PARABOLIC CURVES IN $\mathbb{C}^3$

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We discuss a family of holomorphic self-maps of $\mathbb{C}^3$ tangent to the identity at the origin presenting dynamical phenomena not appearing for lower-dimensional maps.

1. Introduction

The classical Leau-Fatou flower theorem in one-dimensional holomorphic dynamics is the following.

Theorem 1.1 (Leau-Fatou flower theorem [9, 14]). *Let $g(\zeta)=\zeta+a_k\zeta^k+O(\zeta^{k+1})$, with $k\geq 2$ and $a_k\neq 0$, be a holomorphic function fixing the origin. Then, there are $k-1$ disjoint domains $D_1,\dots,D_{k-1}$, with the origin in their boundary, invariant under g (i.e., $g(D_j)\subset D_j$) such that $(g|_{D_j})^n\to 0$ uniformly on compact subsets as $n\to\infty$ for $j=1,\dots,k-1$, where g^n denotes the composition of g with itself n times.*

Any such domain is called a *parabolic domain* for f at the origin, and they are (together with attracting basins, Siegel disks and Hermann rings) among the building blocks of Fatou sets of rational functions (see, e.g., [17] for a modern exposition).

In [3], this theorem has been generalized to any (germ of) holomorphic self-map f of $\mathbb{C}^2$ fixing the origin and tangent to the identity, that is, such that $f(O)=O$ and $df_O=\mathrm{id}$. To precisely describe the statement, we need a couple of definitions.

Let f be a germ of holomorphic self-map of $\mathbb{C}^n$ fixing the origin and tangent to the identity. Writing $f=(f_1,\dots,f_n)$, let $f_j=z_j+P_{j,\nu_j}+P_{j,\nu_j+1}+\cdots$ be the homogeneous expansion of f in series of homogeneous polynomial, where $\deg P_{j,k}=k$ (or $P_{j,k}\equiv 0$) and $P_{j,\nu_j}\not\equiv 0$. The *order* $\nu(f)$ of f at the origin is defined by $\nu(f)=\min\{\nu_1,\dots,\nu_n\}$.

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A *parabolic curve* for f at the origin is an injective holomorphic map $\varphi : \Delta \to \mathbb{C}^n$ satisfying the following properties:

(i) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;
(ii) φ is continuous at the origin, and $\varphi(0) = O$;
(iii) $\varphi(\Delta)$ is invariant under f, and $(f|_{\varphi(\Delta)})^n \to O$ as $n \to \infty$.

Furthermore, if $[\varphi(\zeta)] \to [v] \in \mathbb{P}^{n-1}$ as $\zeta \to 0$ (where $[\cdot]$ denotes the canonical projection of $\mathbb{C}^n \setminus \{O\}$ onto $\mathbb{P}^{n-1}$), we say that φ is *tangent to* $[v]$ at the origin.

Then, in [3], the following theorem was proved.

Theorem 1.2. *Let f be a (germ of) holomorphic self-map of $\mathbb{C}^2$ tangent to the identity such that the origin is an isolated fixed point. Then, there exist (at least) $\nu(f) - 1$ parabolic curves for f at the origin.*

The proof of this theorem was achieved following a path suggested by a problem in continuous holomorphic dynamics, the so-called separatrix problem. It was known since the end of the last century, thanks, for example, to Poincaré [20], that a generic holomorphic vector field with an isolated singularity at the origin in $\mathbb{C}^n$ admits invariant submanifolds (i.e., leaves of the one-dimensional foliation induced by the given vector field) passing through the singularity (separatrix); but it remained unknown for more than one hundred years, even after replacing "submanifold" by "complex analytic subvariety," whether this was true for *any* holomorphic vector field with an isolated singularity. At last, in 1982, Camacho and Sad [6] proved that separatrix always exist through isolated singularities of two-dimensional holomorphic vector fields.

The proof of Camacho-Sad theorem depended on three ingredients: Poincaré's generic result; a canonical reduction of the singularity to simpler, reduced cases via blowups (developed by Briot and Bouquet [5], Dumortier [8], Seidenberg [21], and Van den Essen [22]; see [16] for a good account); and an index theorem for compact smooth leaves.

Accordingly, the proof of Theorem 1.2 depended on three ingredients as well: a generic result due to Hakim [12, 13] (see Section 2 for a precise statement); a reduction of the singularity via blowups, and an index theorem for pointwise fixed one-dimensional compact submanifolds, both developed in [3] (but see [4] for a generalization of the index theorem to not necessarily smooth one-dimensional subvarieties).

These results leave open the problem of what happens in dimensions greater than two. For the separatrix problem, the answer, surprisingly, is negative: Gómez-Mont and Luengo [10] (see also [15, 18, 19] for $n > 3$) found a family of holomorphic vector fields with an isolated singularity at the origin in $\mathbb{C}^3$ and no complex analytic leaf passing through the singularity.

On the other hand, the aim of this paper is to provide an example showing that, in dimensions greater than 2, the discrete case presents behaviors not predicted by the analogy with the continuous case.

An apparently trivial characteristic of complex analytic leaves is that they survive blowups: if S is a one-dimensional leaf, possibly singular, of a holomorphic foliation $\mathcal{F}$ on a complex manifold M and $p \in S$, then the proper transform of S in the blowup $\tilde{M}$ of M at p is still a one-dimensional leaf of the canonical lifting $\tilde{\mathcal{F}}$ of $\mathcal{F}$ to $\tilde{M}$. This characteristic is the cornerstone of Gómez-Mont and Luengo construction of foliations in $\mathbb{C}^3$ with no separatrix through a singular point.

As we prove in Section 2, the parabolic curves constructed in Theorem 1.2 survive blowups too. Indeed, we will show that any such curve $\varphi : \Delta \to \mathbb{C}^2$ admits an *asymptotic expansion* at the origin: there exists a formal power series at the origin asymptotic to φ in Δ. In particular, the strict transform of the image of φ is still a parabolic curve for the lifting of f to the blowup of the origin in $\mathbb{C}^2$, and we can keep blowing up as many times as we want to always obtain a parabolic curve for the corresponding lifting. Such parabolic curves are called *robust*; see Section 2 for a precise definition.

Then, in Section 4, we will be able to prove the following theorem.

Theorem 1.3. *There exists a family of (germs of) holomorphic self-maps of $\mathbb{C}^3$ tangent to the identity and with the origin as isolated fixed point but with no robust parabolic curves at the origin. Nevertheless, all these maps admit parabolic curves at the origin.*

So, if $n = 3$ in the discrete case, there are maps with only "fragile" (i.e., destroyed by repeated blowups) parabolic curves, which is a phenomenon not happening for $n = 2$ and with no clear analogy in the continuous case.

2. Robust parabolic curves

We start recalling a few definitions adapted from [2, 3]. The symbol $\mathbb{O}_n$ denotes the ring of germs of holomorphic functions defined in a neighbourhood of the origin O of $\mathbb{C}^n$. Any $g \in \mathbb{O}_n$ has a *homogeneous expansion* as infinite sum of homogeneous polynomials, $g = P_0 + P_1 + \cdots$, with $\deg P_j = j$ (or $P_j \equiv 0$); the least $j \geq 0$ such that P_j is not identically zero is the *order* $\nu(g)$ of g.

Given a subset S of a complex n-dimensional manifold M, we denote by $\operatorname{End}(M, S)$ the set of germs about S of holomorphic self-maps of M fixing S pointwise. If $S = \{p\}$, we write $\operatorname{End}(M, p)$ instead of $\operatorname{End}(M, \{p\})$. We say that an $f \in \operatorname{End}(M, p)$ is *tangent to the identity* if $df_p = \mathrm{id}$.

Let $f \in \operatorname{End}(\mathbb{C}^n, O)$. We always write $f = (f_1, \dots, f_n)$; furthermore, $f_j = P_{1,j} + P_{2,j} + \cdots$ will be the homogeneous expansions of f_j (in most cases, $P_{1,j}(z) = z_j$). We consistently write $f_j = P_{1,j} + g_j$; furthermore, by definition, the *order* of f is $\nu(f) = \min\{\nu(g_1), \dots, \nu(g_n)\}$. We always assume $\nu(f) < +\infty$, that is, $f \not\equiv \mathrm{id}_{\mathbb{C}^n}$. We also set $\ell = \gcd(g_1, \dots, g_n)$ and $g_j = \ell g_j^o$; both ℓ and the g_j^o's are defined up to units in $\mathbb{O}_n$. In particular, if ℓ is not a unit, then $\ell(z) = 0$ is an (not necessarily reduced) equation of the germ at the origin of the fixed-points set of f; conversely, if the germ at the origin of the fixed-points set of f has dimension $n - 1$, then ℓ is not a unit.

The *pure order* of f at the origin is $\nu_o(f) = \min\{\nu(g_1^o), \dots, \nu(g_n^o)\}$. We say that the origin is *singular* for f if $\nu_o(f) \geq 1$, that is, if $g_1^o, \dots, g_n^o$ vanish at the origin. This happens, for instance, if the fixed-points set of f at the origin has a dimension less than $n-1$ (e.g., if the origin is an isolated fixed point). Furthermore, $g_j^o = P_{0,j}^o + P_{1,j}^o + \cdots$ will be the homogeneous expansion of g_j^o.

Following Hakim [12, 13], we say that $v = [v_1 : \cdots : v_n] \in \mathbb{P}^{n-1}$ is a *characteristic direction* for f at the origin if there exists $\lambda \in \mathbb{C}$ such that $P_{\nu(f),j}(v_1, \dots, v_n) = \lambda v_j$ for $j = 1, \dots, n$; it is a *nondegenerate* characteristic direction if $\lambda \neq 0$, and *degenerate* otherwise.

More generally, if $P = (P_1, \dots, P_n) \in \operatorname{End}(\mathbb{C}^n, O)$ is an n-tuple of homogeneous polynomials of degree ν, a *characteristic direction* for P is a vector $v \in \mathbb{P}^{n-1}$ such that $P(v) = \lambda v$ for a suitable $\lambda \in \mathbb{C}$; again, it is *degenerate* or *nondegenerate* according to λ being zero or nonzero. If v is an isolated characteristic direction of P, its *multiplicity* $\mu_P(v)$ is the local intersection multiplicity (see, e.g., [7] or [11] for definition and properties of the local intersection multiplicity) at v in $\mathbb{P}^{n-1}$ of the polynomials $z_{j_0} P_j - z_j P_{j_0}$ with $j \neq j_0$, where j_0 is any index such that $v_{j_0} \neq 0$ (and $\mu_P(v)$ is clearly independent of j_0).

Lemma 2.1. *Let $P = (P_1, \dots, P_n) \in \operatorname{End}(\mathbb{C}^n, O)$ be an n-tuple of homogeneous polynomials of degree $\nu \geq 2$. Denote by $\pi : \mathbb{P}^n \setminus \{[1:0:\cdots:0]\} \to \mathbb{P}^{n-1}$ the projection $\pi([v_0 : v_1 : \cdots : v_n]) = [v_1 : \cdots : v_n]$, and let $\mathcal{S} \subset \mathbb{P}^n$ be the set of solutions of the system*

$$\begin{gathered} P_1(z) - z_0^{\nu-1} z_1 = 0, \\ \vdots \\ P_n(z) - z_0^{\nu-1} z_n = 0. \end{gathered} \tag{2.1}$$

Then,

(i) *the vector $v = [v_1 : \cdots : v_n] \in \mathbb{P}^{n-1}$ is a characteristic direction for P if and only if $\pi^{-1}(v) \cap \mathcal{S}$ is not empty. More precisely, v is a degenerate characteristic direction if and only if $\pi^{-1}(v) \cap \mathcal{S} = \{[0 : v_1 : \cdots : v_n]\}$, and it is a nondegenerate characteristic direction if and only if $\pi^{-1}(v) \cap \mathcal{S}$ contains exactly $\nu - 1$ elements all with nonzero first coordinate;*

(ii) *if v is a nondegenerate isolated characteristic direction, then its multiplicity $\mu_P(v)$ is equal to the multiplicity of any element in $\pi^{-1}(v) \cap \mathcal{S}$ as a solution of (2.1); on the other hand, if v is a degenerate isolated characteristic direction, then the multiplicity of $[0 : v_1 : \cdots : v_n]$ as a solution of (2.1) is $(\nu - 1)\mu_P(v)$;*

(iii) *the number of characteristic directions of P, counted according to their multiplicities, if finite is given by $(\nu^n - 1)/(\nu - 1)$.*

Proof. (i) This is obvious.

(ii) Without loss of generality, we can assume that $v = [0 : \cdots : 0 : 1]$ and fix $\tilde{v} \in \pi^{-1}(v) \cap \mathcal{S}$. In the usual local coordinates of the subset $\{z_n \neq 0\} \subset \mathbb{P}^n$, the point $\tilde{v}$ is represented by $(\lambda, 0, \dots, 0)$, where $\lambda = 0$ if and only if v is degenerate. Analogously, the local coordinates of the subset $\{z_n \neq 0\} \subset \mathbb{P}^{n-1}$ are centered in v. So, we have

$$\mu_P(v) = I\big(P_1(z',1) - z_1 P_n(z',1), \dots, P_{n-1}(z',1) - z_{n-1}P_n(z',1); O\big), \tag{2.2}$$

while the multiplicity of $\tilde{v}$ as a solution of (2.1) is

$$\begin{aligned} \tilde{\mu} = I\big(&P_1(z',1) - z_0^{\nu-1} z_1, \dots, P_{n-1}(z',1) \\ &- z_0^{\nu-1} z_{n-1}, P_n(z',1) - z_0^{\nu-1}; (\lambda, 0, \dots, 0)\big), \end{aligned} \tag{2.3}$$

where I denotes the local intersection multiplicity and $z' = (z_1, \dots, z_{n-1})$. The standard properties of I immediately yields

$$\begin{aligned} \tilde{\mu} = I\big(&P_1(z',1) - z_1 P_n(z',1), \dots, P_{n-1}(z',1) \\ &- z_{n-1}P_n(z',1), P_n(z',1) - z_0^{\nu-1}; (\lambda, 0, \dots, 0)\big). \end{aligned} \tag{2.4}$$

Set $Q_j(z') = P_j(z',1) - z_j P_n(z',1)$ for $j = 1, \dots, n-1$. Now, if v is degenerate, that is, if $\lambda = 0$, the ring $\mathbb{O}_n/(Q_1, \dots, Q_{n-1}, P_n(z',1) - z_0^{\nu-1})$ is generated by $1, z_0, \dots, z_0^{\nu-2}$ on the ring $\mathbb{O}_{n-1}/(Q_1, \dots, Q_{n-1})$; therefore,

$$\tilde{\mu} = (\nu - 1)\mu_P(v) \tag{2.5}$$

as claimed.

On the other hand, if $\lambda \neq 0$, we can translate the coordinates obtaining

$$\tilde{\mu} = I\big(Q_1, \dots, Q_{n-1}, P_n(z',1) - (z_0 + \lambda)^{\nu-1}; O\big). \tag{2.6}$$

Now, $P_n(O',1) = \lambda^{\nu-1}$; therefore, there is $Q_n \in \mathbb{O}_{n-1}$ such that $Q_n(O') = \lambda$ and

$$P_n(z',1) - (z_0 + \lambda)^{\nu-1} = \big(Q_n(z') - z_0 - \lambda\big) R(z_0, z') \tag{2.7}$$

in $\mathbb{O}_n$, with $R(O) \neq 0$. Indeed, it suffices to take $Q_n \in \mathbb{O}_{n-1}$ so that $Q_n(O') = \lambda$ and $Q_n^{\nu-1} = P_n(z',1)$ in $\mathbb{O}_{n-1}$. But then,

$$\tilde{\mu} = I\big(Q_1, \dots, Q_{n-1}, Q_n - \lambda - z_0; O\big), \tag{2.8}$$

and thus, the argument used before yields $\tilde{\mu} = \mu_P(v)$.

(iii) By Bezout's theorem, we know that (2.1) has exactly (infinite or) ν^n solutions, counted according to their multiplicities. The solution $[1:0:\cdots:0]$, which is the only one not generating a characteristic direction of P, has multiplicity 1. Thus, we are left with $\nu^n - 1$ solutions, and the assertion follows from (i) and (ii). □

A characteristic direction for f at the origin is a characteristic direction for $P_f = (P_{\nu(f),1}, \dots, P_{\nu(f),n})$. Similarly, a *singular direction* for f at the origin is a characteristic direction for $P^o_f = (P^o_{\nu_o(f),1},\dots,P^o_{\nu_o(f),n})$. Since $P_{\nu(f),j} = P^o_{\nu_o(f),j}R_\kappa$, where R_κ is the first nonzero term in the homogeneous expansion of ℓ (and we have $\nu(f) = \nu_o(f)+\kappa$), it is clear that every nondegenerate characteristic direction is a singular direction, and that every singular direction is a characteristic direction.

The set of singular directions is clearly an algebraic subvariety of $\mathbb{P}^{n-1}$. If the maximal dimension of the irreducible components of this subvariety is k, we say that the origin is *k-dicritical* for f; if $k = 0$ (i.e., if there is only a finite number of singular directions), we say that the origin is *nondicritical* for f.

We now recall some basic definitions and results on blowing up maps, referring to [2] for details. Let M be a complex n-manifold and $p \in M$. The *blowup* of M at p is the set $\tilde{M} = (M \setminus \{p\}) \cup \mathbb{P}(T_pM)$, endowed with the manifold structure we presently describe, together with the projection $\pi : \tilde{M} \to M$ given by $\pi|_{M\setminus\{p\}} = \mathrm{id}_{M\setminus\{p\}}$ and $\pi|_{\mathbb{P}(T_pM)} \equiv p$. The set $S = \mathbb{P}(T_pM) = \pi^{-1}(p)$ is the *exceptional divisor* of the blowup.

Fix a chart $\varphi = (z_1,\dots,z_n) : U \to \mathbb{C}^n$ of M centered at p. Set $U_j = (U \setminus \{z_j = 0\}) \cup (S \setminus \mathrm{Ker}(dz_j|_p))$, and let $\chi_j : U_j \to \mathbb{C}^n$ be given by

$$\chi_j(q)_h = \begin{cases} z_j(q), & \text{if } j = h,\ q \in U \setminus \{z_j = 0\}, \\ \dfrac{z_h(q)}{z_j(q)}, & \text{if } j \neq h,\ q \in U \setminus \{z_j = 0\}, \\ \dfrac{d(z_h)_p(q)}{d(z_j)_p(q)}, & \text{if } j \neq h,\ q \in S \setminus \mathrm{Ker}\,(dz_j|_p), \\ 0, & \text{if } j = h,\ q \in S \setminus \mathrm{Ker}\,(dz_j|_p). \end{cases} \tag{2.9}$$

Then, the charts (U_j,χ_j), together with an atlas of $M \setminus \{p\}$, endow $\tilde{M}$ with a structure of n-dimensional complex manifold such that the projection π is holomorphic everywhere and given by

$$[\varphi \circ \pi \circ \chi_j^{-1}(w)]_h = \begin{cases} w_j, & \text{if } j = h, \\ w_j w_h, & \text{if } j \neq h. \end{cases} \tag{2.10}$$

In the sequel, we refer to these charts (or to charts obtained by these composing with a translation) as *canonical charts*.

Let $f \in \mathrm{End}(M,p)$ be tangent to the identity. Then, (see [2]) there exists a unique map $\tilde{f} \in \mathrm{End}(\tilde{M},S)$, the *blowup* of f at p, such that $\pi \circ \tilde{f} = f \circ \pi$. The action of $\tilde{f}$ on S is induced by the action of df_p on $\mathbb{P}(T_pM)$; in particular, $\tilde{f}|_S = \mathrm{id}_S$.

Lemma 2.2. *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent to the identity, and $\tilde{f}$ its blowup at O. Assume that O is not $(n-1)$-dicritical. Then, a direction $v_0 \in \mathbb{P}^{n-1}$ is singular for f if and only if it is a singular point for $\tilde{f}$.*

Proof. Without loss of generality, we can assume that $v_0 = [1:0:\cdots:0]$. Writing

$$f_j(z) = z_j + \ell(z)\big(P^o_{\nu,j}(z) + R_{\nu+1}\big), \tag{2.11}$$

where $\nu = \nu_o(f)$ and $R_{\nu+1}$ denotes a remainder term of order at least $\nu+1$, in the canonical chart centered at v_0, the blowup $\tilde{f}$ is represented by

$$\tilde{f}_j(w) = \begin{cases} w_1 + \ell(w_1, w_1w')w_1^{\nu}\big[P^o_{\nu,1}(1,w') + O(w_1)\big], & \text{for } j = 1, \\ w_j + \phi_j(w)\ell(w_1, w_1w')w_1^{\nu-1} & \\ \qquad \times\big[P^o_{\nu,j}(1,w') - w_jP^o_{\nu,1}(1,w') + O(w_1)\big], & \text{for } 2 \le j \le n, \end{cases} \tag{2.12}$$

where $\phi_2, \dots, \phi_n$ are units and $w' = (w_2, \dots, w_n)$. Now, since O is not $(n-1)$-dicritical, we must have $P^o_{\nu,j}(1,w') - w_jP^o_{\nu,1}(1,w') \not\equiv 0$ for at least one $j \ge 2$; therefore, arguing as in [3, Lemma 2.1 and Corollary 2.1], we see that the origin (i.e., v_0) is a singular point for $\tilde{f}$ if and only if $P^o_{\nu,j}(1,O') = 0$ for $2 \le j \le n$, that is, if and only if v_0 is a singular direction for f. □

A *parabolic curve* at p for f is an injective holomorphic map $\varphi : \Delta \to M \setminus \{p\}$ such that

(i) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;
(ii) φ is continuous at the origin, and $\varphi(0) = p$;
(iii) $\varphi(\Delta)$ is invariant under f, and $(f|_{\varphi(\Delta)})^k \to p$ as $k \to \infty$.

Remark 2.3. Since φ is injective and $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$, there is a unique holomorphic function $f_o : \Delta \to \Delta$ such that $f \circ \varphi = \varphi \circ f_o$. Clearly, 0 is the Wolff point of f_o; therefore, Wolff's lemma implies that $f_o(\Delta_r) \subseteq \Delta_r$ for all $r > 0$, where Δ_r is the horocycle in Δ of center 0 and radius r (see, e.g., [1] for Wolff's lemma and the definition of horocycles). In particular, $\varphi|_{\Delta_r}$ is still a parabolic curve for f at the origin for any $r > 0$.

Let $\varphi : \Delta \to M$ be a parabolic curve for f at p. If there exists $v \in \mathbb{P}(T_pM)$ such that $\tilde{\varphi} = \pi^{-1} \circ \varphi$ is a parabolic curve at v for $\tilde{f}$ (where $\pi : \tilde{M} \to M$ is the blowup of M at p, and $\tilde{f}$ is the blowup of f), then we say that φ is *tangent* to v at p, and that $\tilde{\varphi}$ is the *strict transform* of φ. We explicitly remark that since the image of φ does not contain p, the curve $\tilde{\varphi}$ is always well defined and $\tilde{\varphi}(\Delta)$ is $\tilde{f}$-invariant; however, $\tilde{\varphi}$ is a parabolic curve for $\tilde{f}$ only if φ is tangent to some direction in p.

The main results of [3, 12, 13] can then be summarized as follows.

Theorem 2.4. *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent at the identity. Then,*

(i) *if f admits a parabolic curve at the origin tangent to a direction v, then v is a characteristic direction of f [12];*

(ii) *if v is a nondegenerate characteristic direction for f at the origin, then f admits at least $\nu(f)-1$ parabolic curves at the origin tangent to v* [12, 13];
(iii) *if $n=2$ and O is an isolated fixed point of f, then f always admits at least $\nu(f)-1$ parabolic curves tangent to some singular direction* [3].

The importance of Theorem 2.4(iii) is that it is easy to find examples of maps tangent to the identity with *no* nondegenerate characteristic directions, where we cannot apply Theorem 2.4(ii). On the other hand, the techniques in [3] do not yet allow to prove the existence of parabolic curves tangent to any given nondegenerate characteristic direction, not even for $n=2$: nondegenerate characteristic directions with positive rational residual index are left out (see [3, Corollary 3.1] for details).

As mentioned in the introduction, parabolic curves are the moral analogue of separatrix. Now, separatrix, being analytic subvarieties, survive blowups: the strict transform of an analytic subvariety is still an analytic subvariety. This is not always the case for parabolic curves; for instance, if φ is a parabolic curve provided by Theorem 2.4(ii) and tangent to a nondegenerate characteristic direction with residual index equal to 1 (see [3] for the definition), then, after a finite number of blowups, the strict transform of φ is not anymore defined.

On the other hand, the main goal of this section is to prove that the parabolic curves given by Theorem 2.4(iii) do survive blowups. Even better, they are essentially defined by a power series.

To get a precise statement, we need a few more definitions. We say that we can *blow up at level* 1 a parabolic curve φ if there exists $r_0>0$ such that $\varphi|_{\Delta_{r_0}}$ is tangent to some direction $v\in\mathbb{P}(T_pM)$, where Δ_{r_0} is the horocycle centered at the origin of radius r_0 in the domain Δ of φ. Let φ^1 denote the strict transform of $\varphi|_{\Delta_{r_0}}$; if we can blow up φ^1 at level 1, we say that we can *blow up φ at level* 2, and we denote by φ^2 the parabolic curve so obtained (defined on a possibly smaller horocycle). In an inductive way, we say that we can *blow up φ at level h* if we can blow up φ^{h-1} at level 1. We then say that φ is *robust* if the following two conditions are satisfied:

(a) we can blow up φ at level h for any $h\geq 1$;
(b) there is a formal power series $\mathfrak{Q}\in(\mathbb{C}[[\zeta]])^n$ such that, for every $h\geq 1$, there is $r_h>0$ such that $\varphi-\mathfrak{Q}_h=O(\zeta^{h+1})$ in Δ_{r_h}, where $\mathfrak{Q}_h$ denotes the truncation at degree h of $\mathfrak{Q}$.

The main goal of this section is to prove that the parabolic curves whose existence is predicted by Theorem 2.4(iii) are robust. To do so, we need the following.

Lemma 2.5. *Given $\delta>0$ and $m\in\mathbb{N}^*$, set $D_{\delta,m}=\{\zeta\in\mathbb{C}\mid|\zeta^m-\delta|<\delta\}$, and let Δ be any one of the m connected components of $D_{\delta,m}$. Then, the horocycles centered at the origin of Δ are all of the form $D_{\delta',m}\cap\Delta$ for a suitable $0<\delta'<\delta$.*

Proof. The domain Δ is sent biholomorphically onto $D_{\delta,1}$ by the map $\zeta \mapsto \zeta^r$. In turn, the domain $D_{\delta,1}$ is sent biholomorphically onto the right half-plane $H_\delta = \{\operatorname{Re}\zeta > 1/2\delta\}$ by the map $\zeta \to 1/\zeta$. The horocycles of H_δ centered at ∞ are exactly the half-planes $H_{\delta'}$ with $0 < \delta' < \delta$, and the assertion follows. □

THEOREM 2.6. *Let $f \in \operatorname{End}(\mathbb{C}^2, O)$ be tangent at the identity, and assume that the origin is an isolated fixed point of f. Then, f admits at least $\nu(f) - 1$ robust parabolic curves.*

Proof. It suffices to show that the parabolic curves obtained in [3] are robust. First of all, in [3], we showed that, after a finite number of blowups and affine changes of variables, we can assume that f is of the form

$$\begin{aligned} f_1(z_1, z_2) &= z_1 - z_1^{m+1} + O(z_1^{m+2}, z_1^{m+1}z_2), \\ f_2(z_1, z_2) &= z_2(1 - \lambda z_1^m + O(z_1^{m+1}, z_1^m z_2)) + z_1^{m+2}\psi_{m+1}(z_1), \end{aligned} \tag{2.13}$$

with $\operatorname{Re}\lambda < 0$ and $m + 1 \geq \nu(f)$. Since affine changes of variables and blowdowns send robust parabolic curves in robust parabolic curves, it suffices to prove the assertion when f is of the form (2.13).

We make the change of variables

$$\begin{aligned} Z_1 &= z_1, \\ Z_2 &= z_2 - \frac{\psi_{m+1}(0)}{\lambda - 1} z_1^2. \end{aligned} \tag{2.14}$$

In the new coordinates, the map f is represented by

$$\begin{aligned} f_1(Z_1, Z_2) &= Z_1 - Z_1^{m+1} + O(Z_1^{m+2}, Z_1^{m+1}Z_2), \\ f_2(Z_1, Z_2) &= Z_2(1 - \lambda Z_1^m + O(Z_1^{m+1}, Z_1^m Z_2)) + Z_1^{m+3}\psi_{m+2}(Z_1). \end{aligned} \tag{2.15}$$

Let Δ be one of the m connected (and simply connected) components of $D_{\delta,m}$, where $\delta > 0$ will be chosen later, and set

$$\mathcal{F}_m(\delta) = \{u \in \operatorname{Hol}(\Delta, \mathbb{C}) \mid u(\zeta) = \zeta^2 u^o(\zeta),\ \|u^o\|_\infty \leq 1,\ |u'(\zeta)| \leq |\zeta|\}. \tag{2.16}$$

Then, in [3, Theorem 3.1], following [12, 13], we proved that, for every δ small enough, there is a *unique* $u \in \mathcal{F}_m(\delta)$ such that $\varphi_u(\zeta) = (\zeta, u(\zeta))$ is a parabolic curve for f at the origin. Our aim now is to exploit the uniqueness of u to show that φ_u is robust. Since different components of $D_{\delta,m}$ give distinct parabolic curves, we conclude the assertion.

We first prove that condition (a) of the definition is satisfied. Blowing up f and setting $Z_1 = w_1$ and $Z_2 = w_1 w_2$, we get

$$\begin{aligned} \tilde{f}_1(w_1, w_2) &= w_1 - w_1^{m+1} + O(w_1^{m+2}, w_1^{m+2}w_2), \\ \tilde{f}_2(w_1, w_2) &= w_2(1 - (\lambda - 1)w_1^m + O(w_1^{m+1}, w_1^{m+1}w_2)) + w_1^{m+2}\tilde{\psi}_{m+2}(w_1), \end{aligned} \tag{2.17}$$

where $\tilde{\psi}_{m+2}(w_1) - \psi_{m+2}(w_1) = O(w_1^m)$; in particular, $\tilde{\psi}_{m+2}(0) = \psi_{m+2}(0)$. Now, making the change of variables

$$\hat{w}_1 = w_1, \qquad \hat{w}_2 = w_2 - \frac{\psi_{m+2}(0)}{\lambda - 2} w_1^2, \tag{2.18}$$

we get

$$\begin{aligned} \hat{f}_1(\hat{w}_1, \hat{w}_2) &= \hat{w}_1 - \hat{w}_1^{m+1} + O(\hat{w}_1^{m+2}, \hat{w}_1^{m+2}\hat{w}_2), \\ \hat{f}_2(\hat{w}_1, \hat{w}_2) &= \hat{w}_2(1 - (\lambda - 1)\hat{w}_1^m + O(\hat{w}_1^{m+1}, \hat{w}_1^{m+1}\hat{w}_2)) + \hat{w}_1^{m+3}\hat{\psi}_{m+2}(\hat{w}_1), \end{aligned} \tag{2.19}$$

which is of the form (2.15). Thus, we get $\hat{\delta} > 0$, $\hat{u} \in \mathcal{F}_m(\hat{\delta})$ such that $\varphi_{\hat{u}}$ is a parabolic curve for $\hat{f}$ at the origin. Set then

$$u_1(\zeta) = \zeta \hat{u}(\zeta) + \frac{\psi_{m+2}(0)}{\lambda - 2} \zeta^3. \tag{2.20}$$

By construction, φ_{u_1} is a parabolic curve for f; we claim that $u_1 \in \mathcal{F}_m(\delta')$ for δ' small enough.

We can write $u_1 = \zeta^2 u_1^o$ with $u_1^o(\zeta) = \zeta \hat{u}^o(\zeta) + c\zeta$, where $c = \psi_{m+2}(0)/(\lambda - 2)$. Therefore,

$$\begin{aligned} |u_1^o(\zeta)| &\leq (1 + |c|)|\zeta|, \\ |u_1'(\zeta)| = |\zeta^2 \hat{u}^o(\zeta) + \zeta \hat{u}'(\zeta) + 3c\zeta^2| &\leq (2 + 3|c|)|\zeta|^2; \end{aligned} \tag{2.21}$$

so, if $\delta' > 0$ is such that $|\zeta| \leq (2 + 3|c|)^{-1}$ for all $\zeta \in \Delta \cap D_{\delta',m}$, we get $u_1 \in \mathcal{F}_m(\delta')$ as desired. The uniqueness of u then implies that u_1 is the restriction of u to $\Delta \cap D_{\delta',m}$, and thus, by Lemma 2.5, we have proved that we can blow up φ_u at level 1. But clearly, the same proof works for $\varphi_{\hat{u}}$, which means that we can blow up φ_u at level 2. Arguing by induction, we immediately see that condition (a) of the definition of robust parabolic curves is satisfied.

We are left to verify condition (b). We prove, by induction on h, that there is a unique polynomial Q_h of degree at most h such that $u(\zeta) - Q_h(\zeta) = O(\zeta^{h+1})$ on $\Delta \cap D_{\delta',m}$ for all δ' small enough (depending on h). The uniqueness of the Q_h for all h will then imply the existence of a well-defined formal power series $Q \in \mathbb{C}[[\zeta]]$ such that $\mathcal{Q} = (\zeta, Q(\zeta))$ satisfies condition (b) of the definition of robust parabolic curves.

Since $u(\zeta) = \zeta^2 u^o(\zeta)$, for $h = 1$ the only choice is $Q_1 \equiv 0$. Actually, (2.20) shows that, in a possibly smaller horocycle, we have $u(\zeta) = \zeta^3 u^o(\zeta)$, and thus, $Q_2 \equiv 0$ too. So, now, we assume that the claim is true for $h - 1 \geq 2$; in particular, we get a polynomial $\hat{Q}_{h-1}$ of degree at most $h - 1$ such that $\hat{u}(\zeta) - \hat{Q}_{h-1}(\zeta) = O(\zeta^h)$ in a sufficiently small horocycle. But then, setting

$$Q_h(\zeta) = \zeta \hat{Q}_{h-1}(\zeta) + \frac{\psi_{r+2}(0)}{\lambda - 2} \zeta^3 \tag{2.22}$$

and recalling (2.20), we immediately see that $u(\zeta) - Q_h(\zeta) = O(\zeta^{h+1})$ in a sufficiently small horocycle as desired.

Finally, if Q'_h is another polynomial of degree, at most, h such that $u(\zeta) - Q'_h(\zeta) = O(\zeta^{h+1})$ in $\Delta \cap D_{\delta',m}$, we have

$$Q_h(\zeta) - Q'_h(\zeta) = \big(u(\zeta) - Q'_h(\zeta)\big) - \big(u(\zeta) - Q_h(\zeta)\big) = O(\zeta^{h+1}), \tag{2.23}$$

and thus, $Q_h \equiv Q'_h$. □

Remark 2.7. The formal power series so obtained is a polynomial if and only if, after enough blowups, the second component of the blowup map is divisible by w_2. Indeed, the procedure described in the proof stops if and only if we get a blowup map of the form (2.15) with $\psi_{m+2} \equiv 0$. In this case, the function u at that level is identically zero, and thus, blowing down, we see that the parabolic curves we get are restrictions of a holomorphic curve defined in a whole neighbourhood of the origin.

3. Singular points

As mentioned in the introduction, to understand the dynamical behavior of maps tangent to the identity, we need to blow up points. The aim of this section is to prove that, for maps obtained with such a procedure, only singular points are dynamically interesting.

Proposition 3.1. *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be of the form*

$$f_j(z) = \begin{cases} z_j + z_j\left(\prod_{h=1}^{r} z_h^{\nu_h}\right) g_j(z), & \text{for } 1 \le j \le r,\\ z_j + \left(\prod_{h=1}^{r} z_h^{\nu_h}\right) g_j(z), & \text{for } r+1 \le j \le n, \end{cases} \tag{3.1}$$

for suitable $1 \le r < n$, $\nu_1, \ldots, \nu_r \ge 1$, $g_1, \ldots, g_n \in \mathbb{O}_n$. Assume that $g_{j_0}(O) \neq 0$ for some $r+1 \le j_0 \le n$. Then, no infinite orbit can stay arbitrarily close to O, that is, there is a neighbourhood U of the origin such that, for every $q \in U$, there is $n_0 \in \mathbb{N}$ such that $f^{n_0}(q) \notin U$ or $f^{n_0}(q) \in \mathrm{Fix}(f)$.

Remark 3.2. We will see in the next section that all maps in which we are interested are of the form (3.1), possibly with $r = n$. Notice that if $g_{j_0}(O) \neq 0$ for some $r+1 \le j_0 \le n$, then O is not singular for f.

Proof. Without loss of generality, we can assume $j_0 = n$, and, after a linear change of coordinates, we can also assume that $g_n(O) = 1$. Write $g_j(z) = a_j + A_j(z)$ for $j = 1, \ldots, n$ with $\nu(A_j) \ge 1$; we then make the following change of coordinates:

$$Z_j = \begin{cases} z_j\big(1 + A_n(z)\big)^{1/r\nu_j}, & \text{for } 1 \le j \le r,\\ z_j, & \text{for } r+1 \le j \le n. \end{cases} \tag{3.2}$$

In the new coordinates, the map is expressed by

$$F_j(Z) = \begin{cases} Z_j + Z_j\left(\prod_{h=1}^{r} Z_h^{\nu_h}\right)\tilde{g}_j(Z), & \text{for } 1 \le j \le r, \\ Z_j + \left(\prod_{h=1}^{r} Z_h^{\nu_h}\right)\left(a_j + \tilde{A}_j(Z)\right), & \text{for } r+1 \le j \le n-1, \\ Z_n + \prod_{h=1}^{r} Z_h^{\nu_h}, & \text{for } j = n. \end{cases} \tag{3.3}$$

The only nontrivial formula here is the first one, which is obtained as follows:

$$\begin{aligned} F_j(Z) &= f_j(z)\left[1 + A_n(f(z))\right]^{1/r\nu_j} = z_j\left[1 + \left(\prod_{h=1}^{r} z_h^{\nu_h}\right)g_j(z)\right]\left[1 + A_n(f(z))\right]^{1/r\nu_j} \\ &= Z_j\left[1 + \left(\prod_{h=1}^{r} Z_h^{\nu_h}\right)\frac{g_j(z)}{1 + A_n(z)}\right]\left[1 + \frac{A_n(f(z)) - A_n(z)}{1 + A_n(z)}\right]^{1/r\nu_j} \\ &= Z_j\left[1 + \left(\prod_{h=1}^{r} Z_h^{\nu_h}\right)\left(a_j + B_j(z)\right)\right]\left[1 + \left(\prod_{h=1}^{r} Z_h^{\nu_h}\right)C(z)\right] \\ &= Z_j\left[1 + \left(\prod_{h=1}^{r} Z_h^{\nu_h}\right)\tilde{g}_j(Z)\right], \end{aligned} \tag{3.4}$$

for suitable holomorphic functions B_j, C, and $\tilde{g}_j$, where we used the fact that $\prod_{h=1}^{r} z_h^{\nu_h}$ divides each component of $f(z) - z$, and thus, $A_n(f(z)) - A_n(z)$ is divisible by $\prod_{h=1}^{r} Z_h^{\nu_h}$.

Set $Z^{(k)} = F^k(Z)$ and $W^{(k)} = \prod_{h=1}^{r}(Z_h^{(k)})^{\nu_h}$; in particular,

$$Z_n^{(k)} - Z_n^{(0)} = \sum_{l=0}^{k-1}\left(F_n(Z^{(l)}) - Z_n^{(l)}\right) = \sum_{l=0}^{k-1} W^{(l)}. \tag{3.5}$$

Now, for $j = 1,\dots,r$, we have

$$(Z_j^{(1)})^{\nu_j} = Z_j^{\nu_j}\left[1 + W^{(0)}\tilde{g}_j(Z)\right]^{\nu_j}; \tag{3.6}$$

therefore,

$$\begin{aligned} \frac{1}{W^{(1)}} &= \frac{1}{W^{(0)}}\prod_{h=1}^{r}\frac{1}{\left[1 + W^{(0)}\tilde{g}_h(Z)\right]^{\nu_h}} \\ &= \frac{1}{W^{(0)}}\prod_{h=1}^{r}\left[1 - \nu_h W^{(0)}\tilde{g}_h(Z) + O\left(\left(W^{(0)}\right)^2\right)\right] = \frac{1}{W^{(0)}} + a(Z), \end{aligned} \tag{3.7}$$

for a suitable holomorphic function $a(Z)$. In particular, if $P(\rho_1,\dots,\rho_n) = \{|Z_j| < \rho_j,\ j = 1,\dots,n\}$ is a small enough polydisk centered at the origin, we find $M > 0$

such that

$$\left| \frac{1}{W^{(1)}} - \frac{1}{W^{(0)}} \right| \leq M, \tag{3.8}$$

for all $Z \in P(\rho_1, \ldots, \rho_n)$. If $\rho_1, \ldots, \rho_r$ are small enough, we can assume that $M|W^{(0)}| < 1$ for all $Z \in P(\rho_1, \ldots, \rho_n)$. Choose then $0 < \rho < \min\{(2M)^{-1}\log 2, \rho_n\}$, and set $U = P(\rho_1, \ldots, \rho_{n-1}, \rho)$; we claim that no point in $U \setminus \mathrm{Fix}(F)$ can have an orbit completely contained in $U \setminus \mathrm{Fix}(F)$.

Suppose, by contradiction, that $Z^{(0)} \in U \setminus \mathrm{Fix}(F)$ is such that $Z^{(k)} = F^k(Z^{(0)}) \in U \setminus \mathrm{Fix}(F)$ for all $k \in \mathbb{N}$. In particular, $W^{(k)} \neq 0$ for all $k \geq 0$, and so, $|(1/W^{(k)}) - (1/W^{(0)})| \leq kM$. Hence,

$$\left| \frac{W^{(0)}}{W^{(k)}} - 1 \right| \leq kM |W^{(0)}|, \tag{3.9}$$

for all $k \geq 0$. This implies that if $kM|W^{(0)}| < 1$ then $W^{(k)}/W^{(0)}$ belongs to the disk having the segment $[(1 + kM|W^{(0)}|)^{-1}, (1 - kM|W^{(0)}|)^{-1}]$ as diameter, and thus,

$$\operatorname{Re} \frac{W^{(k)}}{W^{(0)}} \geq \frac{1}{1 + kM|W^{(0)}|}. \tag{3.10}$$

Let $k_0 \geq 2$ be the integer such that $(k_0 - 1)M|W^{(0)}| < 1 \leq k_0 M|W^{(0)}|$. Then,

$$\operatorname{Re} \frac{W^{(l)}}{W^{(0)}} \geq \frac{1}{(k_0 + l)M|W^{(0)}|}, \tag{3.11}$$

for $0 \leq l \leq k_0 - 1$. But this implies

$$\begin{aligned} |Z_n^{(k_0)} - Z_n^{(0)}| &= \left| \sum_{l=0}^{k_0-1} W^{(l)} \right| = |W^{(0)}| \left| \sum_{l=0}^{k_0-1} \frac{W^{(l)}}{W^{(0)}} \right| \\ &\geq |W^{(0)}| \sum_{l=0}^{k_0-1} \operatorname{Re} \frac{W^{(l)}}{W^{(0)}} \geq \sum_{l=0}^{k_0-1} \frac{1}{(k_0 + l)M} \geq \frac{\log 2}{M} > 2\rho, \end{aligned} \tag{3.12}$$

and so $Z^{(k_0)} \notin U$, a contradiction. □

4. The example

We now introduce the kind of singularity we need in our example. Let $f \in \mathrm{End}(M, p)$, where M is a three-dimensional complex manifold and $p \in M$. We say that p is a *simple corner* for f if there are $a, b \in \mathbb{N}^*$, $c \in \mathbb{N}$, $\lambda_1 \in \mathbb{C}^*$, $\lambda_2 \in \mathbb{C} \setminus (\mathbb{Q}^+\lambda_1)$, and local coordinates (z_1, z_2, z_3) centered at p so that we can write

$$f_j(z) = \begin{cases} z_1 + (z_1^a z_2^b z_3^c) z_1 (\lambda_1 + g_1), & \text{for } j = 1, \\ z_2 + (z_1^a z_2^b z_3^c) z_2 (\lambda_2 + g_2), & \text{for } j = 2, \\ z_3 + (z_1^a z_2^b z_3^c) g_3, & \text{for } j = 3, \end{cases} \tag{4.1}$$

with $\nu(g_j) \geq 1$ for $j = 1, \dots, 3$. We moreover require that $z_3 | g_3$ if $c > 0$. Notice that a simple corner is automatically a singular point for f.

The main properties of simple corners are collected in the following proposition.

Proposition 4.1. *Let p be a simple corner for a map $f \in \operatorname{End}(M, p)$, and denote by $\tilde{f} \in \operatorname{End}(\tilde{M}, S)$ the blowup of f at p. Then,*

(i) *p is never two-dicritical;*
(ii) *the singular directions of f are always simple corners of $\tilde{f}$;*
(iii) *if $q \in S$ is nonsingular for $\tilde{f}$, then no infinite orbit of $\tilde{f}$ can stay arbitrarily close to q.*

Proof. We first compute the singular directions of f at p. Choose local coordinates centered at p so that f can be expressed in the form (4.1), and set also

$$g_3(z) = \alpha z_1 + \beta z_2 + \gamma z_3 + g_3^o, \tag{4.2}$$

with $\nu(g_3^o) \geq 2$. Notice that $\alpha = \beta = 0$ and $z_3 | g_3^o$ if $c > 0$.

Now, $v = [v_1 : v_2 : v_3] \in \mathbb{P}^2(\mathbb{C})$ is a singular direction for f if and only if

$$\operatorname{rk} \begin{vmatrix} \lambda_1 v_1 & v_1 \\ \lambda_2 v_2 & v_2 \\ \alpha v_1 + \beta v_2 + \gamma v_3 & v_3 \end{vmatrix} \leq 1, \tag{4.3}$$

that is, if and only if

$$\begin{gathered} (\lambda_1 - \lambda_2) v_1 v_2 = 0, \\ (\alpha v_1 + \beta v_2 + (\gamma - \lambda_1) v_3) v_1 = 0, \\ (\alpha v_1 + \beta v_2 + (\gamma - \lambda_2) v_3) v_2 = 0. \end{gathered} \tag{4.4}$$

Since $\lambda_1 - \lambda_2 \neq 0$ by assumption, we see that the singular directions of f are

$$\begin{aligned} &[0:0:1], [0:\lambda_2 - \gamma : \beta], [1:0:v_3], \quad \text{for any } v_3 \in \mathbb{C}, \text{ if } \lambda_1 = \gamma \neq \lambda_2,\ \alpha = 0; \\ &[0:0:1], [\lambda_1 - \gamma : 0 : \alpha], [0:1:v_3], \quad \text{for any } v_3 \in \mathbb{C}, \text{ if } \lambda_2 = \gamma \neq \lambda_1,\ \beta = 0; \\ &[0:0:1], [0:\lambda_2 - \gamma : \beta], [\lambda_1 - \gamma : 0 : \alpha], \quad \text{otherwise.} \end{aligned} \tag{4.5}$$

In particular, p is never two-dicritical and (i) is proved. By Lemma 2.2, then, the singular points of $\tilde{f}$ belonging to the exceptional divisor are exactly the singular directions of f.

To prove (ii) and (iii), we study $\tilde{f}$. In the canonical coordinates centered in $[1:0:0]$, we have

$$\tilde{f}_j(w) = \begin{cases} w_1 + (w_1^{a+b+c} w_2^b w_3^c) w_1 (\lambda_1 + O(w_1)), & \text{for } j = 1, \\ w_2 + (w_1^{a+b+c} w_2^b w_3^c) w_2 (\lambda_2 - \lambda_1 + O(w_1)), & \text{for } j = 2, \\ w_3 + (w_1^{a+b+c} w_2^b w_3^c) (\alpha + \beta w_2 + (\gamma - \lambda_1) w_3 + O(w_1)), & \text{for } j = 3. \end{cases} \tag{4.6}$$

Furthermore, if $c>0$, then $\alpha=\beta=0$ and the remainder term for $j=3$ is $O(w_1 w_3)$. The exceptional divisor in this chart has equation $w_1 = 0$, and the singular points of $\tilde{f}$ contained in this chart have coordinates $(0,0,\alpha/(\lambda_1-\gamma))$ if $\lambda_1 \neq \gamma$, or $(0,0,w_3)$ if $\lambda_1 = \gamma$ and $\alpha = 0$.

Let $q = (0, q_2, q_3)$ be a point in the exceptional divisor. Then, in the coordinates centered at q obtained by translation, we get

$$\tilde{f}_j(w) = \begin{cases} w_1 + \left(w_1^{a+b+c} (w_2+q_2)^b (w_3+q_3)^c\right) w_1 (\lambda_1 + O(w_1)), & \text{if } j = 1, \\ w_2 + \left(w_1^{a+b+c} (w_2+q_2)^b (w_3+q_3)^c\right) (w_2+q_2)(\lambda_2 - \lambda_1 + O(w_1)), & \text{if } j = 2, \\ w_3 + \left(w_1^{a+b+c} (w_2+q_2)^b (w_3+q_3)^c\right) & \\ \qquad \times (\alpha + \beta q_2 + (\gamma-\lambda_1) q_3 + \beta w_2 + (\gamma - \lambda_1) w_3 + O(w_1)), & \text{if } j = 3. \end{cases} \tag{4.7}$$

Furthermore, if $c > 0$ then $\alpha = \beta = 0$, and moreover, if $q_3 = 0$, then the remainder term for $j = 3$ is $O(w_1 w_3)$.

If $q_2 \neq 0$, we see that $\tilde{f}$ satisfies the hypotheses of Proposition 3.1; therefore, we get (iii) for q. If $q_2 = 0$ and $\alpha + (\gamma - \lambda_1) q_3 \neq 0$, then we can again apply Proposition 3.1; so, we have proven (iii) for all nonsingular q in this chart. Finally, if q is singular, then $\tilde{f}$ is in the form (4.1) because $\lambda_2/\lambda_1 \notin \mathbb{Q}^+$ implies $(\lambda_2 - \lambda_1)/\lambda_1 \notin \mathbb{Q}^+$, and so, we have proved (ii) in this chart.

In the canonical coordinates centered in $[0:1:0]$, we have

$$\tilde{f}_j(w) = \begin{cases} w_1 + (w_1^a w_2^{a+b+c} w_3^c) w_1 (\lambda_1 - \lambda_2 + O(w_2)), & \text{for } j = 1, \\ w_2 + (w_1^a w_2^{a+b+c} w_3^c) w_2 (\lambda_2 + O(w_2)), & \text{for } j = 2, \\ w_3 + (w_1^a w_2^{a+b+c} w_3^c) (\beta + \alpha w_1 + (\gamma - \lambda_2) w_3 + O(w_2)), & \text{for } j = 3, \end{cases} \tag{4.8}$$

where if $c > 0$, then $\alpha = \beta = 0$ and the remainder term for $j = 3$ is $O(w_2 w_3)$. Thus, arguing as before, we get (ii) and (iii) in this chart too.

To end the proof, we must show that $[0:0:1]$ is a simple corner for $\tilde{f}$. Expressing $\tilde{f}$ in the coordinate chart centered in $[0:0:1]$, we get

$$\tilde{f}_j(w) = \begin{cases} w_1 + (w_1^a w_2^b w_3^{a+b+c}) w_1 (\lambda_1 - \gamma - \alpha w_1 - \beta w_2 + O(w_3)), & \text{for } j = 1, \\ w_2 + (w_1^a w_2^b w_3^{a+b+c}) w_2 (\lambda_2 - \gamma - \alpha w_1 - \beta w_2 + O(w_3)), & \text{for } j = 2, \\ w_3 + (w_1^a w_2^b w_3^{a+b+c}) w_3 (\gamma + \alpha w_1 + \beta w_2 + O(w_3)), & \text{for } j = 3; \end{cases} \tag{4.9}$$

thus, up to renumbering the coordinates, it suffices to prove that at least one of the quotients $(\lambda_1 - \gamma)/(\lambda_2 - \gamma)$, $(\lambda_1 - \gamma)/\gamma$, and $(\lambda_2 - \gamma)/\gamma$ does not belong to $\mathbb{Q}^+$. If $\gamma = 0$, there is nothing to prove. If $\gamma \neq 0$, then $(\lambda_1 - \gamma)/\gamma$ and $(\lambda_2 - \gamma)/\gamma \in \mathbb{Q}^+$ would imply $\lambda_2/\lambda_1 \in \mathbb{Q}^+$, a contradiction, and we are done. □

We are finally ready to prove the main result of this paper.

THEOREM 4.2. *Let $f = (f_1, f_2, f_3) \in \operatorname{End}(\mathbb{C}^3, O)$ be of the form*

$$f_j(z) = \begin{cases} z_1 + z_1^2 - 9z_1z_2 - 14z_1z_3 + 6z_2z_3 + a_1z_1^3 + a_2z_2^3 + a_3z_3^3 + O(\|z\|^4), & \text{for } j = 1, \\ z_2 - z_1z_2 + 2z_1z_3 - 3z_2^2 - 10z_2z_3 + b_1z_1^3 + b_2z_2^3 + b_3z_3^3 + O(\|z\|^4), & \text{for } j = 2, \\ z_3 - 3z_1z_2 + 4z_1z_3 - 8z_3^2 + c_1z_1^3 + c_2z_2^3 + c_3z_3^3 + O(\|z\|^4), & \text{for } j = 3, \end{cases} \tag{4.10}$$

with $b_1 \neq c_1$, $a_2 \neq c_2$, and $a_3 \neq c_3$. Then, f is tangent to the identity and with the origin as isolated fixed point, but it has no robust parabolic curves at the origin. Nevertheless, it admits parabolic curves at the origin.

Proof. Our first aim is to compute the characteristic directions of f. This amounts to solving the system

$$\begin{aligned} 2x^2y - 6xy^2 - 2x^2z - 4xyz + 6y^2z &= 0, \\ 3x^2y - 3x^2z - 9xyz - 6xz^2 + 6yz^2 &= 0, \\ 3xy^2 - 5xyz - 3y^2z + 2xz^2 - 2yz^2 &= 0. \end{aligned} \tag{4.11}$$

It is easy to see that the first polynomial is irreducible; therefore, (4.11) cannot have infinitely many solutions, and thus, Lemma 2.1 implies that f has 7 characteristic directions, counted according to their multiplicity. Clearly, $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$ are characteristic directions; since the first two have multiplicity 2 and the third one has multiplicity 3, we have found all of them.

Now, we can prove that O is an isolated fixed point of f. If this is not the case, the fixed-points set of the blowup $\tilde{f}$ of f at the origin must contain a component intersecting the exceptional divisor, and it is not difficult to see that the intersection must be a characteristic direction of f. So, it suffices to prove that the only component of the fixed-points set of $\tilde{f}$ containing a characteristic direction is the exceptional divisor.

In the canonical chart containing $[1:0:0]$, the map $\tilde{f}$ is given by

$$\tilde{f}_j(w) = \begin{cases} w_1 + w_1^2\big(1 + a_1 w_1 - 9w_2 - 14w_3 \\ \qquad\qquad +6w_2w_3 + O\big(w_1^2, w_1w_2^3, w_1w_3^3\big)\big), & \text{for } j = 1, \\ w_2 + w_1\big(b_1w_1 - 2w_2 + 2w_3 + 6w_2^2 \\ \qquad\qquad +4w_2w_3 - 6w_2^2w_3 + O\big(w_1^2, w_1w_2, w_1w_3\big)\big), & \text{for } j = 2, \\ w_3 + w_1\big(c_1w_1 - 3w_2 + 3w_3 + 6w_3^2 \\ \qquad\qquad +9w_2w_3 - 6w_2w_3^2 + O\big(w_1^2, w_1w_2, w_1w_3\big)\big), & \text{for } j = 3. \end{cases} \tag{4.12}$$

We write

$$\tilde{f}_j(w) - w_j = \begin{cases} w_1^2\big(1 + h_1\big), & \text{for } j = 1, \\ w_1\big(b_1w_1 - 2w_2 + 2w_3 + h_2\big), & \text{for } j = 2, \\ w_1\big(c_1w_1 - 3w_2 + 3w_2 + h_3\big), & \text{for } j = 3; \end{cases} \tag{4.13}$$

if we show

$$I\big(w_1\big(1 + h_1\big), b_1w_1 - 2w_2 + 2w_3 + h_2, c_1w_1 - 3w_2 + 3w_2 + h_3; O\big) < +\infty \tag{4.14}$$

as a consequence, we get that the only component of the fixed-points set of $\tilde{f}$ containing $[1:0:0]$ is the exceptional divisor. But, indeed,

$$\begin{aligned} I\big(w_1&\big(1 + h_1\big), b_1w_1 - 2w_2 + 2w_3 + h_2, c_1w_1 - 3w_2 + 3w_2 + h_3; O\big) \\ &= I\big(w_1, b_1w_1 - 2w_2 + 2w_3 + h_2, c_1w_1 - 3w_2 + 3w_2 + h_3; O\big) \\ &= I\big(-2w_2 + 2w_3 + 6w_2^2 + 4w_2w_3 - 6w_2^2w_3, -3w_2 + 3w_3 \\ &\qquad + 6w_3^2 + 9w_2w_3 - 6w_2w_3^2; O\big) \\ &= 3. \end{aligned} \tag{4.15}$$

Similar computations work at $[0:1:0]$ and $[0:0:1]$, and thus, we have proved that O is an isolated fixed point for f. In particular, characteristic directions and singular directions agree.

Now, using (4.12), it is not difficult to see that every point of the exceptional divisor in the canonical chart containing $[1:0:0]$, but $[1:0:0]$ itself, satisfies the assumptions of Proposition 3.1. Furthermore, the singular directions of $\tilde{f}$ at $[1:0:0]$ are $[0:1:1]$ and $[0:2:3]$ (here, we consider that $b_1 \neq c_1$). If $\hat{f}$ is the blowup of $\tilde{f}$ at $[1:0:0]$, the expression of $\hat{f}$ in the canonical chart centered in

$[0:1:0]$ (containing both the singular directions of $\tilde{f}$ at $[1:0:0]$) is

$$\hat{f}_j(x) = \begin{cases} x_1 + (x_1x_2)x_1\big(3 - b_1x_1 - 15x_2 - 2x_3 + O(x_1x_2, x_1x_3, x_2x_3, x_2^2)\big), & \text{for } j = 1,\\ x_2 + (x_1x_2)x_2\big(-2 + b_1x_1 + 6x_2 + 2x_3 + O(x_1x_2, x_2x_3, x_2^2)\big), & \text{for } j = 2,\\ x_3 + (x_1x_2)\big(-3 + c_1x_1 + 5x_3 - 2x_3^2 + O(x_1x_2, x_1x_3, x_2x_3, x_2^2)\big), & \text{for } j = 3. \end{cases} \tag{4.16}$$

Again, it is not difficult to check that every point in the exceptional divisor, but the two singular points, satisfies the assumptions of Proposition 3.1. If we center the coordinates in $[0:1:1]$ via a translation, we get

$$\hat{f}_j(x) = \begin{cases} x_1 + (x_1x_2)x_1(1 + g_1), & \text{for } j = 1,\\ x_2 + (x_1x_2)x_2(0 + g_2), & \text{for } j = 2,\\ x_3 + (x_1x_2)g_3, & \text{for } j = 3, \end{cases} \tag{4.17}$$

with $\nu(g_j) \geq 1$ for $j = 1, 2, 3$. Analogously, if we center the coordinates in $[0:2:3]$, we get

$$\hat{f}_j(x) = \begin{cases} x_1 + (x_1x_2)x_1(0 + g_1'), & \text{for } j = 1,\\ x_2 + (x_1x_2)x_2(1 + g_2'), & \text{for } j = 2,\\ x_3 + (x_1x_2)g_3', & \text{for } j = 3, \end{cases} \tag{4.18}$$

again with $\nu(g_j') \geq 1$ for $j = 1, 2, 3$. In other words, both singular points of $\hat{f}$ are simple corners.

We leave to the reader the corresponding computations in the other charts. In all cases, we find that, after the second blowup, the only singular points are simple corners, and all other points in the exceptional divisor satisfy the assumptions of Proposition 3.1. By Proposition 4.1, this holds true blowing up any finite number of singular points.

Now, let us assume, by contradiction, that f admits a robust parabolic curve φ at the origin. Then, φ^1 is a robust parabolic curve for $\tilde{f}$ at some point of the exceptional divisor; by Proposition 3.1, this point must be a singular point for $\tilde{f}$. If φ^1 is not tangent to the exceptional divisor, φ^2 must be a robust parabolic curve for $\hat{f}$ at a point q which is a smooth point of the total transform of the exceptional divisor at level 1; but since q must be a singular point of $\hat{f}$, by Proposition 3.1, and we saw that all singular points of $\hat{f}$ are simple corners, we get a contradiction.

So, φ^1 is tangent to the exceptional divisor of $\tilde{f}$. But since φ^1 is given by a power series, after a finite number of blowups, we get a φ^k which is not anymore tangent to the corresponding exceptional divisor. In particular, then, φ^{k+1} is a robust parabolic curve at a point q of the exceptional divisor which is not a corner.

But Propositions 4.1 and 3.1 (and the previous computations) imply that q must be a singular point and that the only singular points are corners; therefore, we again have a contradiction.

So, f has no robust parabolic curves at the origin. On the other hand, it is easy to check that $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$ are nondegenerate characteristic directions; therefore, Theorem 2.4(ii) yields a parabolic curve at the origin for each of these directions. □

Remark 4.3. The eigenvalues of the 2×2-matrices associated by Hakim [12, 13] to the characteristic directions of the map f are $\{0,1\}$. Therefore, the parabolic curves whose existence is predicted by Theorem 2.4(ii) are described by power series in z and $z\log z$, and thus they cannot be robust.

Remark 4.4. We chose to define the map f in Theorem 4.2 with actual numbers for the sake of definiteness; however, it is possible to prove similar results for a larger family of maps. Looking carefully at the computations in the proof, it turns out that we actually used only a couple of properties of f, that it had three singular directions with multiplicities respectively 2, 2, and 3, and that each of those had, in turn, only two singular directions, both giving rise to simple corners. Furthermore, the last property is obtained if the linear part of $\tilde{f}-\mathrm{id}$ at each singular point is nondiagonalizable with exactly one nonzero double eigenvalue. For more details, see [10] where similar computations are carried out in the continuous case.

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