A WEAK ERGODIC THEOREM FOR INFINITE PRODUCTS OF LIPSCHITZIAN MAPPINGS

SIMEON REICH AND ALEXANDER J. ZASLAVSKI

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Let K be a bounded, closed, and convex subset of a Banach space. For a Lipschitzian self-mapping A of K, we denote by $\mathrm{Lip}(A)$ its Lipschitz constant. In this paper, we establish a convergence property of infinite products of Lipschitzian self-mappings of K. We consider the set of all sequences $\{A_t\}_{t=1}^{\infty}$ of such self-mappings with the property $\limsup_{t\to\infty} \mathrm{Lip}(A_t) \le 1$. Endowing it with an appropriate topology, we establish a weak ergodic theorem for the infinite products corresponding to generic sequences in this space.

1. Introduction

The asymptotic behavior of infinite products of operators finds applications in many areas of mathematics (see, e.g., [1, 2, 3, 4, 5, 8, 9, 10, 12, 14, 15, 16, 17, 18] and the references therein). Given a bounded, closed, and convex subset K of a Banach space and a sequence $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$ of self-mappings of K, we are interested in the convergence properties of the sequence of products $\{A_n \cdots A_1 x\}_{n=1}^{\infty}$, where $x \in K$. In the special case of a constant sequence $\mathbf{A}$, we are led to study the asymptotic behavior of a single operator. In their seminal paper [7], De Blasi and Myjak show that the powers of a generic nonexpansive self-mapping of K do converge. Such an approach, when a certain property is investigated for a whole space of operators and not just for a single operator, has already been successfully applied in many areas of analysis. For instance, in a recent paper [15], we have extended the De Blasi-Myjak result in several directions to certain sequence spaces of nonexpansive operators. One of these directions has involved weak ergodicity in the sense of population biology (see [6, 11, 13, 15]). More precisely, we have shown that for most (in the sense of Baire's categories) sequences, the distances between the corresponding (random) infinite products with different initial points tend to zero, uniformly on K. The main result of

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the present paper (Theorem 1.1 below) is an extension of [15, Theorem 2.2] to Lipschitzian mappings which are not necessarily nonexpansive.

Assume that $(X, \|\cdot\|)$ is a Banach space and that $K \subset X$ is a bounded, closed, and convex subset of X.

For any $A : K \to X$, define

$$\operatorname{Lip}(A) = \sup\{\|Ax - Ay\|/\|x - y\| : x, y \in K \text{ and } x \neq y\}. \tag{1.1}$$

Denote by $\mathcal{A}$ the set of all sequences $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$, where each $A_t : K \to K$ satisfies $\operatorname{Lip}(A_t) < \infty$, $t = 1, 2, \ldots$, and

$$\limsup_{t\to\infty} \operatorname{Lip}(A_t) \leq 1. \tag{1.2}$$

Set

$$d(K) = \sup\{\|x - y\| : x, y \in K\}. \tag{1.3}$$

For $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$ and $\mathbf{B} = \{B_t\}_{t=1}^{\infty}$ in $\mathcal{A}$, define

$$\begin{aligned} d_s(\mathbf{A}, \mathbf{B}) = \sup\{||A_t x - B_t x|| : t = 1, 2, \ldots \text{ and } x \in K\} \\ + \sup\{\operatorname{Lip}(A_t - B_t) : t = 1, 2, \ldots\}. \end{aligned} \tag{1.4}$$

Clearly, $(\mathcal{A}, d_s)$ is a complete metric space. The metric d_s induces in $\mathcal{A}$ a topology which we call the strong topology. For each $\mathbf{A} = \{A_t\}_{t=1}^{\infty}$ and $\mathbf{B} = \{B_t\}_{t=1}^{\infty}$ in $\mathcal{A}$, we set

$$d_w(\mathbf{A}, \mathbf{B}) = \sup\{||A_t x - B_t x|| : t = 1, 2, \ldots \text{ and } x \in K\}. \tag{1.5}$$

Clearly, $(\mathcal{A}, d_w)$ is also a metric space. The metric d_w induces in $\mathcal{A}$ a topology which we call the weak topology. In the sequel, for each $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{A}$, we denote

$$\operatorname{Lip}(\mathbf{A}) = \sup\{\operatorname{Lip}(A_t) : t = 1, 2, \ldots\}. \tag{1.6}$$

Now we are ready to state our main result. Its proof will be given in Section 3. Section 2 is devoted to two auxiliary assertions.

Theorem 1.1. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$ such that for each $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in \mathcal{F}$ and each $\epsilon > 0$, the following property holds: there exist an open neighborhood $\mathcal{U}$ of $\mathbf{C}$ in $\mathcal{A}$ with the weak topology and a natural number N, such that for each $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{U}$, each $x, y \in K$, each integer $n \geq N$,* *and each injective mapping* $r : \{1,\dots,n\} \to \{1,2,\dots\}$,

$$||B_{r(n)} \cdots B_{r(1)}x - B_{r(n)} \cdots B_{r(1)}y|| < \epsilon. \tag{1.7}$$

A theorem of this type is called a weak ergodic theorem in the population biology literature [6, 11, 13, 15].

2. Two auxiliary assertions

Fix $\theta \in K$. For each $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{A}$ and each $\gamma \in (0,1)$, define $\mathbf{A}_\gamma = \{A_{t\gamma}\}_{t=1}^{\infty} \in \mathcal{A}$ by

$$A_{t\gamma}x = (1-\gamma)A_t x + \gamma\theta, \quad x \in K, \ t = 1,2,\dots. \tag{2.1}$$

It is easy to see that for each $\gamma \in (0,1)$ and each $\mathbf{A} \in \mathcal{A}$,

$$\begin{gathered} d_w(\mathbf{A},\mathbf{A}_\gamma) \le \gamma d(K), \\ d_s(\mathbf{A},\mathbf{A}_\gamma) \le \gamma d(K) + \gamma \sup\{\operatorname{Lip}(A_t) : t = 1,2,\dots\}. \end{gathered} \tag{2.2}$$

The second inequality in (2.2) implies that the set $\{\mathbf{A}_\gamma : \mathbf{A} \in \mathcal{A}, \ \gamma \in (0,1)\}$ is everywhere dense with respect to the strong topology.

Lemma 2.1. *Let* $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{A}$, $\gamma \in (0,1)$, *and* $\epsilon > 0$. *Then, there exists a natural number* $N \ge 4$ *such that for each injective mapping* $r : \{1,\dots,N\} \to \{1,2,\dots\}$ *and each* $x, y \in K$,

$$||A_{r(N)\gamma} \cdots A_{r(1)\gamma}x - A_{r(N)\gamma} \cdots A_{r(1)\gamma}y|| \le \epsilon. \tag{2.3}$$

Proof. Let $x, y \in K$ and let t be a natural number. It follows from (2.1) that

$$||A_{t\gamma}x - A_{t\gamma}y|| \le (1-\gamma)||A_t x - A_t y|| \le (1-\gamma)\operatorname{Lip}(A_t)\|x-y\|. \tag{2.4}$$

Therefore, for each natural number t,

$$\operatorname{Lip}(A_{t\gamma}) \le (1-\gamma)\operatorname{Lip}(A_t). \tag{2.5}$$

Since $\limsup_{t\to\infty}\operatorname{Lip}(A_t) \le 1$, it follows from (2.5) that there exists a natural number n_0 such that

$$\operatorname{Lip}(A_{n\gamma}) \le \left(1 - \frac{\gamma}{2}\right), \quad \text{for each integer } n \ge n_0. \tag{2.6}$$

Choose an integer $n_1 \geq 2$ such that

$$(\operatorname{Lip}(\mathbf{A})+1)^{n_0}\left(1-\frac{\gamma}{2}\right)^{n_1} d(K) \leq \epsilon. \tag{2.7}$$

Set

$$N = n_0 + n_1 + 1. \tag{2.8}$$

Let the mapping $r : \{1,\dots,N\} \to \{1,2,\dots\}$ be injective. Define

$$E_1 = \{t \in \{1,\dots,N\} : r(t) < n_0\}, \qquad E_2 = \{1,\dots,N\} \setminus E_1. \tag{2.9}$$

Since the mapping r is injective, the cardinality

$$\operatorname{Card}(E_1) < n_0. \tag{2.10}$$

By (2.8), (2.9), and (2.10), we have

$$\operatorname{Card}(E_2) > n_1. \tag{2.11}$$

It follows from (1.3), (2.9), (2.6), (2.5), (1.6), (2.10), (2.11), and (2.7) that for each $x, y \in K$,

$$\begin{aligned}
&\|A_{r(N)\gamma}\cdots A_{r(1)\gamma}x - A_{r(N)\gamma}\cdots A_{r(1)\gamma}y\| \\
&\quad \leq \prod_{i=1}^{N} \operatorname{Lip}(A_{r(i)\gamma})\|x-y\| \\
&\quad \leq \prod_{i\in E_1} \operatorname{Lip}(A_{r(i)\gamma}) \prod_{i\in E_2} \operatorname{Lip}(A_{r(i)\gamma}) d(K) \\
&\quad \leq \left(1-\frac{\gamma}{2}\right)^{\operatorname{Card}(E_2)} \operatorname{Lip}(\mathbf{A})^{\operatorname{Card}(E_1)} d(K) \\
&\quad \leq \left(1-\frac{\gamma}{2}\right)^{n_1} (\operatorname{Lip}(\mathbf{A})+1)^{n_0} d(K) \leq \epsilon.
\end{aligned} \tag{2.12}$$

Lemma 2.1 is proved. □

Lemma 2.2. *Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{A}$, $\gamma \in (0,1)$, and $\epsilon > 0$. Then, there exist a natural number $N \geq 4$ and a neighborhood $\mathcal{U}$ of $\mathbf{A}_\gamma$ in the space $\mathcal{A}$ with the weak topology such that for each $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{U}$, each injective mapping $r : \{1,\dots,N\} \to \{1,2,\dots\}$, and each $x, y \in K$,*

$$\|B_{r(N)}\cdots B_{r(1)}x - B_{r(N)}\cdots B_{r(1)}y\| < \epsilon. \tag{2.13}$$

Proof. By Lemma 2.1, there exists a natural number $N \geq 4$ such that for each injective mapping $r : \{1,\dots,N\} \to \{1,2,\dots\}$ and each $x, y \in K$,

$$||A_{r(N)\gamma} \cdots A_{r(1)\gamma}x - A_{r(N)\gamma} \cdots A_{r(1)\gamma}y|| \leq \frac{\epsilon}{8}. \tag{2.14}$$

Choose a positive number

$$\delta < 16^{-1}\epsilon\big(\operatorname{Lip}(\mathbf{A}) + 1\big)^{-N} \tag{2.15}$$

and set

$$\mathcal{U} = \{\mathbf{B} \in \mathcal{A} : d_w(\mathbf{A}_\gamma, \mathbf{B}) \leq \delta\}. \tag{2.16}$$

Assume that $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{U}$ and that the mapping $r : \{1,\dots,N\} \to \{1,2,\dots\}$ is injective. We show, by induction, that for any integer $n \in [1,N]$ and any $z \in K$,

$$||B_{r(n)} \cdots B_{r(1)}z - A_{r(n)\gamma} \cdots A_{r(1)\gamma}z|| \leq \delta\big(\operatorname{Lip}(\mathbf{A}) + 1\big)^{n}. \tag{2.17}$$

First we show that (2.17) holds for $n = 1$. Let $z \in K$. By (2.16) and the definition of d_w,

$$||B_{r(1)}z - A_{r(1)\gamma}z|| \leq \delta, \tag{2.18}$$

so that (2.17) is true for $n = 1$. Let $z \in K$, $i \in \{1,\dots,N-1\}$, and assume that (2.17) holds for $n = i$. When combined with (1.1), (1.5), (2.5), and the definition of d_w, this inductive assumption implies that

$$\begin{aligned}
&||B_{r(i+1)}B_{r(i)} \cdots B_{r(1)}z - A_{r(i+1)\gamma}A_{r(i)\gamma} \cdots A_{r(1)\gamma}z|| \\
&\quad\leq ||B_{r(i+1)}B_{r(i)} \cdots B_{r(1)}z - A_{r(i+1)\gamma}B_{r(i)} \cdots B_{r(1)}z|| \\
&\qquad + ||A_{r(i+1)\gamma}B_{r(i)} \cdots B_{r(1)}z - A_{r(i+1)\gamma}A_{r(i)\gamma} \cdots A_{r(1)\gamma}z|| \\
&\quad\leq \operatorname{Lip}(\mathbf{A}_\gamma)||B_{r(i)} \cdots B_{r(1)}z - A_{r(i)\gamma} \cdots A_{r(1)\gamma}z|| \\
&\qquad + ||B_{r(i+1)}B_{r(i)} \cdots B_{r(1)}z - A_{r(i+1)\gamma}B_{r(i)} \cdots B_{r(1)}z|| \\
&\quad\leq \operatorname{Lip}(\mathbf{A})\delta\big(\operatorname{Lip}(\mathbf{A}) + 1\big)^{i} + \delta \leq \delta\big(\operatorname{Lip}(\mathbf{A}) + 1\big)^{i+1}.
\end{aligned} \tag{2.19}$$

Thus, (2.17) holds for $n = i+1$ too. Therefore, (2.17) holds for $n = N$ and for any $z \in K$. Combined with (2.15), this fact implies that

$$||B_{r(N)} \cdots B_{r(1)}z - A_{r(N)\gamma} \cdots A_{r(1)\gamma}z|| \leq \delta\big(\operatorname{Lip}(\mathbf{A}) + 1\big)^{N} < \frac{\epsilon}{16}. \tag{2.20}$$

Now let $x, y \in K$. It follows from (2.20) and the definition of N (see (2.14)) that

$$\begin{aligned} &||B_{r(N)}\cdots B_{r(1)}x - B_{r(N)}\cdots B_{r(1)}y|| \\ &\quad \le ||A_{r(N)\gamma}\cdots A_{r(1)\gamma}x - A_{r(N)\gamma}\cdots A_{r(1)\gamma}y|| \\ &\qquad + ||B_{r(N)}\cdots B_{r(1)}x - A_{r(N)\gamma}\cdots A_{r(1)\gamma}x|| \\ &\qquad + ||B_{r(N)}\cdots B_{r(1)}y - A_{r(N)\gamma}\cdots A_{r(1)\gamma}y|| \\ &\quad \le \frac{2\epsilon}{16} + \frac{\epsilon}{8} < \epsilon. \end{aligned} \tag{2.21}$$

Lemma 2.2 is proved. □

3. Proof of Theorem 1.1

Let $\mathbf{A} = \{A_t\}_{t=1}^{\infty} \in \mathcal{A}$, $\gamma \in (0,1)$, and let n be a natural number. By Lemma 2.2, there exist an open neighborhood $\mathcal{U}(\mathbf{A}, \gamma, n)$ of $\mathbf{A}_\gamma$ in $\mathcal{A}$ with the weak topology and a natural number $N(\mathbf{A}, \gamma, n)$ such that the following property holds:

(i) for each $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{U}(\mathbf{A}, \gamma, n)$, each injective mapping

$$r : \{1,\ldots,N(\mathbf{A},\gamma,i)\} \longrightarrow \{1,2,\ldots\}, \tag{3.1}$$

and each $x, y \in K$, we have

$$||B_{r(N(\mathbf{A},\gamma,n))}\cdots B_{r(1)}x - B_{r(N(\mathbf{A},\gamma,n))}\cdots B_{r(1)}y|| \le \frac{1}{n}. \tag{3.2}$$

Define

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \cup \{\mathcal{U}(\mathbf{A},\gamma,i) : \mathbf{A} \in \mathcal{A},\ \gamma \in (0,1),\ i \ge n\}. \tag{3.3}$$

Clearly, $\mathcal{F}$ is a countable intersection of open (in the weak topology) everywhere dense (in the strong topology) subsets of $\mathcal{A}$.

Let $\mathbf{C} = \{C_t\}_{t=1}^{\infty} \in \mathcal{F}$ and $\epsilon > 0$. We may assume that $\epsilon < 1$. Choose a natural number

$$q > \frac{8}{\epsilon}. \tag{3.4}$$

By the definition of $\mathcal{F}$, there exist $\mathbf{A} \in \mathcal{A}$, $\gamma \in (0,1)$, and an integer $i \ge q$ such that

$$\mathbf{C} \in \mathcal{U}(\mathbf{A},\gamma,i). \tag{3.5}$$

Let $\mathbf{B} = \{B_t\}_{t=1}^{\infty} \in \mathcal{U}(\mathbf{A},\gamma,i)$. It follows from the definition of $\mathcal{U}(\mathbf{A},\gamma,i)$ and property (i) that for each injective mapping $r : \{1,\ldots,N(\mathbf{A},\gamma,i)\} \to \{1,2,\ldots\}$ and each $x, y \in K$, we have

$$||B_{r(N(\mathbf{A},\gamma,i))}\cdots B_{r(1)}x - B_{r(N(\mathbf{A},\gamma,i))}\cdots B_{r(1)}y|| \le \frac{1}{i} \le \frac{1}{q} < \epsilon. \tag{3.6}$$

This implies that for each $x, y \in K$, each integer $n \geq N(\mathbf{A}, y, i)$, and each injective mapping $r : \{1, \ldots, n\} \to \{1, 2, \ldots\}$, we have

$$||B_{r(n)} \cdots B_{r(1)} x - B_{r(n)} \cdots B_{r(1)} y|| < \epsilon. \tag{3.7}$$

This completes the proof of Theorem 1.1.

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Simeon Reich: Department of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: sreich@tx.technion.ac.il

Alexander J. Zaslavski: Department of Mathematics, The Technion-Israel Institute of Technology, 32000 Haifa, Israel
E-mail address: ajzasl@tx.technion.ac.il