arrow[r, "\psi^k/1"] \arrow[d, "c"] & 1 + \tilde{K}O_G(S^n) \arrow[d, "c"] \\
1 + \tilde{K}_G(S^n) \arrow[r, "\psi^k/1"] & 1 + \tilde{K}_G(S^n).
\end{tikzcd}$$

After inverting 2, c becomes an injection, so that by 10.2, the top map $\psi^k/1$ is an injection, after inverting 2, and in particular, after tensoring with $\hat{\mathbb{Q}}_p$. Thus, since $\psi^k/1 \otimes \hat{\mathbb{Q}}_p$ is a linear transformation between finite dimensional vector spaces of the same dimension, it is an isomorphism. This holds just as well for any subgroup $H \leqslant G$. Since $(B_G(O)_{\otimes 1\hat{p}}))[\frac{1}{p}]$ represents the theory $1 + \tilde{K}O_G(-) \otimes \hat{\mathbb{Q}}_p$, it follows that $\psi^k/1$ induces a weak G-equivalence. □

Our next aim is to show

$$[B_G(O)_{0\hat{p}}, B_G(O)_{\otimes 1\hat{p}}]_G \to [B_G(O)_{0\hat{p}}, (B_G(O)_{\otimes 1\hat{p}})\,[1/p]]_G$$

is an injection. This follows immediately from the universal property of completions if we can show that

$$[B_G(O)_0, B_G(O)_{\otimes 1\hat{p}}]_G \to [B_G(O)_0, (B_G(O)_{\otimes 1\hat{p}})[1/p]]_G$$

is an injection, which we show in Proposition 10.6 below. We will make use of results from Section 2 to describe $B_G(O)_0$ as a colimit of spaces $BO(V)$, and then we will employ the following results on the KO-theory of these spaces.

Suppose V is an orthogonal G-representation. Then G acts on $O(V)$ by conjugation, and hence on $BO(V)$. Let $EO(V) = B(*, O(V), O(V))$. Note that $O(V)$ acts freely on $EO(V)$, with quotient $BO(V)$. Moreover, the G-action and the $O(V)$-action on $EO(V)$ induce an action of the semidirect product $G \rtimes O(V)$. Thus, by [**24**, 2.1] we have

$$\tilde{K}_G(BO(V)) \cong \tilde{K}_{G\rtimes O(V)}(EO(V)).$$

Lemma 10.4. *If $\Lambda \leqslant G \rtimes O(V)$ is in the family $\mathcal{F}$ of subgroups subconjugate to G, then the fixed-point space $EO(V)^\Lambda$ is contractible. Otherwise, $EO(V)^\Lambda$ is empty. Thus $EO(V)$ is the classifying space of the family $\mathcal{F}$.*

Proof. If $H \leqslant G$, then $EO(V)^H = B(*, O(V)^H, O(V)^H) \simeq *$; if $\Lambda = tHt^{-1}$, then $EO(V)^\Lambda \cong EO(V)^H \simeq *$. This proves one direction. Now, let H be the image in G of a subgroup $\Lambda \leqslant G \rtimes O(V)$. Then $EO(V)^\Lambda = B(*, O(V)^H, O(V)^\Lambda)$, where Λ acts on $O(V)$ by the formula $u \cdot (g, s) = g^{-1}ugs$ for $u \in O(V), (g, s) \in G \rtimes O(V)$. Therefore, for $EO(V)^\Lambda$ to be nonempty, $O(V)^\Lambda$ must be nonempty, so there must exist $u \in O(V)$ such that $g^{-1}ugs = u$ for all $(g, s) \in \Lambda$. But then, $(g, s) = u^{-1}gu$ in $G \rtimes O(V)$, so that $\Lambda = u^{-1}Hu$. □

Proposition 10.5. *$\tilde{KO}_G(BO(V))[1/2]$ has no torsion.*

Proof. Since $rc = 2$, $c : \tilde{KO}_G(BO(V)) \to \tilde{K}_G(BO(V))$ becomes injective after inverting 2, so it suffices to show that $\tilde{K}_G(BO(V))$ has no torsion. Let $R = R(G \rtimes O(V))$ be the representation ring of $G \rtimes O(V)$. For each Λ subconjugate to G, let $I_\Lambda = ker(R \to R(\Lambda))$, and let $I = I_G$. By the generalization of the Atiyah-Segal completion theorem to families (see [**5**] or [**19**, XIV.6.1]), $\tilde{K}_{G\rtimes O(V)}(EO(V))$ is the completion of $R = R(G \rtimes O(V))$ with respect to the products of powers of the ideals I_Λ. Since $I \leqslant I_\Lambda$ for all Λ subconjugate to G, this is R_I.

Since $G \rtimes O(V)$ is a compact Lie group, R is Noetherian by [**25**, 3.3], so by [**8**, 10.14], $R_{\hat{I}}$ is flat over R. Since R is torsion free, $R_{\hat{I}}$ is torsion free. □

Proposition 10.6. *The map below is an injection*

$$[B_G(O)_0, B_G(O)_{\otimes 1\hat{p}}]_G \to [B_G(O)_0, (B_G(O)_{\otimes 1\hat{p}})[1/p]]_G\,.$$

Proof. First, by Definition 2.15 and Lemmas 2.19 and 2.20, $B_G(O)_0$ can be realized as a colimit of spaces $BO(V_i)$, $V_i \in \mathcal{V}$. Since the homotopy groups of $B_G(O)_{\otimes 1\hat{p}}$ are all p-complete, it now suffices by Lemma 9.14 to show that the map

$$[BO(V), B_G(O)_{\otimes 1\hat{p}}]_G \to [BO(V), (B_G(O)_{\otimes 1\hat{p}})[1/p]]_G$$

is injective. But this is just the map

$$\tilde{KO}_G(BO(V)) \otimes \hat{\mathbb{Z}}_p \to \tilde{KO}_G(BO(V)) \otimes \hat{\mathbb{Q}}_p,$$

which is an injection by 10.5, since p is odd. $\square$

We conclude:

Corollary 10.7. *The maps ρ^k and σ^k from $B_G(O)\hat{_p}$ to $B_G(O)_{\otimes}\hat{_p}$ become homotopic after restricting to the G-connected cover of $B_G(O)\hat{_p}$.*

Our next goal is to show that the G-connected cover of the equivariant Adams-May square is a pull-back diagram in the homotopy category. We use the theory of λ-rings and γ-rings, which we will use to obtain splittings of the theories corresponding to the spaces in the square; the reader may consult [**9**] for further details. We are especially interested in *p-adic γ-rings*, which exhibit a number of useful properties that we explore. We show that, if G is a p-group, p odd, and $IK_G(X)$ denotes the kernel of the augmentation $K_G(X) \to \mathbb{Z}$, then $IK_G(X)\hat{_p}$ is a p-adic γ-ring. We use this to obtain information about $IKO_G(X)\hat{_p}$. We show that this theory splits into $p-1$ summands, and that $\psi^k - 1$ and $\psi^k/1$ induce equivalences on all but one of these summands, while ρ^k and σ^k induce equivalences on the other. The result will then follow formally.

A λ-ring is a commutative unital ring R, equipped with maps $\lambda^n : R \to R$, for each $n \in \mathbb{N}$, satisfying $\lambda^0(x) = 1, \lambda^1(x) = x$, and $\lambda^n(x+y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$, for all $x, y \in R$. For example, $\mathbb{Z}$ is a λ-ring with $\lambda^n(m) = \binom{n}{m}$ for $m \geqslant 0$. The λ-dimension of an element $x \in R$, if it exists, is the least n such that $\lambda^m(x) = 0$ whenever $m > n$. If X is a compact G-space, then the λ-ring structure on $K_G(X)$ is obtained using exterior powers of bundles. In fact, $K_G(X)$ is a *special* λ-ring, meaning the λ-operations satisfy a certain set of identities, which are in fact motivated by the relations satisfied by exterior powers of vector spaces.

Since Adams operations ψ^k are defined in terms of exterior power operations, it should be no surprise that they can be defined for any λ-ring R. If $\xi_1^k + \cdots \xi_r^k = \nu_k(s_1, \cdots, s_r)$, where s_i is the ith elementary symmetric function in the $\{\xi_j\}$, then $\psi^k(x) = \nu_k(\lambda^1(x), \cdots, \lambda^k(x))$. When R is special, the usual properties of Adams operations hold.

Associated to the λ-operations is a collection of operations γ^n, given by $\gamma^n(x) = \lambda^n(x + n - 1)$. The γ-dimension of an element x is defined in analogy with λ-dimension. Given a λ-homomorphism ϵ, called an augmentation, from a special λ-ring R to $\mathbb{Z}$, the kernel I of ϵ is preserved by the γ-operations, and is called a *special γ-ring*. For example, the dimension of bundles determines an augmentation $K_G(X) \to \mathbb{Z}$, whose kernel, which we denote $IK_G(X)$, is a special γ-ring.

When I is a special γ-ring, let I_n denote the subgroup of I additively generated by the products $\gamma^{n_1}(a_1) \cdots \gamma^{n_r}(a_r)$, with each $a_i \in I$ and $\sum n_i \geqslant n$. This defines a filtration on I, called the γ-filtration. There is also a power filtration $\{I^n\}$ and a p-adic filtration $\{p^n I\}$ for I. When I has a finite number of generators, each of finite γ-dimension, the power filtration and the γ-filtration determine the same topologies on I. For example, this holds for $IK_G(X)$.

When I is finitely generated, and the topology determined by its γ-filtration is finer than that of its p-adic filtration, then we say that $\hat{I}_p = I \otimes \hat{\mathbb{Z}}_p$ is a *p-adic γ-ring*. For example, we show below (10.10) that $IK_G(X)\hat{_p}$ is a p-adic γ-ring when G is a p-group, p odd.

When $A = \hat{I}_p$ is a p-adic γ-ring, the map $\mathbb{Z}^+ \times I \to I$ given by $(k, a) \to \psi^k(a)$ is p-adically continuous ([**9**, I.5.6]). We may then define operations ψ^α on A for all $\alpha \in \hat{\mathbb{Z}}_p$. In particular, when α is a primitive $(p-1)$st root of unity, we obtain an operation $T = \psi^\alpha$ on A with $T^{p-1} = 1$. Thus, A splits as a sum of eigenspaces $\sum_{i=0}^{p-2} A_i$, where $A_i = ker(T - \alpha^i)$.

If I is the kernel of the augmentation $R \to \mathbb{Z}$, and $A = \hat{I}_p$ is a p-adic γ-ring, then we can define a cannibalistic classes $\rho^k : A \to 1 + A$, where $1 + A$ denotes the multiplicative group of elements in R mapping to 1 in $\mathbb{Z}$. Namely, letting $\gamma_t(x)$ denote the power series $1 + \gamma^1(x)t + \gamma^2(x)t^2 + \cdots$, we define $\rho^k(x)$ as the product of the series $\gamma_{u/u-1}(x)$, where u runs over all the kth roots of unity except 1. When $A = IK_G(X)\hat{_p}$, the map ρ^k agrees with the geometric construction given in Section 9. If k is a positive integer which is a topological generator of $\hat{\mathbb{Z}}_p$, and $p \neq 2$, then we have Proposition [**9**, III.4.4]:

Proposition 10.8. *$\rho^k : A_0 \to 1 + A_0$ is an isomorphism.*

We need the following lemma to prove that $IK_G(X)\hat{_p}$ is a p-adic γ-ring.

Lemma 10.9. *Let G be a group of order p^m, p an odd prime. Let ξ be a $(G, U(1))$-bundle on a compact G-space X. Then, for w sufficiently large, ξ^{p^w} is equivalent mod p to the trivial line bundle $X \times \mathbb{C}$.*

Proof. Let $\zeta = \xi^{p^m}$. Then for $x \in X^H$, $\zeta_x = (\xi_x)^{p^m}$, which is a trivial line bundle, since $H \leqslant G$ has order dividing p^m. Since $U(1)$ is abelian, it follows from Proposition 2.10 that there is a nonequivariant $U(1)$-bundle on X/G whose restriction to X is ζ. Let $\beta : X/G \to BU(1)$ be the classifying map for this bundle.

Now, since X is compact, and $BU(1) = \mathbb{C}P^\infty$, the map $\beta : X/G \to \mathbb{C}P^\infty$ factors through $\mathbb{C}P^{n-1}$ for some n. If $[H]$ is the canonical line bundle on $\mathbb{C}P^\infty$, then $([H] - 1)^n$ is trivial on $\mathbb{C}P^{n-1}$, so that $\beta^*(([H] - 1)^n)$ is trivial, whence $(\zeta - 1)^n$ is trivial. So, if w is large enough, then $(\zeta - 1)^{p^{w-m}}$ is zero in $\tilde{K}(X)$, and $\zeta^{p^{w-m}}$ is equivalent to the trivial line bundle modulo p. Thus, ξ^{p^w} is equivalent to the trivial line bundle modulo p for w sufficiently large. □

Proposition 10.10. *If G is a p group, $p \neq 2$, then $IK_G(X)\hat{_p}$ is a p-adic γ-ring.*

Proof. Since the γ-filtration and the power filtration determine the same topology, we must show that some power of $IK_G(X)$ lies in $pIK_G(X)$. Since $IK_G(X)$ is finitely generated, it is enough to show that a sufficiently high power of any element in $IK_G(X)$ is p-divisible. Any such element can be written $\xi - \zeta$ for complex G-vector bundles ξ and ζ of the same dimension l.

By the splitting principle ([**24**, 3.9]), there is a space $F(X)$ and a map $\rho : F(X) \to X$ so that the induced map $\rho^* : K_G(X) \to K_G(F(X))$ is injective and $\rho^*\xi$ breaks into a sum of line bundles: $L_1 \oplus L_2 \oplus \cdots \oplus L_l$. Choose w according to Lemma 10.9 so that $L_i^{p^w} \equiv 1 \mod p$ for $i = 1, 2, \cdots l$. Then,

$$\rho^*(\xi^{p^w}) = \left(\oplus_{i=1}^l L_i\right)^{p^w} \equiv \oplus_{i=1}^l L_i^{p^w} \equiv l \mod p.$$

Since ρ^* is injective, $\xi^{p^w} \equiv l \mod p$. By the same argument, $\zeta^{p^w} \equiv l \mod p$ for w sufficiently large. Thus $(\xi - \zeta)^{p^w} \equiv 0 \mod p$ for w sufficiently large. □

We are really after information about $IKO_G(X)\hat{_p}$. We use the maps c and r induced by complexification and forgetting complex structure. We then have the following formal consequence of the continuity of the Adams operations on $IK_G(X)$.

Lemma 10.11. *The Adams operations on $IKO_G(X)[1/2]$ are p-adically continuous.*

Proof. If $x \in IKO_G(X)$, then since $rc = 2$, we have

$$\psi^{k+s}(x) - \psi^k(x) = \frac{1}{2}rc(\psi^{k+s}(x) - \psi^k(x)) = \frac{1}{2}r(\psi^{k+s}(cx) - \psi^k(cx)).$$

Therefore, a given power of p will divide $\psi^{k+s}(x) - \psi^k(x)$ if a sufficiently large power of p divides s. $\square$

Therefore, the Adams operations extend to the p-adics on $IKO_G(X)\hat{_p}$, and just as above, $IKO_G(X)\hat{_p}$ splits into $p-1$ summands, the eigenspaces for $T = \psi^\alpha$. We denote these $B_i(X)$ (or B_i), for $0 \leqslant i \leqslant p-2$, and let $B_0^\perp$ denote $\oplus_{i=1}^{p-2} B_i$.

Now choose k to be a positive integer, congruent to $\alpha \bmod p$, which is a topological generator of $\hat{\mathbb{Z}}_p$. Since Adams operations commute, $T = \psi^\alpha$ commutes with ψ^k, so that ψ^k induces a map $B_i \to B_i$ for each i. In Proposition 10.13 below, we show that this map is an isomorphism for $i \neq 0$, so that $\psi^k - 1$ induces an automorphism of $B_0^\perp$. We need the following lemma.

Lemma 10.12. *If $\zeta \in IKO_G(X)\hat{_p}$, then p divides $(\psi^\alpha - \psi^k)^{p^s}(\zeta)$ for $s >> 0$.*

Proof. First, by definition of ψ^α, there is an integer l so that $l \equiv k \mod p$ and so that $\psi^\alpha \equiv \psi^l \mod p$. For any w, there is an s so that $l^{p^s} \equiv k^{p^s} \mod p^w$. So, by continuity of Adams operations, we have

$$\psi^{l^{p^s}}(\zeta) \equiv \psi^{k^{p^s}}(\zeta) \mod p$$

for s sufficiently large. Since Adams operations commute,

$$(\psi^l - \psi^k)^{p^s}(\zeta) \equiv \left[(\psi^l)^{p^s} - (\psi^k)^{p^s}\right](\zeta) = [\psi^{l^{p^s}}(\zeta) - \psi^{k^{p^s}}(\zeta)] \equiv 0 \mod p.$$

$\square$

Proposition 10.13. *$\psi^k - 1 : B_i \to B_i$ is an isomorphism for $1 \leqslant i \leqslant p-2$.*

Proof. If $\beta \in \hat{\mathbb{Z}}_p$, let $U_\beta = \beta(\psi^\alpha - \psi^k)$. By the Lemma 10.12, $U_\beta^n(x)$ is divisible by p for n sufficiently large, so that the series $1 + U_\beta + U_\beta^2 + \cdots$ converges in the p-adic topology. Therefore, $1 - U_\beta$ induces an isomorphism on each B_i.

Now, since α is a primitive $(p-1)$st root of unity and $i \neq 0$, $\alpha^i - 1$ reduces to a unit in $\mathbb{Z}/p$. Let $\beta = (\alpha^i - 1)^{-1}$. Since $\psi^\alpha = \alpha^i$ on A_i, we have

$$\psi^k - 1 = (\psi^\alpha - 1) - (\psi^\alpha - \psi^k) = (\alpha^i - 1) - (\psi^\alpha - \psi^k) = (\alpha^i - 1)(1 - U_\beta),$$

which yields the result. $\square$

In proposition 10.15 below, we will generalize 10.8 to show that ρ^k induces an isomorphism from $B_0(X)$ to $1 + B_0(X)$. First, we need the following lemma.

Lemma 10.14. *The inclusion $IKSU_G(X)\hat{_p} \to IKU_G(X)\hat{_p}$ induces an isomorphism when restricted to the kernels of $\psi^\alpha - 1$.*

Proof. The fiber sequence of Lemma 2.17 induces a long exact sequence

$$\cdots \to KS^1_G(\Sigma X) \to IKSU_G(X) \to IKU_G(X) \to KS^1_G(X),$$

where $KS^1_G(X)$ denotes the set of (G, S^1)-bundles, or G-equivariant complex line bundles, over X, with group structure given by tensor product of line bundles. The map $IKU_G(X) \to KS^1_G(X)$ is induced by taking nth exterior powers of n-dimensional line bundles, which extends to virtual bundles since taking the top exterior powers converts sums to products; that is, $\lambda^{n+m}(x+y) = \lambda^n(x)\lambda^m(y)$ when x and y are n and m-dimensional respectively.

After p-completing the groups in the above sequence, the Adams operations extend to the p-adics, yielding compatible actions of ψ^α for $\alpha \in \hat{\mathbb{Z}}_p$. In particular, on $KS^1_G(X)\hat{_p}$, ψ^k acts as multiplication by k, so that ψ^α acts as multiplication by α. Thus, the kernel of $\psi^\alpha - 1$ on $KS^1_G(X)\hat{_p}$ is trivial. The result now follows from the long exact sequence. □

Proposition 10.15. *$\rho^k : B_0(X) \to 1 + B_0(X)$ is an isomorphism.*

Proof. Injectivity of ρ^k follows since 2 is a unit, c is injective and commutes with ψ^α, and $\rho^k_c(cx) = c\rho^k(2x)$ (9.1). Now, if $1 + x \in 1 + B_0$, then $1 + cx \in 1 + A_0$ is $\rho^k_c(y)$ for some $y \in A_0$, by 10.8. By 10.14, y can be represented by an element in $IKSU_G(X)\hat{_p}$, so that $\rho^k_c(y) = c\rho^k ry$, so that $1 + cx = c\rho^k(ry)$. Since c is injective, $1 + x = \rho^k(ry)$, so that ρ^k is surjective. □

Since a direct summand of an exact sequence is exact and $B_i(-)$ takes products to products, $B_i(-)$, considered as a functor on the homotopy category of G-connected based G-CW-complexes, is representable by [**19**, p.134]. Thus, the p-completion of the G-connected cover of $B_G(O)$ splits into a product $W_G \times W_G^\perp$ of G-connected based G-CW-complexes, where W_G represents B_0 and $W_G^\perp$ represents $B_0^\perp = \oplus_{i=1}^{p-2} B_i$.

Corollary 10.16. *After passing to G-connected covers, the Adams-May square becomes a pull-back square in the homotopy category.*

Proof. All the maps in the square commute with ψ^α, and hence take W_G to W_G and $W_G^\perp$ to $W_G^\perp$. Moreover, $\psi^k - 1$ and $\psi^k/1$ induce an equivalence on $W_G^\perp$ by Proposition 10.13, and ρ^k and σ^k induce an equivalence on W_G by 10.7 and 10.15. □

Now let J^k_G and $J^k_{G\otimes}$ be the fibers of the maps $\psi^k - 1$ and $\psi^k/1$ in the Adams-May square. Note that the map $\psi^k - 1 : B_G(O) \to B_G(Spin)$ also has a fiber, which we denote F (we will only need this space briefly). The maps α^k_F and ϵ^k in the diagram below are induced maps of fibers.

$$\begin{tikzcd}
F \arrow[r, "\pi"] \arrow[d, dotted, "\alpha_F^k"'] & B_G(O) \arrow[r, "\psi^k - 1"] \arrow[d, "\gamma^k"] & B_G(Spin) \arrow[d, equal] \\
\Omega B_G(\hat{\mathbb{S}}_p) \arrow[r, "\tau"] \arrow[d, dotted, "\epsilon^k"'] & Fib(Bj_{\hat{p}}) \arrow[r, "q"] \arrow[d, "f"] & B_G(Spin) \arrow[d, "\rho^k"] \\
(J^k_{G\otimes}) \arrow[r, "\pi"] & B_G(O)_{\otimes\hat{p}} \arrow[r, "\psi^k/1"] & B_G(O)_{\otimes\hat{p}}.
\end{tikzcd}$$

The completion map on $B_G(O)$ and the projection from $B_G(Spin)$ to $B_G(O)$ induce a map $F \to J^k_G$, which induces p-completion on all homotopy groups since the map $B_G(Spin) \to B_G(O)$ is an equivalence away from 2. Moreover, by 8.3, the space $\Omega B_G(\hat{\mathbb{S}}_p)$ is equivalent to $\Omega^\infty \hat{S}_p^\times$, so that all of its homotopy groups are p-complete. Therefore the map α^k_F above factors uniquely through a map α^k from J^k_G to the component of 1 in $\Omega^\infty \hat{S}_p$. Letting $\tau^k : J^k_G \to J^k_{G\otimes}$ be the composite $\epsilon^k \circ \alpha^k$, we have completed the construction of the equivariant J-theory diagram.

$$\begin{tikzcd}
J^k_G \arrow[r, "\pi"] \arrow[d, "\tau^k"] & B_G(O)_{\hat{p}} \arrow[r, "\psi^k - 1"] \arrow[d, "\sigma^k"] & B_G(O)_{\hat{p}} \arrow[d, "\rho^k"] \\
J^k_{G\otimes} \arrow[r, "\pi"] & B_G(O)_{\otimes\hat{p}} \arrow[r, "\psi^k/1"] & B_G(O)_{\otimes\hat{p}}.
\end{tikzcd}$$

Theorem 10.17. *The map τ^k induces a weak equivalence on G-connected covers, and therefore yields a splitting of the G-connected cover of $\Omega^\infty \hat{S}_p$.*

Proof. By Proposition 10.3, $\psi^k/1$ (and similarly $\psi^k - 1$) induces an equivalence after inverting p, so that the homotopy groups of J^k_G and $J^k_{G\otimes}$ are finite abelian p-groups. Moreover, the homotopy groups of J^k_G and $J^k_{G\otimes}$ are abstractly isomorphic. Thus, it suffices to check that $\epsilon^k \circ \alpha^k$ induces an injection on all positive homotopy groups.

Suppose $\gamma \in \pi^H_n(J^k_G)$ maps to zero under $\epsilon^k \circ \alpha^k$. Then γ maps to zero under $\sigma^k \circ \pi$ and under $(\psi^k - 1) \circ \pi$. By the pull-back property, γ maps to zero under π and hence lifts to an element $\gamma' \in \pi^H_{n+1}(B_G(O))$. Then $\rho^k(\gamma')$ maps to zero in $\pi^H_n(J^k_{G\otimes})$, and hence lifts to $\pi^H_{n+1}(B_G(O)_\otimes)$. By the pull-back property, γ' is in the image of $(\psi^k - 1)_*$ and hence $\gamma = 0$. □

11. Appendix

In this Appendix, we consider further Construction 2.14. In particular, we show that for an $h\mathcal{V}$-functor A and a complete sequence $\mathbb{V}$ of representations in $\mathcal{V}$, $A(\mathbb{V})$ has the structure of a weak G-Hopf space. By this we mean that there is a product $\mu : A(\mathbb{V}) \times A(\mathbb{V}) \to A(\mathbb{V})$ and a unit $\eta : * \to A(\mathbb{V})$ (given by the basepoint) such that the diagrams below are homotopic when their sources are restricted to finite subcomplexes.

$$\begin{tikzcd}
A(\mathbb{V}) \times A(\mathbb{V}) \times A(\mathbb{V}) \arrow[r, "\mu\times 1"] \arrow[d, "1\times\mu"] & A(\mathbb{V}) \times A(\mathbb{V}) \arrow[d, "\mu"] \\
A(\mathbb{V}) \times A(\mathbb{V}) \arrow[r, "\mu"] & A(\mathbb{V})
\end{tikzcd}
\qquad
\begin{tikzcd}
A(\mathbb{V}) \times * \arrow[r, "1\times\eta"] \arrow[dr, "="'] & A(\mathbb{V}) \times A(\mathbb{V}) \arrow[d, "\mu"] & * \times A(\mathbb{V}) \arrow[l, "\eta\times 1"'] \arrow[dl, "="] \\
& A(\mathbb{V}) &
\end{tikzcd} .$$

In addition, we show that when $\pi_0(A(\mathbb{V})^H)$ is a group for all $H \leqslant G$, then $A(\mathbb{V})$ has a weak homotopy inverse map χ, so that the diagram below commutes when the source is restricted to finite subcomplexes:

$$\begin{tikzcd}
A(\mathbb{V}) \arrow[r, "1\times\chi"] \arrow[dr, "*"'] & A(\mathbb{V}) \times A(\mathbb{V}) \arrow[d, "\mu"] \\
& A(\mathbb{V}).
\end{tikzcd}$$

We first define μ. The maps $A(V_i) \times A(V_i) \to A(V_i \oplus V_i)$ assemble to determine a map $A(\oplus) : A(\mathbb{V}) \times A(\mathbb{V}) \to A(\mathbb{V} \oplus \mathbb{V})$, and the maps $A(\iota_1) : A(V_i) \to A(V_i \oplus V_i)$ induced by the inclusion of V_i as the first summand in $V_i \oplus V_i$ assemble to determine an equivalence $A(\iota_1) : A(\mathbb{V}) \to A(\mathbb{V} \oplus \mathbb{V})$. We let μ be the composite $A(\iota_1)^{-1} \circ A(\oplus)$.

Remark 11.1. Any equivariant isometry $\alpha : V \to V$ is homotopic to the identity, since we can break V up into a sum of irreducibles, and the space of equivariant isomorphisms $W \to W$ is $U(n)$ if W is a sum of n copies of an irreducible complex G-representation. In particular, the transposition $\tau : V \oplus V \to V \oplus V$ is homotopic to the identity, so that $A(\tau) : A(V \oplus V) \to A(V \oplus V)$ is homotopic to the identity. It follows that $A(\iota_1)$ is homotopic to $A(\iota_2)$. We could therefore define μ equivalently using ι_2.

The following diagram commutes by associativity and naturality of c. It follows that μ is homotopy associative when restricted to finite subcomplexes

$$\begin{tikzcd}
A(V) \times A(V) \times A(V) \arrow[r, "c\times 1"] \arrow[d, "1\times c"] & A(V \oplus V) \times A(V) \arrow[d, "c"] & A(V) \times A(V) \arrow[l, "A(\iota_1)\times 1"'] \arrow[d, "c"] \\
A(V) \times A(V \oplus V) \arrow[r, "c"] & A(V \oplus V \oplus V) & A(V \oplus V) \arrow[l, "A(\iota_{1,3})"'] \\
A(V) \times A(V) \arrow[r, "c"] \arrow[u, "1\times A(\iota_1)"'] & A(V \oplus V) \arrow[u, "A(\iota_{1,2})"'] & A(V). \arrow[l, "A(\iota_1)"'] \arrow[u, "A(\iota_1)"']
\end{tikzcd}$$

Here, $\iota_{1,2}$ and $\iota_{1,3}$ are the inclusions $V \oplus V \subseteq V \oplus V \oplus V$ in the first two summands and the first and third summand respectively.

A similar argument, together with Remark 11.1, shows that μ is homotopy unital when restricted to finite complexes.

To obtain the map χ, we need the equivariant Dold theorem (see [**28**, 1.11]), the statement of which requires a bit of terminology.

Definition 11.2. A *numerable G-cover* $\mathcal{C}$ of B is a locally finite cover by invariant sets such that for each $U \in \mathcal{C}$ there is a G-map $\lambda_U : B \to I$ with $U = \lambda_U^{-1}(0, 1]$.

Definition 11.3. A *tube* in a G-space X is a G-subspace of the form $U \simeq G \times_H V$ where V is some H-invariant subspace of U, and H is the isotropy subgroup of some point v in V. The orbit Gv is called a *central orbit* of U.

The following is the equivariant generalization of Dold's theorem.

Theorem 11.4. *Suppose that B has a numerable G-cover $\mathcal{C}$ of open tubes which deform equivariantly to specified central orbits and that $g : p \to q$ is a map of G-fibrations over B which restricts to an equivariant homotopy equivalence on each fiber. Then g is a fiberwise G-homotopy equivalence.*

The nonequivariant proof that any CW complex has a numerable cover of open sets (see, for example, [**13**, pp. 29, 249]) easily generalizes to show that any G-CW complex satisfies the hypothesis in the equivariant Dold theorem. Moreover, all the classifying spaces that we've constructed have the structure of G-CW complexes (see Remark 3.6).

Corollary 11.5. *Suppose X is a (weak) Hopf G-space with product $\mu : X \times X \to X$ such that $\pi_0^H(X)$ is a group for each $H \leqslant G$. Then X has a (weak) homotopy inverse map.*

Proof. Consider the following map of G-fibrations over X, in which π_1 is projection onto the first factor.

$$\begin{tikzcd} X \times X \arrow[rr, "\pi_1 \times \mu"] \arrow[dr, "\pi_1"'] & & X \times X \arrow[dl, "\pi_1"] \\ & X & \end{tikzcd}$$

The restriction of the above map of G-fibrations to the fiber over $x \in X^H$ is the H-map given by $y \to \mu(x, y)$. If x' is in the component of the inverse of x in $\pi_0(X^H)$, then $y \to \mu(x, y)$ is weakly inverse to the map $y \to \mu(x', y)$, and hence a weak equivariant equivalence. Since X has the homotopy type of a G-CW complex, any weak equivariant equivalence is an equivariant equivalence.

Thus, by the G-Dold theorem, there is a map $\pi_1 \times \sigma : X \times X \to X \times X$ which is inverse to $\pi_1 \times \mu$. The composite $\pi_2 \circ (\pi_1 \times \sigma) \circ (\mathrm{id} \times *)$ is a weak homotopy inverse map. □

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