EXISTENCE OF MULTIPLE CRITICAL POINTS FOR AN ASYMPTOTICALLY QUADRATIC FUNCTIONAL WITH APPLICATIONS

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Abstract. Morse theory for isolated critical points at infinity is used for the existence of multiple critical points for an asymptotically quadratic functional. Applications are also given for the existence of multiple nontrivial periodic solutions of asymptotically Hamiltonian systems.

1. Introduction and preliminaries

It is known that the objective of the Morse theory is the relation between the topological type of critical points of a function f and the topological structure of the manifold on which the function is defined. The topological type of a critical point x is described by the critical groups $C_k(f,x)$ for which there have been many known results, cf ([3], [8], [7], etc.). The topological structure of the manifold M is described by its Betti number $\beta_k = \dim H_k(M)$. One can make use of the Morse inequalities to gain the existence of unknown critical points to f once one gets some precise information related to $C_k(f,x)$ and β_k or $H_k(M)$.

In this paper we prove some abstract multiple critical point theorems via Morse theory, and then apply these theorems to the study of the existence of multiple periodic solutions for a second order Hamiltonian system. In this section we state some known results concerned with the Betti number β_k. Let us begin with some notions. Let X be a Hilbert space and $f : X \to \mathbb{R}^1$ be a C^1-function. We write $K = \{x \in X : f'(x) = 0\}$ and $f^a = \{x \in X : f(x) \le a\}$ for $a \in \mathbb{R}^1$. The following definition is due to [1].

Definition 1.1. *Suppose that $f(K)$ is bounded from below by $a \in \mathbb{R}^1$ and that f satisfies $(PS)_c$ for all $c \le a$. Then the group*

$$C_k(f,\infty) := H_k(X; f^a), \quad k \in \mathbb{Z},$$

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is said to be the k-th critical group of f at infinity. Here $H_*(\cdot,\cdot)$ denotes a singular relative homology group with the abelian coefficient group G.

From this definition we see that the topology of the pair (X, f^a) contains all the information about the critical points of f because we require that $f(K)$ be bounded from below by $a \in \mathbb{R}^1$. Therefore we need precise estimates for $C_k(f,\infty)$ (actually the Betti number $\beta_k = \dim H_k(X, f^a)$), for $k \in \mathbb{Z}$. This has been done for some type of an indefinite functional, say, for example, an asymptotically quadratic functional. One can refer to [3] or [1].

In [1], Bartsch and Li given some precise descriptions of the critical group $C_k(f,\infty)$ under the following framework:

(A_∞) $f(x) = \frac{1}{2}\langle Ax, x\rangle + g(x)$, where $A : X \to X$ is a self-adjoint linear operator such that 0 is isolated in the spectrum of A. The map $g \in C^2(X, \mathbb{R}^1)$ satisfies $g''(x) \to 0$ as $\|x\| \to \infty$. Moreover g and g' map bounded sets into bounded sets and g' is compact. f satisfies $(PS)_c$ for $c << 0$ and $f(K)$ is bounded from below.

In this case we may say $A = f''(\infty)$ and f'' is continuous at ∞.

Let (A_∞) hold. Set $V := \mathrm{Ker}A$ and $W = V^\perp$. We split W into $W^+ \oplus W^-$ according to the spectrum of A such that $A|_{W^+}$ (resp. $A|_{W^-}$) is positive (resp. negative) definite. Let $\mu := \dim W^-$ be the Morse index of f at infinity and $\nu := \dim V$ be the nullity of f at infinity.

Proposition 1.1. *If (A_∞) holds then*

$$C_k(f,\infty) = 0 \quad \text{for} \quad k \notin [\mu, \mu+\nu].$$

This is also true if $\mu = \infty$ or $\nu = \infty$. If $\mu < \infty$ and $\nu = 0$, then

$$C_k(f,\infty) \cong \delta_{k\mu} G.$$

Proposition 1.2. *Let f satisfy (A_∞).*

a) *$C_k(f,\infty) \cong \delta_{k\mu} G$ provided f satisfies the following angle condition at infinity:*

(AC_∞^+) There exists $M > 0$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $\langle f'(x), v\rangle \geq 0$ for any $x = v + w \in X = V \oplus W$ with $\|x\| \geq M$ and $\|w\| \leq \|x\| \sin\alpha$.

b) *$C_k(f,\infty) \cong \delta_{k,\mu+\nu} G$ provided f satisfies the following condition at infinity:*

(AC_∞^-) There exists $M > 0$ and $\alpha \left(0, \frac{\pi}{2}\right)$ such that $\langle f'(x), v\rangle \leq 0$ for any $x = v + w \in X = V \oplus W$ with $\|x\| \geq M$ and $\|w\| \leq \|x\| \sin\alpha$.

There are many well-known results related to the critical groups of f at an isolated critical point for which one can refer to [8], [3], [7] or others. In [1] a result similar to Prop. 1.2 was given.

Proposition 1.3. *Suppose that θ is an isolated critical point of f such that 0 is isolated in the spectrum $A_0 := d^2 f(\theta)$. Let μ_0 and ν_0 be the Morse index and nullity of θ respectively and $\nu_0 < \infty$ and $\mu_0 < \infty$. Then*

a) *$C_k(f,\theta) \cong \delta_{k\mu_0} G$ provided f satisfies the following angle condition at θ:*

(AC_0^+) *There exists* $\rho > 0$ *and* $\alpha \in \left(0, \frac{\pi}{2}\right)$ *such that* $\langle f'(x), v\rangle \geq 0$ *for any* $x = v + w \in X = V_0 \oplus W_0$ *with* $\|x\| \leq \rho$ *and* $\|w\| \leq \|x\| \sin\alpha$.

b) $C_k(f, \theta) \cong \delta_{k\mu_0+\nu_0} G$ *provided* f *satisfies the following angle condition at* θ*:*

(AC_0^-) *There exists* $\rho > 0$ *and* $\alpha \in \left(0, \frac{\pi}{2}\right)$ *such that* $\langle f'(x), v\rangle \leq 0$ *for any* $x = v + w \in X = V_0 \oplus W_0$ *with* $\|x\| \leq \rho$ *and* $\|w\| \leq \|x\| \sin\alpha$.

Here, $V_0 := Ker(A_0)$ *and* $W_0 := V_0^{\perp}$.

One can refer to [1] for the details of the proofs of Propositions 1.1–1.3. In applications one has to verify that f satisfies the (PS) condition. We remark that f satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}^1$ provided f satisfies the strong angle conditions at infinity:

(SAC_∞^+) (or (SAC_∞^-)). There exist $M > 0, \beta > 0$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $\langle f'(x), \frac{v}{\|v\|}\rangle \geq \beta > 0$ (or resp. $\langle f'(x), \frac{v}{\|v\|}\rangle \leq -\beta < 0$) for any $x = v + w \in X = V \oplus W$ with $\|x\| \geq M$ and $\|w\| \leq \|x\| \sin\alpha$.

More precisely, we have

Lemma 1.1. *Let* f *satisfy* (A_∞) *and* (SAC_∞^+) *(or* (SAC_∞^-)*). Then* f *satisfies the* $(PS)_c$ *condition at every* $c \in \mathbb{R}^1$, *i.e. any sequence* $\{x_n\} \subset X$ *for which* $f(x_n) \to c$ *and* $f'(x_n) \to 0$ *as* $n \to \infty$ *has a convergent subsequence.*

Proof. Let $\{x_n\} \subset X$ be such that

$$f(x_n) \to c \quad \text{as} \quad n \to \infty, \tag{1.1}$$

$$f'(x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{1.2}$$

We first show that $\{x_n\}$ is bounded. Suppose not, then

$$\|x_n\| \to \infty \quad \text{as} \quad n \to \infty. \tag{1.3}$$

Write $x_n = w_n^+ + v_n + w_n^-$ and $w_n = w_n^+ + w_n^-$ where $w_n^\pm \in W^\pm, v_n \in V$ and $w_n \in W$ respectively.

Now for any $y \in X$, we have

$$\langle f'(x_n), y\rangle = \langle Ax_n, y\rangle + \langle g'(x_n), y\rangle. \tag{1.4}$$

Let $\overline{\lambda}$ be the smallest positive point in the spectrum of A and take $y = w_n^+$ in (1.4) then we get

$$\overline{\lambda}\|w_n^+\|^2 \leq \langle f'(x_n), w_n^+\rangle - \langle g'(x_n), w_n^+\rangle. \tag{1.5}$$

Given $\varepsilon > 0$, using (1.2), (1.3) and $g''(x_n) \to 0$ as $n \to \infty$, it follows that there are some constants $c, d > 0$ such that

$$\overline{\lambda}\|w_n^+\|^2 \leq \varepsilon\|x_n\| \quad \|w_n^+\| + c\|w_n^+\| + d \tag{1.6}$$

for n large enough. Hence from (1.6), (1.3) and the fact that ε was chosen arbitrarily we gain

$$\frac{\|w_n^+\|}{\|x_n\|} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.7}$$

Similarly we can show

$$\frac{\|w_n^-\|}{\|x_n\|} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.8}$$

Hence

$$\frac{\|v_n\|}{\|x_n\|} \to 1 \quad \text{and} \quad \frac{\|w_n\|}{\|x_n\|} \to 0 \quad \text{as} \quad n \to \infty. \tag{1.9}$$

Then for $M > 0$ and $\alpha \in \left(0, \frac{\pi}{2}\right)$ we have

$$\|x_n\| \geq M \quad \text{and} \quad \|w_n\| \leq \|x_n\| \sin\alpha \tag{1.10}$$

for all n large enough. Thus the strong angle conditions yields

$$\left|\langle f'(x_n), \frac{v_n}{\|v_n\|}\rangle\right| \geq \beta \quad \text{for all } n \text{ large enough.}$$

This contradicts to the following

$$\lim_{n\to\infty} \langle f'(x_n), \frac{v_n}{\|v_n\|}\rangle = 0 \tag{1.12}$$

and hence $\{x_n\}$ is bounded. Since g' is compact and V is finite dimensional, by the standard argument we gain the existence of a convergent subsequence of $\{x_n\}$. The proof is complete. ∎

This paper is organized in the following way: In section 2 we prove some abstract critical point theorems by means of Proposition 1.2 and 1.3 and the Morse theory. In section 3, as applications, we deal with the existence of multiple nontrivial periodic solutions for asymptotically Hamiltonian system. As we will see that the main difficulty is to verify the strong angle conditions which can be guaranteed by the so-called "pinching" condition.

2. Some abstract critical point theorems

In this section we give some abstract multiple critical point theorems under the framework constructed in [1]. In the following we will denote by μ_i and ν_i the Morse index and nullity of critical points x_i of a functional f. We first consider the case that θ is a nondegenerate critical point of f.

Theorem 2.1. *Let f satisfy (A_∞) and (SAC_∞^+) (or (SAC_∞^-)) and θ be a nondegenerate critical point of f with Morse index μ_0. If $\mu_0 \neq \mu$ (or $\mu_0 \neq \mu+\nu$) then f has at least one nontrivial critical point $x_1 \neq \theta$. Moreover if $\nu_1 \leq |\mu_0 - \mu|$ (or $\nu_1 \leq |\mu_0 - (\mu+\nu)|$) then f has at least two nontrivial critical points.*

Proof. We would like to point out that the techniques for proving this theorem are essentially the same as those for proving [3, Chapter 2, Corollary 5.2]. Also, see [4]. We only sketch out the proof in the case (SAC_∞^+) holds. It follows from Lemma 1.1 and Proposition 1.2 (a) that f satisfies (PS) condition and

$$C_k(f,\infty) \cong \delta_{k\mu} G, \quad k \in \mathbb{Z}.$$

Since θ is nondegenerate with Morse index μ_0,

$$C_k(f,\theta) \cong \delta_{k\mu_0} G, \quad k \in \mathbb{Z}.$$

Hence we have by $\mu_0 \neq \mu$ that

$$C_k(f,\infty) \ncong C_k(f,\theta), \quad k \in \mathbb{Z}$$

from which we get the first conclusion that f has at least one critical point $x_1 \neq \theta$. By a result due to [5], we get

$$C_k(f,x_1) \cong 0, \quad \text{for} \quad k \notin [\mu_1, \mu_1 + \nu_1].$$

Suppose that f has no more other critical points then the relation between the μ-th Morse type number and the μ-th Betti number (cf. [3]) told us that

$$C_\mu(f,x_1) \ncong 0.$$

Hence

$$\mu_1 \le \mu \le \mu_1 + \nu_1.$$

When $\mu = \mu_1$ or $\mu = \mu_1 + \nu_1$, using the splitting theorem and the critical group characterization of the local minimum and the local maximum (cf. [8, Corollary 8.4]) we get

$$C_k(f,x_1) \cong \delta_{k\mu} G, \quad k \in \mathbb{Z}.$$

The Morse inequality now reads as

$$(-1)^\mu = (-1)^{\mu_0} + (-1)^\mu.$$

This is impossible.

We now consider the case $\mu_1 < \mu < \mu_1 + \nu_1$. Combine with the condition $\nu_1 \le |\mu_0 - \mu|$ we have

$$\mu_0 < \mu_1 \quad \text{or} \quad \mu_0 > \mu_1 + \nu_1.$$

For the case $\mu_0 < \mu_1 < \mu$ the $\mu_0 + 1$-th Morse inequality reads as

$$-1 \ge 0.$$

This is a contradiction. For the case $\mu_0 > \mu_1 + \nu_1 > \mu$, the $\mu_1 + \nu_1$-th Morse inequality reads as

$$\sum_{k=0}^{\mu_1+\nu_1} (-1)^{\mu_1+\nu_1-k} \text{rank } C_k(f,x_1) \ge (-1)^{\mu_1+\nu_1-\mu}, \tag{2.1}$$

and the $\mu_1 + \nu_1 - 1$-th Morse inequality reads as

$$\sum_{k=0}^{\mu_1+\nu_1-1} (-1)^{\mu_1+\nu_1-1-k} \text{ rank } C_k(f,x_1) \ge (-1)^{\mu_1+\nu_1-\mu-1}. \tag{2.2}$$

Keeping in mind that $C_k(f,x_1) = 0$ for $k \le \mu_1$ and $k \ge \mu_1 + \nu_1$, it follows from (2.1) and (2.2) that

$$\sum_{k=\mu_1}^{\mu_1+\nu_1} (-1)^{\mu_1+\nu_1-k} (\text{rank } C_k(f,x_1) - \delta_{k\mu}) = 0.$$

Hence the $\mu_0 + 1$-th Morse inequality now reads as

$$-1 \geq 0.$$

This is also a contradiction. Therefore we gain the conclusion that f has at least two nontrivial critical points. The proof is complete. ∎

The next two theorems are concerned with the case θ is a degenerate critical point of f.

Theorem 2.2. *Let f satisfy (A_∞) and (SAC_∞^+). Let θ be a degenerate critical point and the angle condition (AC_0^+) (or (AC_0^-)) hold. If $\mu_0 \neq \mu$ (or $\mu_0 + \nu_0 \neq \mu$) then f has at least one nontrivial critical point $x_1 \neq \theta$. Moreover if $\nu_1 \leq |\mu_0 - \mu|$ (or $\nu_1 \leq |\mu_0 + \nu_0 - \mu|$) then f has at least two nontrivial critical points.*

Theorem 2.3. *Let f satisfy (A_∞) and (SAC_∞^-). Let θ be a degenerate critical point of f and (AC_0^+) (or (AC_0^-)) hold. If $\mu_0 \neq \mu + \nu$ (or $\mu_0 + \nu_0 \neq \mu + \nu$) then f has at least one nontrivial critical point $x_1 \neq \theta$. Moreover if $\nu_1 \leq |\mu_0 - (\mu + \nu)|$ (or $\nu_1 \leq |(\mu_0 + \nu_0) - (\mu + \nu)|$) then f has at least two nontrivial critical points.*

The proofs of Theorem 2.2 and 2.3 are similar to that of Theorem 2.1 so we omit the details.

Remark 2.1. In [3] or [9] the same conclusion as Theorem 2.1 was obtained under the following framework:

Let $f : X \to \mathbb{R}^1$ be such that $f(x) = \frac{1}{2}\langle Ax, x\rangle + g(x)$ and satisfy

(A_1) $A|_{X_\pm}$ has a bounded inverse on $X_\pm$,

(A_2) $\gamma = \dim(X_- \oplus X_0) < \infty$,

(A_3) $g \in C^2(X, \mathbb{R}^1)$ has a compact, bounded differential g' and $g(x_0) \to -\infty$ as $\|x_0\| \to \infty$ for $x_0 \in X_0$, where $X = X_+ \oplus X_0 \oplus X_-$ according to the spectrum decomposition of the self-adjoint linear operator A.

We would like to point out that the above framework differs from ours for there is a strong assumption requiring g' is bounded which is not need in our case.

3. Applications to a second order hamiltonian system

We consider the following second order Hamiltonian system

$$\begin{cases} -\ddot{x} = k^2 x + F_x'(t,x), \\ x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{3.1}$$

where $k \in \mathbb{N}$ and $F : \mathbb{R}^1 \times \mathbb{R}^N \to \mathbb{R}^1$, is a C^2-function and satisfies

(F_1) $F(t,\theta) = 0, F_x'(t,\theta) = \theta$ and $F(t+2\pi, x) = F(t,x),\ (t,x) \in \mathbb{R}^1 \times \mathbb{R}^N$,

(F_2) $$\lim_{|x|\to\infty} F_x''(t,x) = 0 \quad \text{uniformly in} \quad t \in \mathbb{R}^1.$$

Here and in the following, for $x, y \in \mathbb{R}^N$, the symbol xy will denote the inner product in $\mathbb{R}^N$, similarly if A is a $N \times N$ matrix, Ax will denote the usual matrix product. $|x|$ will denote the $\mathbb{R}^N$-norm for $x \in \mathbb{R}^N$.

It is known that the eigenvalues of the linear problem

$$\begin{cases} -\ddot{x} = \lambda x, & t \in (0, 2\pi), \\ x(0) = x(2\pi), \\ \dot{x}(0) = \dot{x}(2\pi), \end{cases} \tag{3.2}$$

are $m^2, m = 0, 1, 2, \cdots$, and the multiplicity of m^2 for $m \geq 1$ are $2N$ and the corresponding eigenspaces are $\operatorname{span}\{e_j \sin mt, e_j \cos mt, j = 1, 2, \cdots, N\}$ where $(e_1, e_2, \cdots, e_N)$ is the standard basis of $\mathbb{R}^N$.

We are interested in the existence of multiple nontrivial 2π-periodic solution of (3.1).

Theorem 3.1. *Let F satisfy $(F_1), (F_2)$ and the following conditions:*
(F_3) There exists $m \in \mathbb{N}$ with $m \neq k$ such that

$$m^2 I < F_x''(t, \theta) + k^2 I < (m+1)^2 I$$

where I is the $N \times N$ identity matrix.
(F_4) (the pinching condition) There exist $C_1, C_2, R > 0$ and $0 < r < 1$ such that

$$F_x'(t, x)x > 0, \qquad |F_x'(t, x)x| > C_1 |x|^{1+r},$$

$$|F_x'(t, x)| < C_2 |x|^r, \quad \textit{for a.e. } t \in [0, 2\pi] \quad \textit{and} \quad x \in \mathbb{R}^N \quad \textit{with} \quad |x| \geq R.$$

Then (3.1) has at least two nontrivial 2π-periodic solutions.

Let us introduce the Sobolev space

$$X := H^1([0, 2\pi], \mathbb{R}^N) = \left\{ x \in L^2([0, 2\pi], \mathbb{R}^N) \;\middle|\; \begin{array}{l} \dot{x} \in L^2([0, 2\pi], \mathbb{R}^N), \\ x(0) = x(2\pi), \dot{x}(0) = \dot{x}(2\pi) \end{array} \right\}$$

with the usual norm

$$\|x\| = \left(\int_0^{2\pi} |\dot{x}|^2 + |x|^2 \right)^{\frac{1}{2}}, \quad \text{for} \quad x \in X.$$

Then X is a Hilbert space. Define, for $x \in X$, the functional

$$f(x) = \frac{1}{2} \int_0^{2\pi} |\dot{x}|^2 - \frac{1}{2} k^2 \int_0^{2\pi} |x|^2 - \int_0^{2\pi} F(t, x). \tag{3.3}$$

Then $f \in C^2(X, \mathbb{R}^1)$ and its derivative is given by

$$\langle f'(x), y \rangle = \int_0^{2\pi} \dot{x}\dot{y} - k^2 \int_0^{2\pi} xy - \int_0^{2\pi} F_x'(t, x) y \tag{3.4}$$

for $x, y \in X$. Thus finding solutions of (3.1) is equivalent to finding critical points of f in X.

We split X into $V \oplus W^+ \oplus W^- = V \oplus W$ according the eigenvalue k^2 of (3.2) where

$$V := \mathrm{Ker}(-\ddot{x} - k^2 x),$$

$$W^- := \oplus_{j=0}^{k-1}\{\mathrm{Ker}(-\ddot{x} - j^2 x)\},$$

$$W^+ = (W^- \oplus V)^\perp \quad \text{and} \quad W = W^+ \oplus W^-.$$

Lemma 3.1. *Suppose that F satisfies $(F_1), (F_2)$ and (F_4). Then the functional f defined by (3.3) satisfies (SAC_∞^-) hence (AC_∞^-).*

Proof. Since the embedding $X \hookrightarrow Y := C([0, 2\pi], \mathbb{R}^N)$ is continuous and $\dim V < +\infty$, there exist $a, b > 0$ such that

$$\|x\|_Y \leq a\|x\|, \quad \text{for} \quad x \in Y, \tag{3.5}$$

$$\|v\| \leq b\|v\|_Y, \quad \text{for} \quad v \in V. \tag{3.6}$$

Since $V = \mathrm{span}\{(e_1, \cdots, e_N)\sin kt, (e_1, \cdots, e_N)\cos kt\}$ it is obvious that, for any $v \in V\backslash\{\theta\}$,

$$\mathrm{meas}\{t \in [0, 2\pi] | v(t) = \theta\} = 0.$$

Thus, using (3.8) in [2] for any given $\delta > 0$ small, there exists some constant $\alpha(\delta) > 0$ such that

$$\mathrm{meas}\{t \in [0, 2\pi] |\ |v(t)| < \alpha(\delta)\|v\|_Y\} < \delta \quad \text{for} \quad v \in V\backslash\{\theta\}. \tag{3.7}$$

Hence

$$\mathrm{meas}\{t \in [0, 2\pi] |\ |v(t)| > \alpha(\delta)\|v\|_Y\} \geq 2\pi - \delta \quad \text{for} \quad v \in V\backslash\{\theta\}. \tag{3.8}$$

Write

$$\Omega_\delta = \{t \in [0, 2\pi] |\ |v(t)| > \alpha(\delta)\|v\|_Y\}, v \in V\backslash\{\theta\}. \tag{3.9}$$

Let

$$C(M, \varepsilon) = \{x = v + w \in X | \|x\| \geq M, \|w\| \leq \varepsilon\|x\|\}, \tag{3.10}$$

where $M > 0$ and $\varepsilon > 0$ will be chosen below.

It follows from (3.10) that for $x = v + w \in C(M, \varepsilon)$

$$\|w\| \leq \frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\|v\|, \tag{3.11}$$

$$\|x\| \leq \frac{1}{\sqrt{1-\varepsilon^2}}\|v\|. \tag{3.12}$$

Now for $x = v + w \in C(M, \varepsilon)$

$$|w(t)| \leq \|w\|_Y \leq \frac{ab\varepsilon}{\sqrt{1-\varepsilon^2}}\|v\|_Y, \quad t \in [0, 2\pi]. \tag{3.13}$$

$$|w(t)| \leq \|w\|_Y \leq a\varepsilon\|x\|, \quad t \in [0, 2\pi]. \tag{3.14}$$

Hence if we choose ε small enough such that

$$\frac{ab\varepsilon}{\sqrt{1-\varepsilon^2}} << \frac{1}{2}\alpha(\delta), \tag{3.15}$$

then

$$|x(t)| > \frac{1}{2}|v(t)| \geq \frac{1}{2}\alpha(\delta)\|v\|_Y \geq \frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\|x\|, \quad \text{for} \quad t \in \Omega_\delta. \tag{3.16}$$

It follows from (3.16) that

$$|x(t)| > R, \quad \text{for} \quad t \in \Omega_\delta \tag{3.17}$$

if we take M large where R is given in (F_4).

Write $\Omega' = [0, 2\pi] - \Omega_\delta$ and $\Omega' = \Omega'_1 + \Omega'_2$ where

$$\Omega'_1 = \{t \in \Omega' : |x(t)| \geq R\}, \quad \Omega'_2 = \{t \in \Omega' : |x(t)| < R\}.$$

Now for $x \in C(M, \varepsilon)$

$$\int_0^{2\pi} F'_x(t,x)v = \int_0^{2\pi} F'_x(t,x)x - \int_0^{2\pi} F'_x(t,x)w.$$

Using (3.10)–(3.17) and $(F_1), (F_2)$ and (F_4), we have

$$\begin{aligned}
&\int_0^{2\pi} F'_x(t,x)x = \left(\int_{\Omega_\delta+\Omega'_1+\Omega'_2}\right) F'_x(t,x)x \\
&\geq \int_{\Omega_\delta} F'_x(t,x)x - \int_{\Omega'_1} |F'_x(t,x)||x(t)| - \int_{\Omega'_2} |F'_x(t,x)||x(t)| \\
&\geq \int_{\Omega_\delta} C_1|x(t)|^{1+r} - \int_{\Omega'_1} C_2|x(t)|^{1+r} - C \\
&> (2\pi - \delta)C_1 \left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+r} \|x\|^{1+r} \\
&\quad - C_2\delta a^{1+r}(\varepsilon + \alpha(\delta))^{1+r}\|x\|^{1+r} - C \\
&= 2\pi C_1 \left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+r} \|x\|^{1+r} \\
&\quad - \delta\left[C_1\left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+r} + C_2a^{1+r}(\varepsilon+\alpha(\delta))^{1+r}\right]\|x\|^{1+r} - C
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{2\pi} F'_x(t,x)w &\leq \int_{|x(t)|\geq R} |F'_x(t,x)||w(t)| + \int_{|x(t)|\leq R} |F'_x(t,x)||w(t)| \\
&\leq \varepsilon C\|x\|^{1+r} + C.
\end{aligned}$$

Here we use C denotes various constants. Hence

$$\int_0^{2\pi} F'_x(t,x)v > \eta\|x\|^{1+r} - C \tag{3.19}$$

where

$$\eta = 2\pi C_1 \left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+r}$$
$$- \delta \left(C_1 \left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+r} + C_2 a^{1+r}(\varepsilon + \alpha(\delta))^{1+r}\right) - C\varepsilon.$$

If we take $\delta > 0$ small enough hence choose $\varepsilon > 0$ small enough (notice (3.15)) then $\eta > 0$. It follows from (3.19) that for $x = v + w \in C(M, \varepsilon)$

$$\langle f'(x), \frac{v}{\|v\|}\rangle = -\int_0^{2\pi} F'_x(t,x)\frac{v}{\|v\|} < -\eta\|x\|^r + \frac{c}{\|v\|} < -\beta < 0$$

for some $\beta > 0$ if we take $M > 0$ large enough. Hence the strong angle condition (SAC_∞^-) holds if we take $\alpha \in \left(0, \frac{\pi}{2}\right)$ be such that $\sin\alpha = \varepsilon$. The proof is complete. ∎

Now we begin to prove Theorem 3.1.

Let A be the self-adjoint extension of the linear operator $-\frac{d^2}{dt^2} - k^2$, then for $x, y \in X$, we have

$$\langle Ax, y\rangle = \int_0^{2\pi} \dot{x}\dot{y} - k^2 \int_0^{2\pi} xy \tag{3.20}$$

Define a map $g : X \to \mathbb{R}^1$ as follows:

$$g(x) = -\int_0^{2\pi} F(t,x), \quad \text{for} \quad x \in X. \tag{3.21}$$

Then the functional f has the form

$$f(x) = \frac{1}{2}\langle Ax, x\rangle + g(x),\ x \in X. \tag{3.22}$$

Proof of Theorem 3.1. It is easy to see that f satisfies (A_∞) and (SAC_∞^-) by Lemma 3.1 and Lemma 1.1. The Morse index μ and the nullity ν of f at infinity are given by

$$\mu = \dim W^- = 2N(k-1) + N, \quad \nu = \dim V = 2N. \tag{3.23}$$

By (F_1) and (F_3) we see that θ is a nondegenerate critical point of f with Morse index $\mu_0 = 2Nm + N$. From $m \neq k$ we see that $\mu_0 \neq \mu + \nu$. Hence f has at least one critical point $x_1 \neq \theta$ by virtue of Theorem 2.1.

Since

$$\operatorname{Ker}(d^2 f(x_1)) = \{x \in H^1([0, 2\pi], \mathbb{R}^N) | - \ddot{x} = k^2 x + F''(t, x_1)x\},$$

we conclude that

$$\nu_1 = \dim \operatorname{Ker}(d^2 f(x_1)) \leq 2N.$$

Thus

$$|\mu_0 - (\mu + \nu)| = 2N|m - k| \geq 2N \geq \nu_1.$$

So that we gain the conclusion that f has at least two nontrivial critical points by applying Theorem 2.1 once more. That is (3.1) has at least two nontrivial 2π-periodic solution. ■

Remark 3.1. The same conclusion is valid if F satisfies (F_1) $(F_2), (F_3)$ with $m \neq k-1$ and the following
$(F_4)'$ There exist $C_1, C_2, R > 0$ and $0 < r < 1$ such that

$$F'_x(t,x)x < 0, \qquad |F'_x(t,x)x| > C_1|x|^{1+r},$$

$$|F'_x(t,x)| < C_2|x|^r, \quad \text{for } x \in \mathbb{R}^N \text{ with } |x| \geq R \quad \text{and for a.e.} \quad t \in [0, 2\pi].$$

Remark 3.2. We would like to point out that the same conclusion as Theorem 3.1 was obtained in[9] under a different framework [see Remark 2.1]. However, there was an important condition which requiring $|F'_x(t,x)|$ bounded was ignored in [9].

From now on we consider the case that θ is degenerate. For this aim we make the following assumption:
(F_5) There exists some integer $m \in \mathbb{N}$ such that

$$F''_x(t,\theta) = (m^2 - k^2)I.$$

It follows from (F_5) that $F'_x(t,x)$ can be written as

$$F'_x(t,x) = (m^2 - k^2)x + G'_x(t,x)$$

in a neighborhood of θ, where $G : \mathbb{R}^1 \times \mathbb{R}^N \to \mathbb{R}^1$ is of class C^2 and satisfies

$$(G_1) \qquad G(t,\theta) = 0,\ G'_x(t,\theta) = \theta \quad \text{and} \quad G(t + 2\pi, x) = G(t,x)$$

for $(t,x) \in \mathbb{R}^1 \times \mathbb{R}^N$. and

$$(G_2) \qquad |G'_x(t,x)| = \circ(|x|) \quad \text{as} \quad |x| \to 0, x \neq \theta.$$

Therefore (3.4) has the form

$$(3.24) \quad \langle f'(x), y\rangle = \int_0^{2\pi} \dot{x}\dot{y} - m^2 \int_0^{2\pi} xy - \int_0^{2\pi} G'_x(t,x)y \quad \text{for} \quad x, y \in X.$$

near θ.

Now we split X into $V_0 \oplus W_0$ according the eigenvalue m^2 of (3.2) where

$$V_0 = \operatorname{Ker}(-\ddot{x} - m^2 x) \quad \text{and} \quad W_0 = V_0^{\perp}.$$

Theorem 3.2. *Let F satisfy $(F_1), (F_2), (F_4), (F_5)$ with $m \neq k$ and the following:*
(G_3) There exist $C_3, C_4 > 0$ and $s > 1$ such that

$$G'_x(t,x)x > 0, \quad |G'_x(t,x)x| > C_3|x|^{1+s},$$

$$|G'_x(t,x)| < C_4|x|^s, \quad \textit{for a.e.} \quad t \in [0, 2\pi] \quad \textit{and} \quad x \in \mathbb{R}^N \quad \textit{with} \quad |x| \leq 1.$$

Then (3.1) has at least two nontrivial 2π-periodic solutions.

Proof. We only need to verify the angle condition (AC_0^-) of Proposition 1.3. Let $C(\rho,\varepsilon)=\{x=v+w\in X=V_0\oplus W_0 \mid \|x\|\le\rho, \|w\|\le\varepsilon\|x\|\}$ where $\rho>0$, $\varepsilon>0$ will be given below. It is easy to see that by the same reasons we can obtain (3.5)–(3.16) if we replace V by V_0 and W by W_0 and $C(M,\varepsilon)$ by $C(\rho,\varepsilon)$ respectively. Now for $x\in C(\rho,\varepsilon)$ we have

$$|x(t)|\le\|x\|_Y\le a\|x\|\le a\rho,\quad t\in[0,2\pi]. \tag{3.25}$$

Hence if we take ρ small then

$$|x(t)|\le 1\quad\text{for all}\quad x\in C(\rho,\varepsilon)\quad\text{and}\quad t\in[0,2\pi]. \tag{3.26}$$

Thus for $x=v+w\in C(\rho,\varepsilon)$

$$\begin{aligned}
&\int_0^{2\pi}G'_x(t,x)v=\int_0^{2\pi}G'_x(t,x)x-\int_0^{2\pi}G'_x(t,x)w\\
&>C_3\int_0^{2\pi}|x|^{1+s}-C_4\int_0^{2\pi}|x|^s|w|\\
&>C_3\int_{\Omega_\delta}|x|^{1+s}-C_3\int_{\Omega'}|x|^{1+s}-C_4\int_0^{2\pi}|x|^s|w|\\
&>(2\pi-\delta)C_3\left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+s}\|x\|^{1+s}-C_3\delta a^{s+1}(\varepsilon+\alpha(\delta))^{1+s}\|x\|^{1+s}\\
&\quad-2\pi C_4a^{1+s}\varepsilon\|x\|^{1+s}\\
&=\left[2\pi C_3\left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+s}\right.\\
&\quad\left.-\delta C_3\left(\left(\frac{\sqrt{1-\varepsilon^2}}{2b}\alpha(\delta)\right)^{1+s}+a^{1+s}(\varepsilon+\alpha(\delta))^{1+s}\right)-2\pi C_4a^{1+s}\varepsilon\right]\|x\|^{1+s}\\
&=\xi\|x\|^{1+s}.
\end{aligned}$$

Notice (3.15), if we take $\delta>0$ small enough and then take $\varepsilon>0$ small enough then $\xi>0$. Hence

$$\langle f'(x),v\rangle=-\int_0^{2\pi}G'_x(t,x)v<-\xi\|x\|^{1+s}<0$$

for any $x=v+w\in C(\rho,\varepsilon)$. Let $\alpha\in\left(0,\frac{\pi}{2}\right)$ be such that $\sin\alpha=\varepsilon$ then the angle condition (AC_0^-) holds.

Now we can apply Theorem 2.3 by keeping in mind that $m\ne k$ to gain the conclusion. The proof is complete. ∎

Remark 3.3. The same conclusion is also true if we let $(F_1),(F_2),(F_4)$ and (F_5) with $m\ne k+1$ and
$(G_3)'$ There exist $C_3,C_4>0$ and $s>1$ such that

$$G'_x(t,x)x<0,\quad |G'_x(t,x)x|>C_3|x|^{1+s},$$

$$|G'_x(t,x)|<C_4|x|^s,\quad\text{for a.e.}\quad t\in[0,2\pi]\quad\text{and}\quad x\in\mathbb{R}^N\quad\text{with}\quad |x|\le 1.$$

hold.

Remark 3.4. We can mix the hypothesis of the above theorems. In other words, the same conclusion is true under either

(i) $(F_1), (F_2), (F_4)', (F_5)$ with $m \neq k-1$ and (G_3) or
(ii) $(F_1), (F_2), (F_4)', (F_5)$ with $m \neq k$ and $(G_3)'$.

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