STRONG CONVERGENCE OF AN ITERATIVE SEQUENCE FOR MAXIMAL MONOTONE OPERATORS IN A BANACH SPACE

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We first introduce a modified proximal point algorithm for maximal monotone operators in a Banach space. Next, we obtain a strong convergence theorem for resolvents of maximal monotone operators in a Banach space which generalizes the previous result by Kamimura and Takahashi in a Hilbert space. Using this result, we deal with the convex minimization problem and the variational inequality problem in a Banach space.

1. Introduction

Let E be a real Banach space and let $T \subset E \times E^*$ be a maximal monotone operator. Then we study the problem of finding a point $v \in E$ satisfying

$$0 \in Tv. \tag{1.1}$$

Such a problem is connected with the *convex minimization problem*. In fact, if $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, then Rockafellar's theorem [14, 15] ensures that the subdifferential mapping $\partial f \subset E \times E^*$ of f is a maximal monotone operator. In this case, the equation $0 \in \partial f(v)$ is equivalent to $f(v) = \min_{x \in E} f(x)$.

In 1976, Rockafellar [17] proved the following weak convergence theorem.

Theorem 1.1 (Rockafellar [17]). *Let H be a Hilbert space and let $T \subset H \times H$ be a maximal monotone operator. Let I be the identity mapping and let $J_r = (I + rT)^{-1}$ for all $r > 0$. Define a sequence $\{x_n\}$ as follows: $x_1 = x \in H$ and*

$$x_{n+1} = J_{r_n} x_n \quad (n = 1, 2, \dots), \tag{1.2}$$

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$. If $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element of $T^{-1}0$.

This is called the *proximal point algorithm*, which was first introduced by Martinet [11]. If $T = \partial f$, where $f : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function,

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then (1.2) is reduced to

$$x_{n+1} = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2r_n} \|x_n - y\|^2 \right\} \quad (n = 1, 2, \ldots). \tag{1.3}$$

Later, many researchers studied the convergence of the proximal point algorithm in a Hilbert space; see Brézis and Lions [4], Lions [10], Passty [12], Güler [7], Solodov and Svaiter [19] and the references mentioned there. In particular, Kamimura and Takahashi [8] proved the following strong convergence theorem.

Theorem 1.2 (Kamimura and Takahashi [8]). *Let H be a Hilbert space and let $T \subset H \times H$ be a maximal monotone operator. Let $J_r = (I + rT)^{-1}$ for all $r > 0$ and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in H$ and*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n \quad (n = 1, 2, \ldots), \tag{1.4}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n\to\infty} r_n = \infty$. If $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to $P_{T^{-1}0}(x)$, where $P_{T^{-1}0}$ is the metric projection from H onto $T^{-1}0$.

Recently, using the hybrid method in mathematical programming, Kamimura and Takahashi [9] obtained a strong convergence theorem for maximal monotone operators in a Banach space, which extended the result of Solodov and Svaiter [19] in a Hilbert space. On the other hand, Censor and Reich [6] introduced a convex combination which is based on Bregman distance and studied some iterative schemes for finding a common asymptotic fixed point of a family of operators in finite-dimensional spaces.

In this paper, motivated by Censor and Reich [6], we introduce the following iterative sequence for a maximal monotone operator $T \subset E \times E^*$ in a smooth and uniformly convex Banach space: $x_1 = x \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n Jx + (1 - \alpha_n) JJ_{r_n} x_n) \quad (n = 1, 2, \ldots), \tag{1.5}$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0, \infty)$, J is the duality mapping from E into E^*, and $J_r = (J + rT)^{-1} J$ for all $r > 0$. Then we extend Kamimura-Takahashi's theorem to the Banach space (Theorem 3.3). It should be noted that we do not assume the weak sequential continuity of the duality mapping [1, 5, 13]. Finally, we apply Theorem 3.3 to the convex minimization problem and the variational inequality problem.

2. Preliminaries

Let E be a (real) Banach space with norm $\|\cdot\|$ and let E^* denote the Banach space of all continuous linear functionals on E. For all $x \in E$ and $x^* \in E^*$, we denote $x^*(x)$ by $\langle x, x^* \rangle$. We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of all real numbers and the set of all positive integers, respectively. The *duality mapping* J from E into E^* is defined by

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\} \tag{2.1}$$

for all $x \in E$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. We sometimes identify a set-valued mapping $A : E \to 2^{E^*}$ with its graph $G(A) = \{(x, x^*) : x^* \in Ax\}$. An operator $T \subset E \times E^*$ with domain $D(T) = \{x \in E : Tx \neq \emptyset\}$ and range $R(T) = \bigcup\{Tx : x \in D(T)\}$ is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in T$. We denote the set $\{x \in E : 0 \in Tx\}$ by $T^{-1}0$. A monotone operator $T \subset E \times E^*$ is said to be *maximal* if its graph is not properly contained in the graph of any other monotone operator. If $T \subset E \times E^*$ is maximal monotone, then the solution set $T^{-1}0$ is closed and convex. A proper function $f : E \to (-\infty, \infty]$ (which means that f is not identically ∞) is said to be *convex* if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y) \tag{2.2}$$

for all $x, y \in E$ and $\alpha \in (0, 1)$. The function f is also said to be *lower semicontinuous* if the set $\{x \in E : f(x) \leq r\}$ is closed in E for all $r \in \mathbb{R}$. For a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$, the *subdifferential* ∂f of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y)\ \forall y \in E\} \tag{2.3}$$

for all $x \in E$. It is easy to verify that $0 \in \partial f(v)$ if and only if $f(v) = \min_{x \in E} f(x)$. It is known that the subdifferential of the function f defined by $f(x) = \|x\|^2/2$ for all $x \in E$ is the duality mapping J. The following theorem is also well known (see Takahashi [21] for details).

Theorem 2.1. *Let E be a Banach space, let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function, and let $g : E \to \mathbb{R}$ be a continuous convex function. Then*

$$\partial(f + g)(x) = \partial f(x) + \partial g(x) \tag{2.4}$$

for all $x \in E$.

A Banach space E is said to be *strictly convex* if

$$\|x\| = \|y\| = 1, \quad x \neq y \Longrightarrow \left\| \frac{x + y}{2} \right\| < 1. \tag{2.5}$$

Also, E is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \Longrightarrow \left\| \frac{x + y}{2} \right\| \leq 1 - \delta. \tag{2.6}$$

It is also said to be *smooth* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.7}$$

exists for all $x, y \in \{z \in E : \|z\| = 1\}$. We know the following (see Takahashi [20] for details):

(1) if E is smooth, then J is single-valued;

(2) if E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;

(3) if E is reflexive, then J is surjective;
(4) if E is uniformly convex, then it is reflexive;
(5) if E^* is uniformly convex, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth Banach space. We use the following function studied in Alber [1], Kamimura and Takahashi [9], and Reich [13]:

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.8}$$

for all $x, y \in E$. It is obvious from the definition of ϕ that $(\|x\| - \|y\|)^2 \leq \phi(x,y)$ for all $x, y \in E$. We also know that

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle \tag{2.9}$$

for all $x, y, z \in E$. The following lemma was also proved in [9].

Lemma 2.2 (Kamimura-Takahashi [9]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If* $\lim_{n\to\infty}\phi(x_n, y_n) = 0$, *then* $\lim_{n\to\infty}\|x_n - y_n\| = 0$.

Let E be a strictly convex, smooth, and reflexive Banach space, and let $T \subset E \times E^*$ be a monotone operator. Then T is maximal if and only if $R(J + rT) = E^*$ for all $r > 0$ (see Barbu [2] and Takahashi [21]). If $T \subset E \times E^*$ is a maximal monotone operator, then for each $r > 0$ and $x \in E$, there corresponds a unique element $x_r \in D(T)$ satisfying

$$J(x) \in J(x_r) + rTx_r. \tag{2.10}$$

We define the *resolvent* of T by $J_r x = x_r$. In other words, $J_r = (J + rT)^{-1}J$ for all $r > 0$. The resolvent J_r is a single-valued mapping from E into $D(T)$. If E is a Hilbert space, then J_r is *nonexpansive*, that is, $\|J_r x - J_r y\| \leq \|x - y\|$ for all $x, y \in E$ (see Takahashi [20]). It is easy to show that $T^{-1}0 = F(J_r)$ for all $r > 0$, where $F(J_r)$ denotes the set of all fixed points of J_r. We can also define, for each $r > 0$, the *Yosida approximation* of T by $A_r = (J - JJ_r)/r$. We know that $(J_r x, A_r x) \in T$ for all $r > 0$ and $x \in E$. Let C be a nonempty closed convex subset of the space E. By Alber [1] or Kamimura and Takahashi [9], for each $x \in E$, there corresponds a unique element $x_0 \in C$ (denoted by $P_C(x)$) such that

$$\phi(x_0, x) = \min_{y\in C}\phi(y,x). \tag{2.11}$$

The mapping P_C is called the *generalized projection* from E onto C. If E is a Hilbert space, then P_C is coincident with the metric projection from E onto C. We also know the following lemma.

Lemma 2.3 ([1], see also [9]). *Let E be a smooth Banach space, let C be a nonempty closed convex subset of E, and let $x \in E$ and $x_0 \in C$. Then the following are equivalent:*

(1) $\phi(x_0, x) = \min_{y\in C}\phi(y,x)$;
(2) $\langle y - x_0, Jx - Jx_0\rangle \leq 0$ *for all* $y \in C$.

3. Strong convergence theorem

The resolvents of maximal monotone operators have the following property, which was proved in the case of the resolvents of normality operators in Kamimura and Takahashi [9].

Lemma 3.1. *Let E be a strictly convex, smooth, and reflexive Banach space, let $T \subset E \times E^*$ be a maximal monotone operator with $T^{-1}0 \neq \emptyset$, and let $J_r = (J + rT)^{-1}J$ for each $r > 0$. Then*

$$\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x) \tag{3.1}$$

for all $r > 0$, $u \in T^{-1}0$, and $x \in E$.

Proof. Let $r > 0$, $u \in T^{-1}0$, and $x \in E$ be given. By the monotonicity of T, we have

$$\begin{aligned} \phi(u,x) &= \phi(u, J_r x) + \phi(J_r x, x) + 2\langle u - J_r x, JJ_r x - Jx\rangle \\ &= \phi(u, J_r x) + \phi(J_r x, x) + 2r\langle u - J_r x, -A_r x\rangle \\ &\geq \phi(u, J_r x) + \phi(J_r x, x). \end{aligned} \tag{3.2}$$

□

Let E be a strictly convex, smooth, and reflexive Banach space, and let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, and surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{3.3}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous and convex function from E^* into $\mathbb{R}$. We can prove the following lemma.

Lemma 3.2. *Let E be a strictly convex, smooth, and reflexive Banach space, and let V be as in (3.3). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq V(x, x^* + y^*) \tag{3.4}$$

for all $x \in E$ and $x^*, y^* \in E^*$.

Proof. Let $x \in E$ be given. Define $g(x^*) = V(x, x^*)$ and $f(x^*) = \|x^*\|^2$ for all $x^* \in E^*$. Since J^{-1} is the duality mapping from E^* into E, we have

$$\partial g(x^*) = \partial(-2\langle x, \cdot\rangle + f)(x^*) = -2x + 2J^{-1}(x^*) \tag{3.5}$$

for all $x^* \in E^*$. Hence, we have

$$g(x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \leq g(x^* + y^*), \tag{3.6}$$

that is,

$$V(x,x^*)+2\langle J^{-1}(x^*)-x,y^*\rangle \le V(x,x^*+y^*) \tag{3.7}$$

for all $x^*, y^* \in E^*$. □

Now we can prove the following strong convergence theorem, which is a generalization of Kamimura-Takahashi's theorem (Theorem 1.2).

THEOREM 3.3. *Let E be a smooth and uniformly convex Banach space and let $T \subset E\times E^*$ be a maximal monotone operator. Let $J_r = (J+rT)^{-1}J$ for all $r>0$ and let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$x_{n+1} = J^{-1}(\alpha_n Jx + (1-\alpha_n)JJ_{r_n}x_n) \quad (n = 1,2,\ldots), \tag{3.8}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\lim_{n\to\infty} r_n = \infty$. If $T^{-1}0 \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to $P_{T^{-1}0}(x)$, where $P_{T^{-1}0}$ is the generalized projection from E onto $T^{-1}0$.

Proof. Put $y_n = J_{r_n}x_n$ for all $n \in \mathbb{N}$. We denote the mapping $P_{T^{-1}0}$ by P. We first prove that $\{x_n\}$ is bounded. From Lemma 3.1, we have

$$\begin{aligned}
\phi(Px,x_{n+1}) &= \phi(Px,J^{-1}(\alpha_n Jx+(1-\alpha_n)Jy_n))\\
&= V(Px,\alpha_n Jx+(1-\alpha_n)Jy_n)\\
&\le \alpha_n V(Px,Jx)+(1-\alpha_n)V(Px,Jy_n)\\
&= \alpha_n\phi(Px,x)+(1-\alpha_n)\phi(Px,J_{r_n}x_n)\\
&\le \alpha_n\phi(Px,x)+(1-\alpha_n)\phi(Px,x_n)
\end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$. Hence, by induction, we have $\phi(Px,x_n) \le \phi(Px,x)$ for all $n \in \mathbb{N}$. Since $(\|u\|-\|v\|)^2 \le \phi(u,v)$ for all $u,v \in E$, the sequence $\{x_n\}$ is bounded. Since $\phi(Px,y_n) = \phi(Px,J_{r_n}x_n) \le \phi(Px,x_n)$ for all $n \in \mathbb{N}$, $\{y_n\}$ is also bounded. We next prove

$$\limsup_{n\to\infty}\langle x_n - Px, Jx - JPx\rangle \le 0. \tag{3.10}$$

Put $z_n = x_{n+1}$ for all $n \in \mathbb{N}$. Since $\{z_n\}$ is bounded, we have a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\lim_{i\to\infty}\langle z_{n_i} - Px, Jx - JPx\rangle = \limsup_{n\to\infty}\langle z_n - Px, Jx - JPx\rangle \tag{3.11}$$

and $\{z_{n_i}\}$ converges weakly to some $v \in E$. From the definition of $\{x_n\}$, we have

$$Jz_n - Jy_n = \alpha_n(Jx - Jy_n) \tag{3.12}$$

for all $n \in \mathbb{N}$. Since $\{y_n\}$ is bounded and $\alpha_n \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty}\|Jz_n - Jy_n\| = \lim_{n\to\infty}\alpha_n\|Jx - Jy_n\| = 0. \tag{3.13}$$

Since E is uniformly convex, E^* is uniformly smooth, and hence the duality mapping J^{-1} from E^* into E is uniformly norm-to-norm continuous on each bounded subset of E^*. Therefore, we obtain from (3.13) that

$$\lim_{n\to\infty} \|z_n - y_n\| = \lim_{n\to\infty} \|J^{-1}(Jz_n) - J^{-1}(Jy_n)\| = 0. \tag{3.14}$$

This implies that $y_{n_i} \rightharpoonup v$ as $i \to \infty$, where $\rightharpoonup$ implies the weak convergence. On the other hand, from $r_n \to \infty$ as $n \to \infty$, we have

$$\lim_{n\to\infty} \|A_{r_n} x_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Jy_n\| = 0. \tag{3.15}$$

If $(z, z^*) \in T$, then it holds from the monotonicity of T that

$$\langle z - y_{n_i}, z^* - A_{r_{n_i}} x_{n_i} \rangle \geq 0 \tag{3.16}$$

for all $i \in \mathbb{N}$. Letting $i \to \infty$, we get $\langle z - v, z^* \rangle \geq 0$. Then, the maximality of T implies $v \in T^{-1}0$. Applying Lemma 2.3, we obtain

$$\limsup_{n\to\infty} \langle z_n - Px, Jx - JPx \rangle = \lim_{i\to\infty} \langle z_{n_i} - Px, Jx - JPx \rangle = \langle v - Px, Jx - JPx \rangle \leq 0. \tag{3.17}$$

Finally, we prove that $x_n \to Px$ as $n \to \infty$. Let $\varepsilon > 0$ be given. From (3.10), we have $m \in \mathbb{N}$ such that

$$\langle x_n - Px, Jx - JPx \rangle \leq \varepsilon \tag{3.18}$$

for all $n \geq m$. If $n \geq m$, then it holds from (3.18) and Lemmas 3.1 and 3.2 that

$$\begin{aligned}
\phi(Px, x_{n+1}) &= V(Px, \alpha_n Jx + (1-\alpha_n) Jy_n) \\
&\leq V(Px, \alpha_n Jx + (1-\alpha_n) Jy_n - \alpha_n (Jx - JPx)) \\
&\quad - 2\langle J^{-1}(\alpha_n Jx + (1-\alpha_n) Jy_n) - Px, -\alpha_n (Jx - JPx) \rangle \\
&= V(Px, (1-\alpha_n) Jy_n + \alpha_n JPx) + 2\langle x_{n+1} - Px, \alpha_n (Jx - JPx) \rangle \\
&\leq (1-\alpha_n) V(Px, Jy_n) + \alpha_n V(Px, JPx) + 2\alpha_n \langle x_{n+1} - Px, Jx - JPx \rangle \\
&\leq (1-\alpha_n)\phi(Px, y_n) + \alpha_n \phi(Px, Px) + 2\alpha_n \varepsilon \\
&= (1-\alpha_n)\phi(Px, J_{r_n} x_n) + 2\alpha_n \varepsilon \\
&\leq (1-\alpha_n)\phi(Px, x_n) + 2\alpha_n \varepsilon \\
&= 2\varepsilon\{1 - (1-\alpha_n)\} + (1-\alpha_n)\phi(Px, x_n).
\end{aligned} \tag{3.19}$$

Therefore, we have

$$\begin{aligned}
&\phi(Px,x_{n+1}) \\
&\quad\le 2\varepsilon\{1-(1-\alpha_n)\}+(1-\alpha_n)[2\varepsilon\{1-(1-\alpha_{n-1})\}+(1-\alpha_{n-1})\phi(Px,x_{n-1})] \\
&\quad= 2\varepsilon\{1-(1-\alpha_n)(1-\alpha_{n-1})\}+(1-\alpha_n)(1-\alpha_{n-1})\phi(Px,x_{n-1}) \\
&\quad\le\cdots\le 2\varepsilon\left\{1-\prod_{i=m}^{n}(1-\alpha_i)\right\}+\prod_{i=m}^{n}(1-\alpha_i)\phi(Px,x_m)
\end{aligned}\tag{3.20}$$

for all $n \ge m$. Since $\sum_{i=1}^{\infty}\alpha_i=\infty$, we have $\prod_{i=m}^{\infty}(1-\alpha_i)=0$ (see Takahashi [21]). Hence, we have

$$\begin{aligned}
\limsup_{n\to\infty}\phi(Px,x_n) &= \limsup_{l\to\infty}\phi(Px,x_{m+l+1}) \\
&\le \limsup_{l\to\infty}\left[2\varepsilon\left\{1-\prod_{i=m}^{m+l}(1-\alpha_i)\right\}+\prod_{i=m}^{m+l}(1-\alpha_i)\phi(Px,x_m)\right]=2\varepsilon.
\end{aligned}\tag{3.21}$$

This implies $\limsup_{n\to\infty}\phi(Px,x_n)\le 0$ and hence we get

$$\lim_{n\to\infty}\phi(Px,x_n)=0.\tag{3.22}$$

Applying Lemma 2.2, we obtain

$$\lim_{n\to\infty}||Px-x_n||=0.\tag{3.23}$$

Therefore, $\{x_n\}$ converges strongly to $P_{T^{-1}0}(x)$. □

4. Applications

In this section, we first study the problem of finding a minimizer of a proper lower semicontinuous convex function in a Banach space.

Theorem 4.1. *Let E be a smooth and uniformly convex Banach space and let $f : E \to (-\infty,\infty]$ be a proper lower semicontinuous convex function such that $(\partial f)^{-1}(0)\ne\varnothing$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$\begin{aligned}
y_n &= \arg\min_{y\in E}\left\{f(y)+\frac{1}{2r_n}\|y\|^2-\frac{1}{r_n}\langle y,Jx_n\rangle\right\}\quad(n=1,2,\ldots), \\
x_{n+1} &= J^{-1}(\alpha_n Jx+(1-\alpha_n)Jy_n)\quad(n=1,2,\ldots),
\end{aligned}\tag{4.1}$$

where $\{\alpha_n\}\subset[0,1]$ and $\{r_n\}\subset(0,\infty)$ satisfy $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}\alpha_n=\infty$, and $\lim_{n\to\infty}r_n=\infty$. Then the sequence $\{x_n\}$ converges strongly to $P_{(\partial f)^{-1}(0)}(x)$.

Proof. By Rockafellar's theorem [14, 15], the subdifferential mapping $\partial f\subset E\times E^*$ is maximal monotone (see also Borwein [3], Simons [18], or Takahashi [21]). Fix $r>0$, $z\in E$, and let J_r be the resolvent of ∂f. Then we have

$$Jz\in J(J_rz)+r\partial f(J_rz)\tag{4.2}$$

and hence,

$$0 \in \partial f(J_r z) + \frac{1}{r} J(J_r z) - \frac{1}{r} Jz = \partial\Big(f + \frac{1}{2r}\|\cdot\|^2 - \frac{1}{r} Jz\Big)(J_r z). \tag{4.3}$$

Thus, we have

$$J_r z = \arg\min_{y \in E} \Big\{ f(y) + \frac{1}{2r}\|y\|^2 - \frac{1}{r}\langle y, Jz\rangle \Big\}. \tag{4.4}$$

Therefore, $y_n = J_{r_n} x_n$ for all $n \in \mathbb{N}$. Using Theorem 3.3, $\{x_n\}$ converges strongly to $P_{(\partial f)^{-1}(0)}(x)$. □

We next study the problem of finding a solution of a variational inequality. Let C be a nonempty closed convex subset of a Banach space E and let $A : C \to E^*$ be a single-valued monotone operator which is *hemicontinuous,* that is, continuous along each line segment in C with respect to the weak* topology of E^*. Then a point $v \in C$ is said to be a solution of the *variational inequality* for A if

$$\langle y - v, Av\rangle \geq 0 \tag{4.5}$$

holds for all $y \in C$. We denote by $VI(C,A)$ the set of all solutions of the variational inequality for A. We also denote by $N_C(x)$ the *normal cone* for C at a point $x \in C$, that is,

$$N_C(x) = \{x^* \in E^* : \langle y - x, x^*\rangle \leq 0 \ \forall y \in C\}. \tag{4.6}$$

Theorem 4.2. *Let C be a nonempty closed convex subset of a smooth and uniformly convex Banach space E and let $A : C \to E^*$ be a single-valued, monotone, and hemicontinuous operator such that $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in E$ and*

$$\begin{aligned} y_n &= VI\Big(C, A + \frac{1}{r_n}(J - Jx_n)\Big) \quad (n = 1,2,\ldots), \\ x_{n+1} &= J^{-1}(\alpha_n Jx + (1-\alpha_n)Jy_n) \quad (n = 1,2,\ldots), \end{aligned} \tag{4.7}$$

where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\lim_{n\to\infty} r_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $P_{VI(C,A)}(x)$.

Proof. By Rockafellar's theorem [16], the mapping $T \subset E \times E^*$ defined by

$$Tx = \begin{cases} A(x) + N_C(x), & \text{if } x \in C, \\ \emptyset, & \text{otherwise}, \end{cases} \tag{4.8}$$

is maximal monotone and $T^{-1}0 = VI(C,A)$. Fix $r > 0$, $z \in E$, and let J_r be the resolvent of T. Then we have

$$Jz \in J(J_r z) + rT(J_r z) \tag{4.9}$$

and hence,

$$-A(J_r z) + \frac{1}{r}(Jz - J(J_r z)) \in N_C(J_r z). \tag{4.10}$$

Thus, we have

$$\left\langle y - J_r z, A(J_r z) + \frac{1}{r}(J(J_r z) - Jz) \right\rangle \geq 0 \tag{4.11}$$

for all $y \in C$, that is,

$$J_r z = VI\left(C, A + \frac{1}{r}(J - Jz)\right). \tag{4.12}$$

Therefore, $y_n = J_{r_n} x_n$ for all $n \in \mathbb{N}$. Using Theorem 3.3, $\{x_n\}$ converges strongly to $P_{VI(C,A)}(x)$. □

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