DOMAINS WHICH ARE LOCALLY UNIFORMLY LINEARLY CONVEX IN THE KOBAYASHI DISTANCE

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We show a construction of domains in complex reflexive Banach spaces which are locally uniformly convex in linear sense in their Kobayashi distance. We also show connections between norm and Kobayashi distance properties.

1. Introduction

Recently, in [1], it has been proved that if B is an open unit ball in a Cartesian product $l^2 \times l^2$ furnished with the l^p-norm $\|\cdot\|$ and k_B is the Kobayashi distance on B, then the metric space (B, k_B) is locally uniformly convex in linear sense. Our construction of domains, which are locally uniformly convex in their Kobayashi distances, is based on the ideas from [1]. Such domains play an important role in the fixed-point theory of holomorphic mappings (see [1, 2, 4, 13, 14]).

In Section 4, we show connections between norm and Kobayashi distance properties.

2. Preliminaries

Throughout this paper, all Banach spaces X will be complex and reflexive, all domains $D \subset X$ bounded and convex, and k_D will denote the Kobayashi distance on D [6, 7, 9, 10, 11, 12].

We will use the notions and notations from [2]. Here, we recall a few facts only.

The Kobayashi distance k_D is locally equivalent to the norm $\|\cdot\|$ [9]. Indeed, if $\operatorname{dist}_{\|\cdot\|}(x, \partial D)$ denotes the distance in $(X, \|\cdot\|)$ between the point x and the boundary ∂D of the domain D, and $\operatorname{diam}_{\|\cdot\|} D$ is the diameter of D in $(X, \|\cdot\|)$,

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then

$$\operatorname{arg\,tanh}\left(\frac{\|x-y\|}{\operatorname{diam}_{\|\cdot\|} D}\right) \leq k_D(x,y) \tag{2.1}$$

for all $x, y \in D$ and

$$k_D(x,y) \leq \operatorname{arg\,tanh}\left(\frac{\|x-y\|}{\operatorname{dist}_{\|\cdot\|}(x,\partial D)}\right) \tag{2.2}$$

whenever $\|x-y\| < \operatorname{dist}_{\|\cdot\|}(x,\partial D)$.

A subset C of D is said to lie strictly inside D if $\operatorname{dist}_{\|\cdot\|}(C,\partial D) > 0$. We can observe that a subset C of D is k_D-bounded if and only if C lies strictly inside D [9, Proposition 23].

Each open (closed) k_D-ball in the metric space (D,k_D) is convex [15] and if D is strictly convex, then every k_D-ball is also strictly convex in a linear sense [3, 18] (see also [17]).

The metric space (D,k_D) is called a locally uniformly linearly convex space [2] if there exist $w \in D$ and the function

$$\delta(w,\cdot,\cdot,\cdot,\cdot,\cdot) \tag{2.3}$$

such that for all $0 < R_1$, $k_D(w,z) \leq R_1$, $0 < R_2 \leq R \leq R_3$, and $0 < \epsilon_1 \leq \epsilon \leq \epsilon_2 < 2$, we have

$$\delta(w,R_1,R_2,R_3,\epsilon_1,\epsilon_2) > 0,$$

$$\left.\begin{array}{l} k_D(z,x) \leq R \\ k_D(z,y) \leq R \\ k_D(x,y) \geq \epsilon R \end{array}\right\} \Longrightarrow k_D\left(z, \frac{1}{2}x + \frac{1}{2}y\right) \leq \left(1 - \delta(w,R_1,R_2,R_3,\epsilon_1,\epsilon_2)\right)R. \tag{2.4}$$

The function $\delta(w,\cdot,\cdot,\cdot,\cdot,\cdot)$ is called a modulus of linear convexity for the Kobayashi distance k_D.

The open unit ball B_H in a Hilbert space is called the Hilbert ball [5, 7, 8, 14, 16].

For more useful properties of the Kobayashi distance, see [14].

3. Examples of locally uniformly linearly convex domains

The first known domain is the Hilbert ball [13, 14]. Other examples are given in [1]. Namely, if B is the open unit ball in a Cartesian product $l^2 \times l^2$ furnished with the l^p-norm, where $1 < p < \infty$ and $p \neq 2$, then the metric space (B,k_B) is also locally uniformly linearly convex.

Before stating our main result, we prove the following auxiliary lemma.

Lemma 3.1. *Let X be a finite-dimensional Banach space and D a bounded, closed, and strictly convex domain in X. Then, the metric space (D,k_D) is locally uniformly linearly convex.*

Proof. Since D is a bounded and strictly convex domain in X, each k_D-ball is strictly convex in a linear sense. Therefore, using the equivalent definition of the k_D-boundedness and the compactness argument, we see that the metric space (D, k_D) is locally uniformly linearly convex. □

Now, we state the main result of this paper.

THEOREM 3.2. *Let Y be a finite-dimensional subspace of a complex reflexive Banach space X and D a bounded strictly convex domain in X. Suppose that*

(i) *there exists a point $x_0 \in D_0 = D \cap Y$,*

(ii) *there exists a holomorphic retraction $r : D \to D_0$,*

(iii) *for every $R > 0$ and for any three points x, y, and z in the closed k_D-ball $\overline{B}(x_0, R)$, there exists a biholomorphic affine mapping $T : D \to D$ such that $T(x_0) = x_0$ and $T(x), T(y), T(z) \in Y \cap D_0$.*

Then, the metric space (D, k_D) is locally uniformly linearly convex.

Proof. First, observe that D_0 is a strictly convex domain in Y and by (ii),

$$k_{D_0}(u, w) = k_D(u, w) \tag{3.1}$$

for all $u, w \in D_0$. This (combined with assumption (i)) implies that the closed k_{D_0}-ball $\overline{B_0}(x_0, R)$ is equal to $\overline{B}(x_0, R) \cap D_0$.

Let x, y, and z be three arbitrarily chosen points in the closed k_D-ball $\overline{B}(x_0, R)$. By assumption (iii), there exists a biholomorphic affine mapping $T : D \to D$ such that $Tx, Ty, Tz \in Y \cap D_0$ and $Tx_0 = x_0$. Since this biholomorphic mapping is always a k_D-isometry [6, 7, 9, 10, 14], we get

$$\begin{aligned} Tx, Ty, Tz &\in \overline{B}(x_0, R) \cap D_0, \\ k_D(x, y) &= k_{D_0}(Tx, Ty), \\ k_D(x, z) &= k_{D_0}(Tx, Tz), \\ k_D(y, z) &= k_{D_0}(Ty, Tz). \end{aligned} \tag{3.2}$$

Therefore, we may restrict our further considerations to the finite-dimensional Banach space Y. By Lemma 3.1, the metric space (D_0, k_{D_0}) is locally uniformly linearly convex and this implies the same property of (D, k_D). □

Example 3.3. If B is the open unit ball in a Cartesian product $X = \mathbb{C}^n \times l^2$, furnished with the l^p-norm, where $1 < p < \infty$, and in $\mathbb{C}^n$ we have a strictly convex norm (i.e., the open unit ball in this norm is strictly convex), then the metric space (B, k_B) is locally uniformly linearly convex.

Indeed, let $\{e_1, e_2, \dots\}$ be the standard basis in the Hilbert space l^2. For any three points $x = (x_1, x_2)$, $y = (y_1, y_2)$, and $z = (z_1, z_2)$ in $B \subset \mathbb{C}^n \times l^2$, there exists a linear isometry $T_1 : l^2 \to l^2$ such that

$$\tilde{T}x_2, \tilde{T}y_2, \tilde{T}z_2 \in \operatorname{lin}\{e_1, e_2, e_3\}. \tag{3.3}$$

Put

$$Y = \mathbb{C}^n \times \operatorname{lin}\{e_1, e_2, e_3\},$$
$$B_1 = Y \cap B, \tag{3.4}$$
$$T(w_1, w_2) = (w_1, \tilde{T}w_2)$$

for $(w_1, w_2) \in B \subset \mathbb{C}^n \times l^2$. It is obvious that B_1 is the open unit ball in Y and

$$k_B(u, w) = k_{B_1}(u, w) \tag{3.5}$$

for all $u, w \in B_1$. Therefore, we can apply Theorem 3.2.

Example 3.4. In the Cartesian product $X = l^2 \times l^2 \times l^2$, we have the following norm:

$$||(x_1, x_2, x_3)|| = \left[||x_1||^p + \left(||x_2||^q + ||x_3||^q \right)^{p/q} \right]^{1/p}, \tag{3.6}$$

where $1 < p, q < \infty$, $p, q \neq 2$, $p \neq q$, and $(x_1, x_2, x_3) \in X$. Let B be the open unit ball in X. The metric space (B, k_B) is locally uniformly linearly convex. The proof of this fact is similar to that given in Example 3.3.

Example 3.5. Let X be the Hilbert space l^2 with the standard orthonormal basis $\{e_1, e_2, \dots\}$. Let D_0 be an arbitrary bounded strictly convex domain in $\operatorname{lin}\{e_1\}$. Let ∂D_0 denote the boundary of D_0 in $\operatorname{lin}\{e_1\}$. A strictly convex domain $D \in X$, generated by D_0, is defined as follows:

$$D = \left\{ z + w : z \in D_0,\ w \in \operatorname{lin}\{e_2, e_3, \dots\},\ \|w\| < \sqrt{\operatorname{dist}(z, \partial D_0)} \right\}. \tag{3.7}$$

It is easy to check that we may apply Theorem 3.2, and therefore the metric space (D, k_D) is locally uniformly linearly convex.

Remark 3.6. A construction of more complicated examples is obvious.

4. Connections between norm and Kobayashi distance properties

There is some connection between the local uniform convexity in linear sense of the unit ball (B, k_B) and the uniform convexity of the whole Banach space. Namely, the following theorem is valid.

Theorem 4.1. *Let $(X, \|\cdot\|)$ be a complex Banach space and B the open unit ball in $(X, \|\cdot\|)$. If (B, k_B) is locally uniformly convex in linear sense, then the Banach space $(X, \|\cdot\|)$ is uniformly convex.*

Proof. It is sufficient to show that the ball $B(0, 1/2)$ in $(X, \|\cdot\|)$ is uniformly convex. Let

$$\begin{aligned} \|x\| = \|y\| &= \frac{1}{2}, \\ \|x - y\| &\geq \frac{1}{2}\epsilon. \end{aligned} \tag{4.1}$$

We know that the norm $\|\cdot\|$ and the Kobayashi distance are locally equivalent and, additionally, we have

$$\begin{aligned} k_B(0, x) = k_B(0, y) &= \operatorname{arg\,tanh}\left(\frac{1}{2}\right) = R, \\ k_B(x, y) \geq \operatorname{arg\,tanh}\left(\frac{\|x - y\|}{2}\right) &\geq \frac{\operatorname{arg\,tanh}((1/4)\epsilon)}{R} R = \eta R. \end{aligned} \tag{4.2}$$

Hence, by the local uniform convexity in linear sense of the unit ball (B, k_B), we get

$$\begin{aligned} k_B\left(0, \frac{1}{2}x + \frac{1}{2}y\right) &\leq (1 - \delta(0, R, R, R, \eta, \eta))R \\ &= (1 - \delta(0, R, R, R, \eta, \eta)) \operatorname{arg\,tanh}\left(\frac{1}{2}\right) \\ &= \operatorname{arg\,tanh}\left((1 - \delta^*)\frac{1}{2}\right) \end{aligned} \tag{4.3}$$

and therefore

$$\left\|\frac{1}{2}x + \frac{1}{2}y\right\| \leq (1 - \delta^*)\frac{1}{2}, \tag{4.4}$$

where

$$\delta^* = 1 - 2\tanh\left((1 - \delta(0, R, R, R, \eta, \eta)) \operatorname{arg\,tanh}\left(\frac{1}{2}\right)\right). \tag{4.5}$$

□

Remark 4.2. There is the following open problem. Does the uniform convexity of the complex Banach space $(X, \|\cdot\|)$ imply the local uniform convexity in linear sense of (B, k_B), where B is the open unit ball in $(X, \|\cdot\|)$?

It is worth recalling here two facts about strict convexity. As we mentioned in Section 2, the strict convexity of the domain D implies that every k_D-ball is also strictly convex in a linear sense [3, 18] (see also [17]). It is natural to ask whether the strict convexity of (D, k_D) implies the strict convexity of D. The answer is, no, as the following example shows.

Example 4.3 (see [4]). Consider the domain

$$D = \Delta \cap \left\{ z \in \mathbb{C} : \operatorname{Re} z < \frac{1}{\sqrt{2}} \right\} \tag{4.6}$$

in the complex plane $\mathbb{C}$. Then, every k_D-ball is strictly convex in a linear sense but D is not a strictly convex set.

On the other hand, in the case of the open unit ball, we have the positive answer to the above question.

Theorem 4.4. *Let $(X, \|\cdot\|)$ be a complex Banach space and B the open unit ball in $(X, \|\cdot\|)$. The Banach space $(X, \|\cdot\|)$ is strictly convex if and only if (B, k_B) is strictly convex in linear sense.*

Proof. We know that the strict convexity of the ball B implies that every k_B-ball is also strictly convex in a linear sense [3, 18] (see also [17]). Now, if each k_B-ball is strictly convex in a linear sense, then we can repeat the method of the proof of Theorem 4.1 to get the strict convexity of the Banach space $(X, \|\cdot\|)$. □

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