EXISTENCE OF SOLUTIONS OF MINIMIZATION PROBLEMS WITH AN INCREASING COST FUNCTION AND POROSITY

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We consider the minimization problem $f(x) \to \min$, $x \in K$, where K is a closed subset of an ordered Banach space X and f belongs to a space of increasing lower semicontinuous functions on K. In our previous work, we showed that the complement of the set of all functions f, for which the corresponding minimization problem has a solution, is of the first category. In the present paper we show that this complement is also a σ-porous set.

1. Introduction

The study of a generic existence of solutions in optimization has recently been a rapidly growing area of research (see [1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15] and the references mentioned there). Instead of considering the existence of solutions for a single cost function, we study it for a space of all such cost functions equipped with an appropriate complete uniformity and show that a solution exists for most of these functions. Namely, we show that in the space of functions, there exists a subset which is a countable intersection of open everywhere dense sets such that for each cost function in this subset, the corresponding minimization problem has a unique solution. This approach allows us to establish the existence of solutions of minimization problems without restrictive assumptions on the functions and on their domains.

Let K be a nonempty closed subset of a Banach ordered space $(X, \|\cdot\|, \geq)$. A function $f : K \to \mathbb{R}^1 \cup \{+\infty\}$ is called increasing if

$$f(x) \leq f(y) \quad \forall x, y \in K \text{ such that } x \leq y. \tag{1.1}$$

Increasing functions are considered in many models of mathematical economics. As a rule, both utility and production functions are increasing with respect to natural order relations.

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In this paper, we study the existence of a solution of the minimization problem

$$f(x) \longrightarrow \min, \quad x \in K, \tag{1.2}$$

where $f : K \to \mathbb{R}^1 \cup \{+\infty\}$ is an increasing lower semicontinuous function. In [10, 12], it was established the generic existence of solutions of problem (1.2) for certain classes of increasing lower semicontinuous functions f. Note that the perturbations which are usually used to obtain a generic existence result are not suitable for these classes since they break the monotonicity. In [10], we proposed the new kind of perturbations which allowed us to establish the generic existence of solutions for certain classes of increasing lower semicontinuous functions. In the present paper, we show that the complement of the set of all functions f, for which the corresponding minimization problem has a solution, is not only of the first category but also σ-porous.

Before we continue, we briefly recall the concept of porosity [2, 4]. As a matter of fact, several different notions of porosity have been used in the literature. In the present paper, we will use porosity with respect to a pair of metrics, a concept which was introduced in [15].

When (Y, d) is a metric space, we denote by $B_d(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$. Assume that Y is a nonempty set and $d_1, d_2 : Y \times Y \to [0, \infty)$ are two metrics which satisfy $d_1(x, y) \le d_2(x, y)$ for all $x, y \in Y$. A subset $E \subset Y$ is called porous in Y with respect to the pair (d_1, d_2) (or just porous in Y if the pair of metrics is understood) if there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$, there exists $z \in Y$ for which $d_2(z, y) \le r$ and $B_{d_1}(z, \alpha r) \cap E = \emptyset$. A subset of the space Y is called σ-porous in Y with respect to (d_1, d_2) (or just σ-porous in Y if the pair of metrics is understood) if it is a countable union of porous (with respect to (d_1, d_2)) subsets of Y. Note that if $d_1 = d_2$, then by [15, Proposition 1.1] our definitions reduce to those in [2, 4]. We use porosity with respect to a pair of metrics because in applications a space is usually endowed with a pair of metrics and one of them is weaker than the other. Note that porosity of a set with respect to one of these two metrics does not imply its porosity with respect to the other metric. However, it is shown in [15, Proposition 1.2] that if a subset $E \subset Y$ is porous in Y with respect to (d_1, d_2), then E is porous in Y with respect to any metric which is weaker than d_2 and stronger than d_1.

We obtain our main results as a realization of a general variational principle which is established in Section 3.

2. Well-posedness of optimization problems with increasing cost functions

In this paper, we use the following notations and definitions. Let $(X, \|\cdot\|, \ge)$ be a Banach ordered space and $X_+ = \{x \in X : x \ge 0\}$ the cone of its positive elements. Assume that X_+ is a closed convex cone such that $\|x\| \le \|y\|$ for each $x, y \in X_+$ satisfying $x \le y$. We assume that the cone X_+ has the following property:

(A) if $\{x_i\}_{i=1}^{\infty} \subset X$, $x_{i+1} \le x_i$, for all integers $i \ge 1$ and $\sup\{\|x_i\| : i = 1, 2, \dots\} < \infty$, then the sequence $\{x_i\}_{i=1}^{\infty}$ converges.

The property (A) is well known in the theory of ordered Banach spaces (see, e.g., [7, 10, 11]). Recall that the cone X_+ has the property (A) if the space X is reflexive. The property (A) also holds for the cone of nonnegative functions (with respect to usual order relation) in the space L_1 of all integrable on a measure space functions.

Assume that K is a closed subset of X. For each function $f : Y \to [-\infty, +\infty]$, where Y is a nonempty set, we define

$$\operatorname{dom}(f) = \{y \in Y : |f(y)| < \infty\}, \qquad \inf(f) = \inf\{f(y) : y \in Y\}. \tag{2.1}$$

We use the convention that $\infty - \infty = 0$, $\infty/\infty = 1$, and $\ln(\infty) = \infty$.

Assume that $\mathcal{A}$ is a nonempty set and $d_w, d_s : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ are two metrics which satisfy $d_w(a, b) \le d_s(a, b)$ for all $a, b \in \mathcal{A}$. We assume that the metric space $(\mathcal{A}, d_s)$ is complete. The topology induced in $\mathcal{A}$ by the metric d_s is called the strong topology and the topology induced in $\mathcal{A}$ by the metric d_w is called the weak topology.

We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function $f_a : K \to [-\infty, +\infty]$ is associated and f_a is not identically ∞ for all $a \in \mathcal{A}$.

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on K is strongly well posed with respect to $(\mathcal{A}, d_w)$ if the following assertions hold:

(1) the infimum $\inf(f_a)$ is finite and attained at a point $x^{(a)} \in K$ such that for each $x \in K$ satisfying $f_a(x) = \inf(f_a)$, the inequality $x \le x^{(a)}$ holds;
(2) for any $\epsilon > 0$, there exist $\delta > 0$ and a neighborhood U of a in $\mathcal{A}$ with the weak topology such that for each $b \in U$, $\inf(f_b)$ is finite; and if $x \in K$ satisfies $f_b(x) \le \inf(f_b) + \delta$, then $|f_a(x^{(a)}) - f_b(x)| < \epsilon$ and there is $u \in X$ such that $\|u\| < \epsilon$ and $x \le x^{(a)} + u$.

Note that if $X_+ = \{0\}$, then our definition reduces to those in [6, 13].

For each integer $n \ge 1$, denote by $\mathcal{A}_n$ the set of all $a \in \mathcal{A}$ which have the following property:

(P1) there exist $x \in K$ and positive numbers r, η, and c such that

$$-\infty < f_a(x) < \inf(f_a) + \frac{1}{n}, \tag{2.2}$$

and for each $b \in \mathcal{A}$ satisfying $d_w(a, b) < r$, $\inf(f_b)$ is finite; and if $z \in K$ satisfies $f_b(z) \le \inf(f_b) + \eta$, then $\|z\| \le c$, $|f_b(z) - f_a(x)| \le 1/n$, and there is $u \in X$ such that $\|u\| \le 1/n$ and $z \le x + u$.

Proposition 2.1. *Assume that $a \in \cap_{n=1}^{\infty} \mathcal{A}_n$. Then the minimization problem for f_a on K is strongly well posed with respect to $(\mathcal{A}, d_w)$.*

Proof. By (P1) for each integer $n \geq 1$, there exist $x_n \in K$, $r_n > 0$, $\eta_n > 0$, and $c_n > 0$ such that

$$-\infty < f_a(x_n) < \inf(f_a) + 2^{-n} \tag{2.3}$$

and the following property holds:

(P2) for each $b \in \mathcal{A}$ satisfying $d_w(a,b) < r_n$, $\inf(f_b)$ is finite; and if $z \in K$ satisfies $f_b(z) \leq \inf(f_b) + \eta_n$, then there exists $u \in X$ such that

$$\|u\| \leq 2^{-n}, \qquad z \leq x_n + u, \qquad \|z\| \leq c_n, \qquad |f_b(z) - f_a(x_n)| \leq 2^{-n}. \tag{2.4}$$

We may assume without loss of generality that for all integers $n \geq 1$,

$$\eta_n, r_n < 4^{-n-1}, \qquad \eta_n < \eta_1. \tag{2.5}$$

There exists a strictly increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ such that

$$4 \cdot 2^{-k_{n+1}} < \eta(k_n) \quad \text{for all integers } n \geq 1. \tag{2.6}$$

Let $n \geq 1$ be an integer. Inequality (2.3) implies that

$$-\infty < f_a(x_{k_{n+1}}) < \inf(f_a) + 2^{-k_{n+1}} < \inf(f_a) + \eta(k_n). \tag{2.7}$$

By (2.7), (2.5), and the definition of c_1,

$$\|x_{k_{n+1}}\| \leq c_1. \tag{2.8}$$

It follows from (2.7), (P2) (see (2.4)), and the definitions of x_{k_n} and η_{k_n} that there exists $u_n \in X$ such that

$$\|u_n\| \leq 2^{-k_n}, \qquad x_{k_{n+1}} \leq x_{k_n} + u_n, \tag{2.9}$$

$$|f_a(x_{k_{n+1}}) - f_a(x_{k_n})| \leq 2^{-k_n}. \tag{2.10}$$

Set

$$y_n = x_{k_n} + \sum_{i=n}^{\infty} u_i. \tag{2.11}$$

Clearly, the sequence $\{y_n\}_{n=1}^{\infty}$ is well defined. By (2.11) and (2.9), for each integer $n \geq 1$,

$$y_{n+1} - y_n = x_{k_{n+1}} + \sum_{i=n+1}^{\infty} u_i - \left(x_{k_n} + \sum_{i=n}^{\infty} u_i\right) = x_{k_{n+1}} - x_{k_n} - u_n \leq 0. \tag{2.12}$$

Equation (2.11) and inequalities (2.9) and (2.8) imply that

$$\sup\{||y_n|| : n = 1, 2, \dots\} < \infty. \tag{2.13}$$

It follows from (2.13), (2.12), and the property (A) that there is $x^{(a)} = \lim_{n\to\infty} y_n$. Combined with (2.11) and (2.9), this equality implies that

$$x^{(a)} = \lim_{n\to\infty} x_{k_n}. \tag{2.14}$$

By (2.14), (2.3), and the lower semicontinuity of f_a,

$$f_a(x^{(a)}) = \inf(f_a). \tag{2.15}$$

Assume now that $x \in K$ and $f_a(x) = \inf(f_a)$. By the definition of x_{k_n}, $n = 1, 2, \dots$ (see the property (P2)), for each integer $n \geq 1$, there is $v_n \in X$ such that

$$||v_n|| \leq 2^{-k_n}, \qquad x \leq x_{k_n} + v_n. \tag{2.16}$$

These inequalities and (2.14) imply that $x \leq x^{(a)}$.

By (2.15) and the property (P2), for all integers $n \geq 1$,

$$|f_a(x^{(a)}) - f_a(x_n)| \leq 2^{-n}. \tag{2.17}$$

Let $\epsilon > 0$. Choose a natural number m for which

$$||x^{(a)} - x_{k_m}|| < 4^{-1}\epsilon, \qquad 2^{-k_m} < 4^{-1}\epsilon. \tag{2.18}$$

Assume that $b \in B_w(a, r_{k_m}/2)$, $x \in K$, and

$$f_b(x) \leq \inf(f_b) + \eta_{k_m}. \tag{2.19}$$

By (2.19) and the definitions of η_{k_m}, r_{k_m}, and x_{k_m} (see the property (P2)),

$$|f_b(x) - f_a(x_{k_m})| \leq 2^{-k_m} \tag{2.20}$$

and there is $v \in X$ such that

$$||v|| < 2^{-k_m}, \qquad x \leq x_{k_m} + v. \tag{2.21}$$

It follows from (2.21) and (2.18) that

$$\begin{aligned} x &\le x_{k_m} + v = x^{(a)} + \left(x_{k_m} - x^{(a)} + v\right), \\ \left\|x_{k_m} - x^{(a)} + v\right\| &\le \left\|x^{(a)} - x_{k_m}\right\| + \|v\| < \frac{\epsilon}{2}. \end{aligned} \tag{2.22}$$

Inequalities (2.20), (2.17), and (2.18) imply that

$$\begin{aligned} \left| f_a\left(x^{(a)}\right) - f_b(x) \right| &\le \left| f_a\left(x^{(a)}\right) - f_a\left(x_{k_m}\right) \right| + \left| f_a\left(x_{k_m}\right) - f_b(x) \right| \\ &\le 2^{-k_m} + 2^{-k_m} < \frac{\epsilon}{2}. \end{aligned} \tag{2.23}$$

This completes the proof of Proposition 2.1. □

An element $x \in K$ is called minimal if for each $y \in K$ satisfying $y \le x$, the equality $x = y$ is true. Denote by $K_{\min}$ the set of all minimal elements of K.

For each integer $n \ge 1$, denote by $\tilde{\mathcal{A}}_n$ the set of all $a \in \mathcal{A}$ which has the property (P1) with $x \in K_{\min}$.

Analogously to the proof of Proposition 2.1, we can prove the following result.

Proposition 2.2. *Assume that the set $K_{\min}$ is a closed subset of the Banach space X and $a \in \cap_{n=1}^{\infty} \tilde{\mathcal{A}}_n$. Then the minimization problem for f_a on K is strongly well posed with respect to $(\mathcal{A}, d_w)$ and $\inf(f_a)$ is attained at a unique point.*

In the proof of Proposition 2.2, we choose $x_n \in K_{\min}$, $n = 1, 2, \ldots$. This implies that $\inf(f_a)$ is attained at the unique point $x^{(a)} \in K_{\min}$ (see (2.13)).

Remark 2.3. Note that assertion (1) in the definition of a strongly well-posed minimization problem for f_a can be represented in the following way: $\inf(f_a)$ is finite and the set

$$\operatorname*{argmin}_{x \in K} f_a = \{x \in K : f_a(x) = \inf(f_a)\} \tag{2.24}$$

has the largest element.

We construct an example of an increasing function h for which the set $\operatorname{argmin}(h)$ is not a singleton and has the largest element. Define a continuous increasing function $\psi : [0, \infty) \to \mathbb{R}^1$ by

$$\psi(t) = 0, \quad t \in \left[0, \frac{1}{2}\right], \quad \psi(t) = 2t - 1, \quad t \in \left(\frac{1}{2}, \infty\right). \tag{2.25}$$

Let n be a natural number and consider the Euclidean space $\mathbb{R}^n$. Let $K = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \ge 0, i = 1, \ldots, n\}$. Define a function $h : K \to \mathbb{R}^1$ by

$$h(x) = \psi\left(\max\{x_i : i = 1, \ldots, n\}\right), \quad x \in K. \tag{2.26}$$

It is easy to see that h is a continuous increasing function:

$$\inf\{h(x) : x \in K\} = 0 \tag{2.27}$$

and the set

$$\{x \in K : h(x) = 0\} = \left\{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \in \left[0, \frac{1}{2}\right],\ i = 1, \dots, n\right\} \tag{2.28}$$

is not a singleton and has the largest element $(1/2, \dots, 1/2)$.

Remark 2.4. The following example shows that in some cases the sets $\mathcal{A}_n$ can be empty. Let $\mathcal{A} = K = \mathbb{R}^1$. For each $a \in \mathbb{R}^1$, consider the function $f_a : K \to \mathbb{R}^1$, where $f_a = 0$ for any $x \le a$ and $f_a(x) > 0$ for any $x > a$. It is easy to see that the set $\mathcal{A}_n$ is empty for any natural number n.

3. Variational principles

We use the notations and definitions introduced in Section 2. The following are the basic hypotheses about the functions:

(H1) for each $a \in \mathcal{A}$, $\inf(f_a)$ is finite;

(H2) for each $\epsilon > 0$ and each integer $m \ge 1$, there exist numbers $\delta > 0$ and $r_0 > 0$ such that the following property holds:

(P3) for each $a \in \mathcal{A}$ satisfying $\inf(f_a) \le m$ and each $r \in (0, r_0]$, there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K$, and $\bar{d} > 0$ such that

$$d_s(a, \bar{a}) \le r, \qquad \inf(f_{\bar{a}}) \le m + 1, \qquad f_{\bar{a}}(\bar{x}) \le \inf(f_{\bar{a}}) + \epsilon, \tag{3.1}$$

and if $x \in K$ satisfies

$$f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + \delta r, \tag{3.2}$$

then $\|x\| \le \bar{d}$ and there exists $u \in X$ for which $\|u\| \le \epsilon$ and $x \le \bar{x} + u$;

(H3) for each integer $m \ge 1$, there exist $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$, each $a_1, a_2 \in \mathcal{A}$ satisfying $d_w(a_1, a_2) \le \alpha r$, and each $x \in K$ satisfying $\min\{f_{a_1}(x), f_{a_2}(x)\} \le m$, the inequality $|f_{a_1}(x) - f_{a_2}(x)| \le r$ is valid.

Theorem 3.1. *Assume that (H1), (H2), and (H3) hold. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and for each $a \in \mathcal{F}$, the minimization problem for f_a on K is strongly well posed with respect to $(\mathcal{A}, d_w)$.*

Proof. Recall that for each integer $n \ge 1$, $\mathcal{A}_n$ is the set of all $a \in \mathcal{A}$ which has the property (P1). By Proposition 2.1, in order to prove the theorem, it is sufficient to show that the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) for any integer $n \ge 1$. Then the theorem is true with $\mathcal{F} = \cap_{n=1}^{\infty} \mathcal{A}_n$.

Let $n \geq 1$ be an integer. We will show that the set $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s). To meet this goal, it is sufficient to show that for each integer $m \geq 1$, the set

$$\Omega_{nm} := \{a \in \mathcal{A} \setminus \mathcal{A}_n : \inf(f_a) \leq m\} \tag{3.3}$$

is porous in $\mathcal{A}$ with respect to (d_w, d_s).

Let $m \geq 1$ be an integer. By (H3), there exist

$$\alpha_1 \in (0,1), \qquad r_1 \in \left(0, \frac{1}{2}\right) \tag{3.4}$$

such that for each $r \in (0, r_1]$, each $a_1, a_2 \in \mathcal{A}$ satisfying $d_w(a_1, a_2) \leq \alpha_1 r$, and each $x \in K$ satisfying

$$\min\{f_{a_1}(x), f_{a_2}(x)\} \leq m + 4, \tag{3.5}$$

the inequality $|f_{a_1}(x) - f_{a_2}(x)| \leq r$ holds.

By (H2), there exist $\alpha_2, r_2 \in (0,1)$ such that the following property holds:

(P4) for each $a \in \mathcal{A}$ satisfying $\inf(f_a) \leq m+2$ and each $r \in (0, r_2]$, there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K$, and $\bar{d} > 0$ such that

$$d_s(a, \bar{a}) \leq r, \qquad \inf(f_{\bar{a}}) \leq m+3, \qquad f_{\bar{a}}(\bar{x}) \leq \inf(f_{\bar{a}}) + (2n)^{-1}, \tag{3.6}$$

and if $x \in K$ satisfies $f_{\bar{a}}(x) \leq \inf(f_{\bar{a}}) + 4r\alpha_2$, then $\|x\| \leq \bar{d}$ and there exists $u \in X$ for which $\|u\| \leq (2n)^{-1}$ and $x \leq \bar{x} + u$.

Choose

$$\bar{\alpha} \in \left(0, \frac{\alpha_1 \alpha_2}{16}\right), \qquad \bar{r} \in \left(0, \frac{r_1 r_2 \bar{\alpha}}{n}\right). \tag{3.7}$$

Let $a \in \mathcal{A}$ and $r \in (0, \bar{r}]$. There are two cases

$$B_{d_s}\left(a, \frac{r}{4}\right) \cap \{\xi \in \mathcal{A} : \inf(f_\xi) \leq m+2\} = \emptyset, \tag{3.8}$$

$$B_{d_s}\left(a, \frac{r}{4}\right) \cap \{\xi \in \mathcal{A} : \inf(f_\xi) \leq m+2\} \neq \emptyset. \tag{3.9}$$

Assume that (3.8) holds. We will show that for each $\xi \in B_{d_w}(a, \bar{r})$, the inequality $\inf(f_\xi) > m$ is valid. Assume the contrary. Then there exists $\xi \in \mathcal{A}$ such that

$$d_w(\xi, a) \leq \bar{r}, \qquad \inf(f_\xi) \leq m. \tag{3.10}$$

There exists $y \in K$ such that

$$f_\xi(y) \leq m + \frac{1}{2}. \tag{3.11}$$

It follows from the definitions of α_1, r_1 (see (3.4), (3.5)), (3.11), (3.10), and (3.7) that

$$|f_a(y) - f_\xi(y)| \le \alpha_1^{-1}\bar{r} \le \frac{1}{4}. \tag{3.12}$$

This inequality and (3.11) imply that

$$\inf(f_a) \le f_a(y) \le f_\xi(y) + \frac{1}{4} \le m+1, \tag{3.13}$$

a contradiction (see (3.8)). Therefore

$$B_{d_w}(a,\bar{r}) \subset \{\xi \in \mathcal{A} : \inf(f_\xi) > m\} \tag{3.14}$$

and by (3.3),

$$B_{d_w}(a,\bar{r}) \cap \Omega_{nm} = \emptyset. \tag{3.15}$$

Thus, we have shown that (3.8) implies (3.15).

Assume that (3.9) holds. Then there exists $a_1 \in \mathcal{A}$ such that

$$d_s(a,a_1) \le \frac{r}{4}, \qquad \inf(f_{a_1}) \le m+2. \tag{3.16}$$

By the definitions of α_2, r_2, the property (P4), (3.16), and (3.7), there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K$, $\bar{d} > 0$ such that

$$d_s(a_1,\bar{a}) \le \frac{r}{4}, \qquad \inf(f_{\bar{a}}) \le m+3, \qquad f_{\bar{a}}(\bar{x}) \le \inf(f_{\bar{a}}) + (2n)^{-1} \tag{3.17}$$

and that the following property holds:

(P5) if $x \in K$ satisfies $f_{\bar{a}}(x) \le \inf(f_{\bar{a}}) + r\alpha_2$, then $\|x\| \le \bar{d}$ and there exists $u \in X$ for which $\|u\| \le (2n)^{-1}$ and $x \le \bar{x} + u$.

Inequalities (3.17) and (3.16) imply that

$$d_s(a,\bar{a}) \le \frac{r}{2}. \tag{3.18}$$

Assume that

$$\xi \in B_{d_w}(\bar{a},\bar{\alpha}r). \tag{3.19}$$

By (3.17),

$$\inf(f_{\bar{a}}) = \inf\left\{f_{\bar{a}}(z) : z \in K,\ f_{\bar{a}}(z) \le m + \frac{7}{2}\right\}. \tag{3.20}$$

Let $x \in K$ satisfy

$$f_{\bar{a}}(x) \le m + \frac{7}{2}. \tag{3.21}$$

It follows from (3.19), (3.21), (3.7), and the definitions of α_1, r_1 (see (3.4), (3.5)) that

$$|f_{\bar{a}}(x) - f_\xi(x)| \le \bar{\alpha} r \alpha_1^{-1} \le \frac{\alpha_2 r}{16}. \tag{3.22}$$

Since these inequalities hold for any $x \in K$ satisfying (3.21), the relation (3.20) implies that

$$\begin{aligned} \inf(f_\xi) &\le \inf\left\{f_\xi(x) : x \in K,\ f_{\bar{a}}(x) \le m + \frac{7}{2}\right\} \\ &\le \inf\left\{f_{\bar{a}}(x) + \frac{\alpha_2 r}{16} : x \in K,\ f_{\bar{a}}(x) \le m + \frac{7}{2}\right\} \\ &= \frac{\alpha_2 r}{16} + \inf(f_{\bar{a}}). \end{aligned} \tag{3.23}$$

Moreover, since (3.21) holds with $x = \bar{x}$ (see (3.17)), we obtain that $|f_{\bar{a}}(\bar{x}) - f_\xi(\bar{x})| \le \alpha_2 r/16$. Thus

$$\inf(f_\xi) \le \inf(f_{\bar{a}}) + \frac{\alpha_2 r}{16}, \qquad |f_{\bar{a}}(\bar{x}) - f_\xi(\bar{x})| \le \frac{\alpha_2 r}{16}. \tag{3.24}$$

Let $x \in K$ satisfy

$$f_\xi(x) \le \inf(f_\xi) + \frac{1}{4}. \tag{3.25}$$

Inequalities (3.25), (3.24), (3.17) and (3.7) imply that $f_\xi(x) \le m + 7/2$. It follows from this inequality, (3.19), (3.7), and the definitions of α_1, r_1 (see (3.4), (3.5)) that

$$|f_{\bar{a}}(x) - f_\xi(x)| \le \bar{\alpha} r \alpha_1^{-1} \le \frac{\alpha_2 r}{16}. \tag{3.26}$$

Thus, the following property holds:

(P6) if $x \in K$ satisfies (3.25), then $|f_{\bar{a}}(x) - f_\xi(x)| \le \alpha_2 r/16$.

The property (P6) implies that

$$\begin{aligned} \inf(f_{\bar{a}}) &\le \inf\left\{f_{\bar{a}}(x) : x \in K,\ f_\xi(x) \le \inf(f_\xi) + \frac{1}{4}\right\} \\ &\le \inf\left\{f_\xi(x) + \frac{\alpha_2 r}{16} : x \in K,\ f_\xi(x) \le \inf(f_\xi) + \frac{1}{4}\right\} \\ &= \frac{\alpha_2 r}{16} + \inf(f_\xi). \end{aligned} \tag{3.27}$$

Therefore

$$\inf(f_{\bar{a}}) \le \inf(f_\xi) + \frac{\alpha_2 r}{16}. \tag{3.28}$$

Combined with (3.24) and (3.17), this inequality implies that

$$\left|\inf\left(f_{\bar{a}}\right)-\inf\left(f_{\xi}\right)\right| \leq \frac{\alpha_2 r}{16}, \qquad f_{\xi}(\bar{x}) \leq \inf\left(f_{\xi}\right)+\frac{\alpha_2 r}{8}+(2n)^{-1}. \tag{3.29}$$

Assume that $x \in K$ and

$$f_s(x) \leq \inf\left(f_{\xi}\right)+\frac{\alpha_2 r}{16}. \tag{3.30}$$

By (P6),

$$\left|f_{\bar{a}}(x)-f_{\xi}(x)\right| \leq \frac{\alpha_2 r}{16}. \tag{3.31}$$

Inequalities (3.30), (3.29), (3.17) and (3.7) imply that

$$\begin{aligned}\left|f_{\xi}(x)-f_{\bar{a}}(\bar{x})\right| &\leq \left|f_{\xi}(x)-\inf\left(f_{\xi}\right)\right|+\left|\inf\left(f_{\xi}\right)-\inf\left(f_{\bar{a}}\right)\right| \\ &\quad+\left|\inf\left(f_{\bar{a}}\right)-f_{\bar{a}}(\bar{x})\right| \\ &\leq \frac{\alpha_2 r}{16}+\frac{\alpha_2 r}{16}+(2n)^{-1}<n^{-1},\end{aligned} \tag{3.32}$$

$$\left|f_{\xi}(x)-f_{\bar{a}}(\bar{x})\right|<n^{-1}. \tag{3.33}$$

It follows from (3.31), (3.30), and (3.29) that

$$f_{\bar{a}}(x) \leq f_{\xi}(x)+\frac{r\alpha_2}{16} \leq \inf\left(f_{\xi}\right)+\frac{\alpha_2 r}{8} \leq \inf\left(f_{\bar{a}}\right)+\frac{3\alpha_2 r}{16}, \tag{3.34}$$

$$f_{\bar{a}}(x) \leq \inf\left(f_{\bar{a}}\right)+\frac{3\alpha_2 r}{16}. \tag{3.35}$$

It follows from (3.35) and the property (P5) that $\|x\| \leq \bar{d}$ and there is $u \in X$ such that $\|u\| \leq (2n)^{-1}$ and $x \leq u+\bar{x}$. Therefore, if $x \in K$ satisfies (3.30), then (3.33) is valid, $\|x\| \leq \bar{d}$, and there is $u \in X$ for which $\|u\| \leq (2n)^{-1}$ and $x \leq u+\bar{x}$.

Thus, we have shown that for each $\xi \in B_{d_w}(\bar{a}, \bar{\alpha} r)$, the inequalities (3.29) are true and if $x \in K$ satisfies (3.30), then (3.33) is valid, $\|x\| \leq \bar{d}$, and there is $u \in X$ for which $\|u\| \leq (2n)^{-1}$ and $x \leq u+\bar{x}$.

By the definition of $\mathcal{A}_n$ and (3.3),

$$B_{d_w}\left(\bar{a}, \frac{\bar{\alpha} r}{2}\right) \subset \mathcal{A}_n \subset \mathcal{A} \setminus \Omega_{nm}. \tag{3.36}$$

Since (3.8) implies (3.15), we obtain that, in both cases, $B_{d_w}(\bar{a}, \bar{\alpha} r/2) \cap \Omega_{nm}=\varnothing$ with $\bar{a} \in \mathcal{A}$ satisfying (3.18). (Note that if (3.8) is valid, then $\bar{a}=a$.) Hence, the set Ω_{nm} is porous in $\mathcal{A}$ with respect to (d_w, d_s). This implies that $\mathcal{A} \setminus \mathcal{A}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) for all integers $n \geq 1$. Therefore, $\mathcal{A} \setminus (\cap_{n=1}^{\infty} \mathcal{A}_n)$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s). This completes the proof of Theorem 3.1. □

We also use the following hypotheses about the functions:

(H4) for each $\epsilon > 0$ and each integer $m \geq 1$, there exist numbers $\delta > 0$ and $r_0 > 0$ such that the following property holds:

(P7) for each $a \in \mathcal{A}$ satisfying $\inf(f_a) \leq m$ and each $r \in (0, r_0]$, there exist $\bar{a} \in \mathcal{A}$, $\bar{x} \in K_{\min}$, and $\bar{d} > 0$ such that (3.1) is true; and if $x \in K$ satisfies (3.2), then $\|x\| \leq \bar{d}$ and there exists $u \in X$ for which $\|u\| \leq \epsilon$, $x \leq \bar{x} + u$.

Theorem 3.2. *Assume that (H1), (H3), and (H4) hold and $K_{\min}$ is a closed subset of the Banach space X. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and that for each $a \in \mathcal{F}$ the following assertions hold:*

(1) *the minimization problem for f_a on K is strongly well posed with respect to $(\mathcal{A}, d_w)$,*

(2) *the infimum $\inf(f_a)$ is attained at a unique point.*

We can prove Theorem 3.2 analogously to the proof of Theorem 3.1. Recall that for each integer $n \geq 1$, $\tilde{A}_n$ is the set of all $a \in \mathcal{A}$ which have the property (P1) with $x \in K_{\min}$. Set $\mathcal{F} = \cap_{n=1}^{\infty} \tilde{A}_n$. By Proposition 2.2 for each $a \in \mathcal{F}$, assertions (1) and (2) hold. Therefore, in order to prove Theorem 3.2, it is sufficient to show that for each integer $n \geq 1$, the set $\mathcal{A} \setminus \tilde{\mathcal{A}}_n$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s). We can show this fact analogously to the proof of Theorem 3.1.

4. Spaces of increasing functions

In the sequel, we use the functional $\lambda : X \to \mathbb{R}^1$ defined by

$$\lambda(x) = \inf\{\|y\| : y \geq x\}, \quad x \in X. \tag{4.1}$$

The function λ has the following properties (see [10, Proposition 6.1]):

(i) the function λ is sublinear. Namely,

$$\begin{aligned} \lambda(\alpha x) &= \alpha\lambda(x) \quad \forall \alpha \geq 0 \text{ and all } x \in X, \\ \lambda(x_1 + x_2) &\leq \lambda(x_1) + \lambda(x_2) \quad \forall x_1, x_2 \in X, \end{aligned} \tag{4.2}$$

(ii) $\lambda(x) = 0$ if $x \leq 0$,

(iii) if $x_1, x_2 \in X$ and $x_1 \leq x_2$, then $\lambda(x_1) \leq \lambda(x_2)$,

(iv) $0 \leq \lambda(x) \leq \|x\|$ for all $x \in X$.

Clearly, $|\lambda(x) - \lambda(y)| \leq \|x - y\|$ for each $x, y \in X$.

Denote by $\mathcal{M}$ the set of all increasing lower semicontinuos bounded-from-below functions $f : K \to \mathbb{R}^1 \cup \{+\infty\}$ which are not identically $+\infty$. For each $f, g \in \mathcal{M}$, set

$$\begin{aligned} \tilde{d}_s(f,g) &= \sup\{|f(x) - g(x)| : x \in X\}, \\ d_s(f,g) &= \tilde{d}_s(f,g)(1 + \tilde{d}_s(f,g))^{-1}. \end{aligned} \tag{4.3}$$

It is not difficult to see that the metric space $(\mathcal{M}, d_s)$ is complete. Denote by $\mathcal{M}_v$ the set of all finite-valued functions $f \in \mathcal{M}$ and by $\mathcal{M}_c$ the set of all finite-valued continuous functions $f \in \mathcal{M}$. Clearly, $\mathcal{M}_v$ and $\mathcal{M}_c$ are closed subsets of the metric space $(\mathcal{M}, d_s)$.

We say that the set K has property (C) if $K_{\min}$ is a closed subset of K and for each $x \in K$, there is $y \in K_{\min}$ such that $y \le x$.

Denote by $\mathcal{M}_g$ the set of all $f \in \mathcal{M}$ such that $f(x) \to \infty$ as $\|x\| \to \infty$. Clearly, $\mathcal{M}_g$ is a closed subset of the metric space $(\mathcal{M}, d_s)$. Set $\mathcal{M}_{gc} = \mathcal{M}_g \cap \mathcal{M}_c$ and $\mathcal{M}_{gv} = \mathcal{M}_g \cap \mathcal{M}_v$.

It is easy to see that

$$\mathcal{M}_c \subset \mathcal{M}_v \subset \mathcal{M}, \qquad \mathcal{M}_{gc} \subset \mathcal{M}_{gv} \subset \mathcal{M}_g \subset \mathcal{M}. \tag{4.4}$$

Remark 4.1. Let $K = X_+$ and define

$$\begin{gathered} f_1(x) = \|x\|, \quad x \in K, \qquad f_2(x) = \|x\|, \quad x \in K \setminus \{0\}, \qquad f_2(0) = -1, \\ f_3(x) = \|x\| \quad \text{if } x \in K,\ \|x\| \le 1, \qquad f_3(x) = +\infty \quad \text{if } x \in K,\ \|x\| > 1. \end{gathered} \tag{4.5}$$

Clearly,

$$f_1 \in \mathcal{M}_{gc}, \quad f_2 \in \mathcal{M}_{gv} \setminus \mathcal{M}_{gc}, \quad f_3 \in \mathcal{M}_g \setminus \mathcal{M}_{gv}. \tag{4.6}$$

Theorem 4.2. *Assume that $\mathcal{A}$ is either $\mathcal{M}_g$, $\mathcal{M}_{gv}$, or $\mathcal{M}_{gc}$ and that $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_s, d_s) and that for each $f \in \mathcal{F}$ the minimization problem for f on K is strongly well posed with respect to $(\mathcal{A}, d_s)$. If K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is attained at a unique point.*

Proof. By Theorems 3.1 and 3.2, we need to show that (H1), (H2), and (H3) hold and that the property (C) implies (H4). Clearly, (H1) holds. For each $f, g \in \mathcal{M}$, we have that

$$\tilde{d}_s(f,g) = d_s(f,g)\left(1 - d_s(f,g)\right)^{-1} \tag{4.7}$$

and that if $d_s(f,g) \le 1/2$, then $\tilde{d}_s(f,g) \le 2d_s(f,g)$. Combined with (4.7), this property implies (H3).

We will show that (H2) holds and that the property (C) implies (H4).

Let $f \in \mathcal{A}$, $\epsilon \in (0,1)$, and $r \in (0,1]$. Choose $\bar{x} \in K$ such that

$$f(\bar{x}) \le \inf(f) + \frac{\epsilon r}{8}. \tag{4.8}$$

If K has the property (C), then we assume that $\bar{x}$ is a minimal element of K. Define

$$\bar{f}(x) = f(x) + 2^{-1} r \min\{1, \lambda(x - \bar{x})\} \quad \forall x \in K. \tag{4.9}$$

Evidently, $\bar{f} \in \mathcal{A}$, $d_s(f,\bar{f}) \le \tilde{d}_s(f,\bar{f}) \le r/2$, and

$$\inf(\bar{f}) \le \bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + \frac{\epsilon r}{8}. \tag{4.10}$$

Let $x \in K$ and $\bar{f}(x) \le \inf(\bar{f}) + \epsilon r/8$. Then by (4.9) and (4.8),

$$\begin{aligned} f(x) + 2^{-1} r \min\{1, \lambda(x-\bar{x})\} &= \bar{f}(x) \le \inf(\bar{f}) + \frac{\epsilon r}{8} \le \bar{f}(\bar{x}) + \frac{\epsilon r}{8} \\ &= f(\bar{x}) + \frac{\epsilon r}{8} \le f(x) + \frac{\epsilon r}{4}, \\ \min\{1, \lambda(x-\bar{x})\} \le \frac{\epsilon}{2}, &\qquad \lambda(x-\bar{x}) \le \frac{\epsilon}{2}. \end{aligned} \tag{4.11}$$

By (4.1), there exists $u \in X$ such that $x \le \bar{x} + u$ and $\|u\| < \epsilon$. Since $\bar{f}(y) \to \infty$ as $\|y\| \to \infty$, we obtain that $\|x\| \le \bar{d}$, where $\bar{d} > 0$ is a constant which depends only on $\bar{f}$. Thus, (H2) is true and if K has the property (C), then (H4) holds. Theorem 4.2 is proved. □

Theorem 4.3. *Assume that there exists $\bar{z} \in X$ such that $\bar{z} \le x$ for all $x \in K$, that a space $\mathcal{A}$ is either $\mathcal{M}$, $\mathcal{M}_v$, or $\mathcal{M}_c$, and that $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_s, d_s) and that for each $f \in \mathcal{F}$, the minimization problem for f on K is strongly well posed with respect to $(\mathcal{A}, d_s)$. If K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is attained at a unique point.*

Proof. We can prove Theorem 4.3 analogously to the proof of Theorem 4.2. The existence of a constant $\bar{d}$ is obtained in the following manner. Let $x \in K$, $u \in X$, $x \le \bar{x} + u$, and $\|u\| < \epsilon$. Then

$$\begin{gathered} \|x\| \le \|x - \bar{z}\| + \|\bar{z}\| \le \|\bar{z}\| + \|\bar{x} + u - \bar{z}\| \le 2|\bar{z}\| + \|\bar{x}\| + \epsilon, \\ \|x\| \le \bar{d}, \end{gathered} \tag{4.12}$$

where $\bar{d} = 2\|\bar{z}\| + \|\bar{x}\| + \epsilon$. □

Denote by $\mathcal{M}^+$ the set of all $f \in \mathcal{M}$ such that $f(x) \ge 0$ for all $x \in K$. Clearly, $\mathcal{M}^+$ is a closed subset of the metric space $(\mathcal{M}, d_s)$. Define

$$\begin{gathered} \mathcal{M}_v^+ = \mathcal{M}^+ \cap \mathcal{M}_v, \qquad \mathcal{M}_c^+ = \mathcal{M}^+ \cap \mathcal{M}_c, \qquad \mathcal{M}_g^+ = \mathcal{M}^+ \cap \mathcal{M}_g, \\ \mathcal{M}_{gv}^+ = \mathcal{M}^+ \cap \mathcal{M}_{gv}, \qquad \mathcal{M}_{gc}^+ = \mathcal{M}^+ \cap \mathcal{M}_{gc}. \end{gathered} \tag{4.13}$$

For each $f, g \in \mathcal{M}^+$, set

$$\tilde{d}_w(f,g) = \sup\{|\ln(f(z)+1) - \ln(g(z)+1)| : z \in K\}, \tag{4.14}$$

$$d_w(f,g) = \tilde{d}_w(f,g)\left(1 + \tilde{d}_w(f,g)\right)^{-1}. \tag{4.15}$$

It is not difficult to see that the metrtic space $(\mathcal{M}^+, d_w)$ is complete and that $\mathcal{M}_v^+$, $\mathcal{M}_c^+$, $\mathcal{M}_g^+$, $\mathcal{M}_{gv}^+$, and $\mathcal{M}_{gc}^+$ are closed subsets of $(\mathcal{M}^+, d_w)$. Clearly, $d_w(f,g) \le d_s(f,g)$ for all $f, g \in \mathcal{M}^+$.

Theorem 4.4. *Assume that one of the following cases holds:*

(1) *the space $\mathcal{A}$ is either $\mathcal{M}_g^+$, $\mathcal{M}_{gv}^+$, or $\mathcal{M}_{gc}^+$;*

(2) *there is $\bar{z} \in X$ such that $\bar{z} \le x$ for all $x \in K$ and that $\mathcal{A}$ is either $\mathcal{M}^+$, $\mathcal{M}_v^+$, or $\mathcal{M}_c^+$.*

Let $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_s) and that for each $f \in \mathcal{F}$, the minimization problem for f on K is strongly well posed with respect to $(\mathcal{A}, d_w)$. If K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is attained at a unique point.

Proof. By Theorems 3.1 and 3.2, we need to show that (H1), (H2), and (H3) hold and that the property (C) implies (H4). Clearly, (H1) holds. Analogously to the proofs of Theorems 4.2 and 4.3, we can show that (H2) is true. Therefore, in order to prove Theorem 4.4, it is sufficient to show that (H3) holds.

Let $m \ge 1$ be an integer. Choose $\alpha \in (0,1)$ such that

$$\alpha < 4^{-1}\left(2e(m+1)\right)^{-1}. \tag{4.16}$$

Let $r \in (0,1]$, $x \in K$, $f_1, f_2 \in \mathcal{A}$,

$$d_w(f_1, f_2) \le \alpha r, \qquad \min\{f_1(x), f_2(x)\} \le m. \tag{4.17}$$

We may assume without loss of generality that $f_1(x) \le f_2(x)$. Then $\tilde{d}_w(f_1, f_2) \le 2d_w(f_1, f_2) \le 2\alpha r$,

$$\begin{aligned} \ln\left(f_2(x)+1\right) - \ln\left(f_1(x)+1\right) &\le 2\alpha r, \\ f_2(x)+1 &\le \left(f_1(x)+1\right)e^{2\alpha r}, \\ \left|f_2(x) - f_1(x)\right| &\le \left(f_1(x)+1\right)\left(e^{2\alpha r}-1\right) \\ &\le (m+1)\left(e^{2\alpha r}-1\right) = 2\alpha r(m+1)e^{r_1}, \end{aligned} \tag{4.18}$$

with $r_1 \in [0, 2\alpha r]$. Then by (4.16),

$$\left|f_2(x) - f_1(x)\right| \le 2(m+1)\alpha r e^{2\alpha r} \le 2\alpha e(m+1)r < r. \tag{4.19}$$

Thus, (H3) holds and Theorem 4.4 is proved. □

Fix a number $c_0 > 0$ and denote by $\mathcal{M}^{(\mathrm{co})}$ the set of all convex functions $f \in \mathcal{M}$ such that $f(x) \ge c_0\|x\|$ for all $x \in K$. Clearly, $\mathcal{M}^{(\mathrm{co})}$ is a closed subset of the metric space $(\mathcal{M}^+, d_w)$. Set $\mathcal{M}_v^{(\mathrm{co})} = \mathcal{M}^{(\mathrm{co})} \cap \mathcal{M}_v^+$ and $\mathcal{M}_c^{(\mathrm{co})} = \mathcal{M}^{(\mathrm{co})} \cap \mathcal{M}_c^+$. Clearly, $\mathcal{M}_v^{(\mathrm{co})}$ and $\mathcal{M}_c^{(\mathrm{co})}$ are closed subsets of the metric space $(\mathcal{M}^+, d_w)$.

Theorem 4.5. *Assume that a space $\mathcal{A}$ is either $\mathcal{M}^{\mathrm{co}}$, $\mathcal{M}_v^{(\mathrm{co})}$, or $\mathcal{M}_c^{(\mathrm{co})}$ and that $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_w, d_w) and that for each $f \in \mathcal{F}$, the minimization problem for f on K is strongly well posed with respect to $(\mathcal{A}, d_w)$. If the set K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is obtained at a unique point.*

Proof. By Theorems 3.1 and 3.2, we need to show that (H1), (H2), and (H3) hold and that the property (C) implies (H4). Clearly, (H1) holds. Analogously to the proof of Theorem 4.4, we can show that (H3) holds.

Now, we show that (H2) holds and that the property (C) implies (H4).

Let $f \in \mathcal{A}$, $\epsilon \in (0,1)$, and $m \geq 1$ an integer. Choose a natural number m_0 and a number $r_0 > 0$ such that

$$m_0 > (m + c_0 + 2)(\min\{c_0, 1\})^{-1}, \qquad r_0 < (m_0 + 2)^{-1}. \tag{4.20}$$

Choose a positive number

$$\delta < 8^{-1}\epsilon \min\{1, c_0\}. \tag{4.21}$$

Assume that $f \in \mathcal{A}$,

$$r \in (0, r_0], \qquad \inf(f) \leq m. \tag{4.22}$$

There exists $\bar{x} \in K$ such that

$$f(\bar{x}) \leq \inf(f) + 4^{-1}\delta r(m+1)^{-1}. \tag{4.23}$$

If K has the property (C), we assume that $\bar{x}$ is a minimal element of K. Define

$$\bar{f}(x) = f(x) + 4^{-1} r \min\{1, c_0\}(m+1)^{-1}\lambda(x - \bar{x}), \quad x \in K. \tag{4.24}$$

It is easy to see that $\bar{f} \in \mathcal{A}$. It follows from (4.23) and (4.22) that

$$c_0\|\bar{x}\| \leq f(\bar{x}) \leq \inf(f) + 1 \leq m + 1, \qquad \|\bar{x}\| \leq (m+1)c_0^{-1}. \tag{4.25}$$

By (4.14) and (4.24),

$$\begin{aligned}
d_w(f, \bar{f}) \leq \tilde{d}_w(f, \bar{f}) &= \sup\{\,|\ln(\bar{f}(x) + 1) - \ln(f(x) + 1)\,| : x \in K\} \\
&= \sup_{x \in K}\{\ln(1 + (\bar{f}(x) - f(x))(f(x) + 1)^{-1})\} \\
&\leq \sup_{x \in K}\{\ln(1 + (4^{-1} r \min\{1, c_0\}\lambda(x - \bar{x}))((1 + c_0\|x\|)(m+1))^{-1})\} \\
&\leq \sup_{x \in K}\{4^{-1} r \min\{1, c_0\}\lambda(x - \bar{x})((1 + c_0\|x\|)(m+1))^{-1}\} \\
&\leq 4^{-1} r \sup_{x \in K}\{\min\{1, c_0\}\|x - \bar{x}\|((1 + c_0\|x\|)(m+1))^{-1}\} \\
&\leq 4^{-1} r \sup_{x \in K}\{c_0\|x\|(1 + c_0\|x\|)^{-1} \\
&\qquad + \|\bar{x}\| \min\{1, c_0\}(m+1)^{-1}\} \leq 2^{-1} r.
\end{aligned} \tag{4.26}$$

Thus

$$d_w(f, \bar{f}) \leq 2^{-1} r. \tag{4.27}$$

Equation (4.24) and inequality (4.23) imply that

$$\bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + 4^{-1}\delta r(m+1)^{-1} \le \inf(\bar{f}) + 4^{-1}\delta r(m+1)^{-1}. \tag{4.28}$$

Then by (4.22),

$$\inf(\bar{f}) \le \bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + 4^{-1}\delta r(m+1)^{-1} \le m+1. \tag{4.29}$$

Let $x \in K$ and

$$\bar{f}(x) \le \inf(\bar{f}) + 4^{-1}\delta r(m+1)^{-1}. \tag{4.30}$$

Inequality (4.30) and equation (4.29) imply that

$$c_0\|x\| \le m+2, \qquad \|x\| \le c_0^{-1}(m+2). \tag{4.31}$$

By (4.24), (4.30), (4.23), and (4.21),

$$\begin{aligned} &f(x) + 4^{-1}r(m+1)^{-1}\min\{1, c_0\}\lambda(x-\bar{x}) \\ &\quad = \bar{f}(x) \le \bar{f}(\bar{x}) + 4^{-1}\delta r(m+1)^{-1} \\ &\quad = f(\bar{x}) + 4^{-1}\delta r(m+1)^{-1} \le f(x) + 2^{-1}\delta r(m+1)^{-1}, \\ &\min\{1, c_0\}\lambda(x-\bar{x}) \le 2\delta, \qquad \lambda(x-\bar{x}) \le 2\delta(\min\{1, c_0\})^{-1} < 4^{-1}\epsilon. \end{aligned} \tag{4.32}$$

Therefore, there exists $u \in X$ such that $\|u\| < 4^{-1}\epsilon$ and $x \le \bar{x} + u$. Hence, (H2) is true and if K has the property (C), then (H4) holds. This completes the proof of Theorem 4.5. □

A function $f : K \to \mathbb{R}^1 \cup \{\infty\}$ is called quasiconvex if the set $\{x \in K : f(x) \le \alpha\}$ is convex for any $\alpha \in \mathbb{R}^1$. Denote by $\mathcal{M}^{(\mathrm{qu})}$ the set of all quasiconvex functions $f \in \mathcal{M}$. Clearly, $\mathcal{M}^{(\mathrm{qu})}$ is a closed subset of the metric space $(\mathcal{M}, d_s)$. Set

$$\begin{gathered} \mathcal{M}_v^{(\mathrm{qu})} = \mathcal{M}^{(\mathrm{qu})} \cap \mathcal{M}_v, \qquad \mathcal{M}_c^{(\mathrm{qu})} = \mathcal{M}^{(\mathrm{qu})} \cap \mathcal{M}_c, \qquad \mathcal{M}_g^{(\mathrm{qu})} = \mathcal{M}^{(\mathrm{qu})} \cap \mathcal{M}_g, \\ \mathcal{M}_{gv}^{(\mathrm{qu})} = \mathcal{M}^{(\mathrm{qu})} \cap \mathcal{M}_{gv}, \qquad \mathcal{M}_{gc}^{(\mathrm{qu})} = \mathcal{M}^{(\mathrm{qu})} \cap \mathcal{M}_{gc}. \end{gathered} \tag{4.33}$$

Theorem 4.6. *Assume that one of the following cases holds:*

(1) *a space $\mathcal{A}$ is either $\mathcal{M}_g^{(\mathrm{qu})}$, $\mathcal{M}_{gv}^{(\mathrm{qu})}$, or $\mathcal{M}_{gc}^{(\mathrm{qu})}$;*
(2) *there is $\bar{z} \in X$ such that $\bar{z} \le x$ for all $x \in K$ and a space $\mathcal{A}$ is either $\mathcal{M}^{(\mathrm{qu})}$, $\mathcal{M}_v^{(\mathrm{qu})}$, or $\mathcal{M}_c^{(\mathrm{qu})}$.*

Let $f_a = a$ for all $a \in \mathcal{A}$. Then there exists a set $\mathcal{F} \subset \mathcal{A}$ such that the complement $\mathcal{A} \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}$ with respect to (d_s, d_s) and that for each $f \in \mathcal{F}$, the minimization problem for f on K is strongly well posed with respect to $(\mathcal{A}, d_s)$. If K has the property (C), then for each $f \in \mathcal{F}$, $\inf(f)$ is attained at a unique point.

Proof. By Theorems 3.1 and 3.2, we need to show that (H1), (H2), and (H3) hold and that the property (C) implies (H4). Clearly, (H1) holds. Analogously to the proof of Theorem 4.2, we can show that (H3) is true (see (4.7)). Now, we show that (H2) holds and that the property (C) implies (H4).

Let $f \in \mathcal{A}$, $\epsilon \in (0,1]$, and $r \in (0,1]$. Choose $\bar{x} \in K$ such that

$$f(\bar{x}) \le \inf(f) + \frac{\epsilon r}{8}. \tag{4.34}$$

If K has the property (C), then we assume that $\bar{x}$ is a minimial element of K. Define

$$\bar{f}(x) = \max\{f(x), f(\bar{x}) + 2^{-1} r \min\{\lambda(x-\bar{x}), 1\}\}, \quad x \in K. \tag{4.35}$$

Clearly, $\bar{f} \in \mathcal{A}$,

$$\begin{gathered} d_s(f,\bar{f}) \le \tilde{d}_s(f,\bar{f}) \le r, \\ \bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + \frac{\epsilon r}{8} \le \inf(\bar{f}) + \frac{\epsilon r}{8}, \\ \inf(\bar{f}) \le \bar{f}(\bar{x}) = f(\bar{x}) \le \inf(f) + 1. \end{gathered} \tag{4.36}$$

Let $x \in K$ and

$$\bar{f}(x) \le \inf(\bar{f}) + \frac{\epsilon r}{8}. \tag{4.37}$$

Then by (4.35) and (4.37),

$$\begin{gathered} 2^{-1} r \min\{\lambda(x-\bar{x}), 1\} + f(\bar{x}) \le \bar{f}(x) \le \bar{f}(\bar{x}) + \frac{\epsilon r}{8} = f(\bar{x}) + \frac{\epsilon r}{8}, \\ \min\{\lambda(x-\bar{x}), 1\} \le \frac{\epsilon}{4}, \qquad \lambda(x-\bar{x}) \le \frac{\epsilon}{4}, \end{gathered} \tag{4.38}$$

and there exists $u \in X$ such that

$$\|u\| < \frac{\epsilon}{2}, \qquad x \le \bar{x} + u. \tag{4.39}$$

In the first case, we choose $\bar{d} > 0$ such that $\bar{f}(y) > \inf(\bar{f}) + 1$ for all $y \in K$ satisfying $\|y\| > \bar{d}$ and obtain that $\|x\| \le \bar{d}$. In the second case,

$$\begin{gathered} \bar{z} \le x \le \bar{x} + u, \\ \|x\| \le \|\bar{z}\| + \|x - \bar{z}\| \le \|\bar{z}\| + \|\bar{x} + u - \bar{z}\| \\ \le 2\|\bar{z}\| + \|\bar{x}\| + \|u\| \le 2\|\bar{z}| + \|\bar{x}\| + 1, \end{gathered} \tag{4.40}$$

and $\|x\| \le \bar{d} := 2\|\bar{z}\| + \|\bar{x}\| + 1$. Therefore, in both cases, (H2) is true and if the set K has the property (C), then (H4) holds. Theorem 4.6 is proved. □

Theorem 4.7. *Let $\mathcal{A}$ be defined as in Theorem 4.6 and let $\mathcal{A}^+$ be the set of all $f \in \mathcal{A}$ such that $f(x) \geq 0$ for all $x \in K$. Assume that $f_a = a$ for all $a \in \mathcal{A}^+$. Then the metric spaces $(\mathcal{A}^+, d_s)$ and $(\mathcal{A}^+, d_w)$ are complete and there exists a set $\mathcal{F} \subset \mathcal{A}^+$ such that the complement $\mathcal{A}^+ \setminus \mathcal{F}$ is σ-porous in $\mathcal{A}^+$ with respect to (d_w, d_s) and that for each $f \in \mathcal{F}$, the minimization problem for f on K is strongly well posed with respect to $(\mathcal{A}^+, d_w)$. If the set K has the property (C), then for each $f \in \mathcal{F}$,* $\inf(f)$ *is attained at a unique point.*

Proof. By Theorems 3.1 and 3.2, we need to show that (H1), (H2), and (H3) hold and that the property (C) implies (H4). Clearly, (H1) holds. Analogously to the proof of Theorem 4.6, we can show that (H2) is true and that the property (C) implies (H4). We can prove (H3) as in the proof of Theorem 4.4. □

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