EXACT SOLUTIONS OF THE SEMI-INFINITE TODA LATTICE WITH APPLICATIONS TO THE INVERSE SPECTRAL PROBLEM

E. K. IFANTIS AND K. N. VLACHOU

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Several inverse spectral problems are solved by a method which is based on exact solutions of the semi-infinite Toda lattice. In fact, starting with a well-known and appropriate probability measure μ, the solution $\alpha_n(t)$, $b_n(t)$ of the Toda lattice is exactly determined and by taking $t=0$, the solution $\alpha_n(0)$, $b_n(0)$ of the inverse spectral problem is obtained. The solutions of the Toda lattice which are found in this way are finite for every $t>0$ and can also be obtained from the solutions of a simple differential equation. Many other exact solutions obtained from this differential equation show that there exist initial conditions $\alpha_n(0)>0$ and $b_n(0)\in\mathbb{R}$ such that the semi-infinite Toda lattice is not integrable in the sense that the functions $\alpha_n(t)$ and $b_n(t)$ are not finite for every $t>0$.

1. Introduction

We write the semi-infinite Toda lattice as follows:

$$\frac{d\alpha_n(t)}{dt} = \alpha_n(t)\big(b_{n+1}(t)-b_n(t)\big), \tag{1.1}$$

$$\frac{db_n(t)}{dt} = 2\big(\alpha_n^2(t)-\alpha_{n-1}^2(t)\big), \quad t\geq 0,\ n=1,2,\dots \tag{1.2}$$

and we ask for solutions which satisfy the initial conditions

$$\alpha_n(0)=\alpha_n, \qquad b_n(0)=b_n, \tag{1.3}$$

where α_n, b_n are real sequences with $\alpha_n>0$. In an attempt to compute the functions $b_n(t)$ and $\alpha_n(t)$ in some problems where the existence and uniqueness of a solution is proved, we observed that many solutions of the initial value problem (1.1), (1.2), (1.3) satisfy the relation

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = \delta b_1(t)+c, \tag{1.4}$$

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where δ, c are real numbers with $\delta \geq 0$ and the dot means differentiability with respect to t. Then it follows easily that

$$\alpha_n^2(t) = \left(\frac{n(n-1)}{2}\delta + n\right)\alpha_1^2(t),$$
$$b_n(t) = \left[(n-1)\delta + 1\right]b_1(t) + (n-1)c. \tag{1.5}$$

An example, where relation (1.4) holds and the functions $\alpha_1(t)$ and $b_1(t)$ can be exactly determined, has been published among others in [7]. Here, we present many examples starting from the well-known probability measures which determine uniquely the sequences in (1.3). All these examples provide alternative solutions of important well-known inverse spectral problems. (For details about the inverse spectral problem, see Section 3.) In other words, the solution of many well-known and important inverse spectral problems is obtained from one source, a class of exactly solvable Toda lattices.

The solutions of the Toda lattice which are found in this way are finite for every $t > 0$. These global solutions can also be obtained from the solutions of the differential equation

$$\ddot{b}_1(t) = 2(\delta b_1(t) + c)\dot{b}_1(t). \tag{1.6}$$

Any solution of this equation satisfies (1.4). Given the initial conditions $b_1(0)$ and $\alpha_1(0) > 0$ of the Toda lattice, we find the solution $b_1(t)$ of (1.6) which satisfies these conditions. We have the possibility to choose δ and c according to the form of the solution we want to construct. Moreover, from (1.6), we can find solutions of the Toda lattice with poles. Thus, many solutions of (1.6) show that there exist initial conditions $\alpha_n(0)$ and $b_n(0)$ such that the Toda lattice is not global integrable in the sense that the functions $\alpha_n(t)$ and $b_n(t)$ are not finite for every $t > 0$ (see Example 2.1 and Remark 4.10). In Section 2, we give the proofs of (1.5) and (1.6) and we obtain from (1.5) and (1.6) several forms of exact solutions of the Toda lattice. In Section 3, we define the inverse spectral problem and present preliminary results which we need. For many measures whose support is not bounded from above, the standard method of determining the function $b_1(t)$, and consequently $\alpha_1(t)$, $b_2(t)$, and so on, fails. With respect to the inverse spectral problem, we avoid this difficulty by using Theorem 3.2 (see also Remark 3.3). In Section 4, we give five examples of inverse spectral problems which can be solved by the method which is based on the determination of exact solutions of the Toda lattice. Note that all the exact solutions obtained in Section 2, by solving (1.6), and in Section 4, by solving several inverse spectral problems, are solutions with unbounded initial conditions.

2. Solutions of $\ddot{b}_1(t) = 2(\delta b_1(t) + c)\dot{b}_1(t)$

First, we give the proofs of relations (1.5) and (1.6).

From (1.1) (for $n = 1$) and (1.4), we have

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = \delta b_1(t) + c = b_2(t) - b_1(t) \tag{2.1}$$

or

$$b_2(t) = (\delta + 1)b_1(t) + c. \tag{2.2}$$

From (2.2) and (1.2), we find

$$\dot{b}_2(t) = (\delta+1)\dot{b}_1(t) = 2\left(\alpha_2^2(t) - \alpha_1^2(t)\right). \tag{2.3}$$

Since, from (1.2), we have

$$\dot{b}_1(t) = 2\alpha_1^2(t), \tag{2.4}$$

we obtain

$$\alpha_2^2(t) = (\delta+2)\alpha_1^2(t). \tag{2.5}$$

Continuing in this way, we find $b_3(t) = (2\delta+1)b_1(t) + 2c$ and $\alpha_3^2(t) = (3\delta+3)\alpha_1^2(t)$. Formulas (1.5) are obtained by induction. From (2.4), we obtain

$$\ddot{b}_1(t) = 4\alpha_1(t)\dot{\alpha}_1(t). \tag{2.6}$$

Thus from (2.4), (2.6), and (1.4), we obtain (1.6).

Equation (1.6) can be easily integrated. We present below several forms of the obtained solutions. The forms of the solutions depend on the initial conditions $\alpha_1(0)$ and $b_1(0)$ and the values of δ and c.

Case 1 ($\delta = 0$, $c = 0$). In this case, the solution of (1.6) is $b_1(t) = 2\alpha_1^2(0)t + b_1(0)$. From this and (2.4), we find $\alpha_1(t) = \alpha_1(0)$ and from (1.5), we obtain

$$\alpha_n(t) = \sqrt{n}\alpha_1(0), \quad b_n(t) = b_1(t) = 2\alpha_1^2(0)t + b_1(0), \quad n = 1,2,\ldots. \tag{2.7}$$

The solution in this case is finite for every $t > 0$ (global solution).

Case 2 ($\delta = 0$, $c \neq 0$). In this case, the solution of (1.6) is

$$b_1(t) = \frac{\alpha_1^2(0)}{c}\left(e^{2ct} - 1\right) + b_1(0) \tag{2.8}$$

and the solution $\alpha_n(t)$, $b_n(t)$ of the system (1.1), (1.2) is

$$\begin{aligned} \alpha_n(t) &= \sqrt{n}\alpha_1(0)e^{ct}, \\ b_n(t) &= \frac{\alpha_1^2(0)}{c}\left(e^{2ct} - 1\right) + b_1(0) + (n-1)c. \end{aligned} \tag{2.9}$$

Case 3 ($\delta > 0$, $c = 0$). The solution of (1.6) depends on the value

$$A_0 = 2\alpha_1^2(0) - \delta b_1^2(0). \tag{2.10}$$

If $A_0 = 0$, the global solution of (1.6) has the form

$$b_1(t) = \frac{b_1(0)}{1 - \delta b_1(0)t}, \quad b_1(0) < 0 \tag{2.11}$$

and the solution $\alpha_n(t)$, $b_n(t)$ of the system (1.1), (1.2) is given by

$$\begin{aligned}\alpha_n(t) &= \sqrt{n(n\delta-\delta+2)}\frac{-\sqrt{\delta}b_1(0)}{2(1-\delta b_1(0)t)},\\ b_n(t) &= n\frac{\delta b_1(0)}{1-\delta b_1(0)t}+\frac{(1-\delta)b_1(0)}{1-\delta b_1(0)t}.\end{aligned} \tag{2.12}$$

If $A_0 > 0$, the solution of (1.6) is

$$b_1(t) = \sqrt{\frac{A_0}{\delta}}\tan\left(\sqrt{\delta A_0}t+\Gamma_1\right), \tag{2.13}$$

$$\begin{aligned}\alpha_n(t) &= \sqrt{n(n\delta-\delta+2)A_0}\frac{1}{2\cos\left(\sqrt{\delta A_0}t+\Gamma_1\right)},\\ b_n(t) &= \left(n+\frac{1}{\delta}-1\right)\sqrt{A_0\delta}\tan\left(\sqrt{A_0\delta}t+\Gamma_1\right),\end{aligned} \tag{2.14}$$

where $\Gamma_1 = \arctan(\sqrt{\delta/A_0}b_1(0))$. In order to have a global solution of the form (2.14), the condition $\cos(\sqrt{A_0\delta}t+\Gamma_1) \neq 0$, for all $t > 0$, should hold.

If $A_0 < 0$, the solution of (1.6) is given by

$$b_1(t) = \frac{\Gamma_2(-\sqrt{-A_0\delta})e^{-2t\sqrt{-A_0\delta}}-\sqrt{-A_0\delta}}{\delta\left(1-\Gamma_2 e^{-2t\sqrt{-A_0\delta}}\right)} \tag{2.15}$$

and the functions $\alpha_n(t)$ and $b_n(t)$ have the form

$$\begin{aligned}\alpha_n(t) &= \sqrt{n(n\delta-\delta+2)\Gamma_2(-A_0)}\frac{e^{-t\sqrt{-A_0\delta}}}{1-\Gamma_2 e^{-2t\sqrt{-A_0\delta}}},\\ b_n(t) &= [(n-1)\delta+1]\frac{\Gamma_2(-\sqrt{-A_0\delta})e^{-2t\sqrt{-A_0\delta}}-\sqrt{-A_0\delta}}{\delta\left(1-\Gamma_2 e^{-2t\sqrt{-A_0\delta}}\right)},\end{aligned} \tag{2.16}$$

where $\Gamma_2 = (\delta b_1(0)+\sqrt{-A_0\delta})/(\delta b_1(0)-\sqrt{-A_0\delta})$. We have a global solution if and only if

$$1-\frac{\delta b_1(0)+\sqrt{-A_0\delta}}{\delta b_1(0)-\sqrt{-A_0\delta}}e^{-2t\sqrt{-A_0\delta}} \neq 0, \quad t \geq 0. \tag{2.17}$$

Case 4 ($\delta > 0$, $c \neq 0$). In this case, the solution of (1.6) and consequently the form of the solution of the system (1.1), (1.2) depends on the value

$$A = (2\alpha_1^2(0)-\delta b_1^2(0)-2cb_1(0))\delta-c^2. \tag{2.18}$$

If $A = 0$, the solution of (1.6) and the functions $\alpha_n(t)$ and $b_n(t)$ have the form

$$b_1(t) = \frac{2cB_1 e^{2ct}}{\delta(1 - B_1 e^{2ct})}, \tag{2.19}$$

$$\begin{aligned} \alpha_n(t) &= \sqrt{\frac{n(n\delta - \delta + 2)B_1}{\delta}} \frac{ce^{ct}}{1 - B_1 e^{2ct}}, \\ b_n(t) &= nc\frac{1 + B_1 e^{2ct}}{1 - B_1 e^{2ct}} + \frac{cB_1 e^{2ct}(2 - \delta) - c\delta}{\delta(1 - B_1 e^{2ct})}, \end{aligned} \tag{2.20}$$

where $B_1 = \delta b_1(0)/(\delta b_1(0) + 2c)$. In order to construct a global solution of the form (2.20), we must choose δ, c, $b_1(0)$, and $\alpha_1(0)$ such that $\delta b_1(0) + 2c \neq 0$, $A = (2\alpha_1^2(0) - \delta b_1^2(0) - 2cb_1(0))\delta - c^2 = 0$, and $1 - B_1 e^{2ct} \neq 0$, for every $t > 0$.

If $A > 0$, we have

$$b_1(t) = \frac{\sqrt{A}}{\delta} \tan(\sqrt{A}t + B_2) - \frac{c}{\delta}, \tag{2.21}$$

$$\begin{aligned} \alpha_n(t) &= \sqrt{\frac{n(n\delta - \delta + 2)A}{\delta}} \frac{1}{2\cos(\sqrt{A}t + B_2)}, \\ b_n(t) &= \left(n + \frac{1}{\delta} - 1\right)\sqrt{A}\tan(\sqrt{A}t + B_2) - \frac{c}{\delta}, \end{aligned} \tag{2.22}$$

where $B_2 = \arctan((\delta b_1(0) + c)/\sqrt{A})$. In order to have a global solution of the form (2.22), the condition $\cos(\sqrt{A}t + B_2) \neq 0$, for all $t > 0$, should hold. Finally, for $A < 0$, the solutions are exponential with

$$b_1(t) = \frac{B_3(c - \sqrt{-A})e^{-2t\sqrt{-A}} - \sqrt{-A} - c}{\delta(1 - B_3 e^{-2t\sqrt{-A}})}, \tag{2.23}$$

$$\begin{aligned} \alpha_n(t) &= \sqrt{\frac{n(n\delta - \delta + 2)B_3(-A)}{\delta}} \frac{e^{-t\sqrt{-A}}}{1 - B_3 e^{-2t\sqrt{-A}}}, \\ b_n(t) &= [(n-1)\delta + 1]b_1(t) + (n-1)c, \end{aligned} \tag{2.24}$$

where $B_3 = (\delta b_1(0) + c + \sqrt{-A})/(\delta b_1(0) + c - \sqrt{-A})$ and $b_1(t)$ is given by (2.23). In order to have a solution without poles, the condition

$$1 - \frac{\delta b_1(0) + c + \sqrt{-A}}{\delta b_1(0) + c - \sqrt{-A}} e^{-2t\sqrt{-A}} \neq 0, \quad t \geq 0, \tag{2.25}$$

must be satisfied.

Example 2.1. Taking $\delta = 2$, $c = 1 - \lambda$, $b_1(0) = 1$, and $\alpha_1(0) = \sqrt{\lambda}$, $0 < \lambda < 1$, we find $A = -(\lambda - 9)(\lambda - 1) < 0$ for $0 < \lambda < 1$, $B_3 = (3 - \lambda + \sqrt{-A})/(3 - \lambda - \sqrt{-A}) > 1$. This means that condition (2.25) is not satisfied for every $t > 0$. We conclude that the Toda lattice, with initial conditions $b_1(0) = 1$, $\alpha_1(0) = \sqrt{\lambda}$, and $b_n(0)$, $\alpha_n(0)$ given by (1.5) for $t = 0$, $\delta = 2$, and $c = 1 - \lambda$, is not integrable in the sense that the functions $\alpha_n(t)$ and $b_n(t)$ are not finite for every $t > 0$.

Example 2.2. Taking $\delta = 2$, $c = 1-\lambda$, $b_1(0) = -1$, $\alpha_1(0) = \sqrt{\lambda}$, $0 < \lambda < 1$, $\alpha_n(0) = n\sqrt{\lambda}$, and $b_n(0) = -(1+\lambda)n+\lambda$, we have $A = -(1-\lambda)^2$, $B_3 = \lambda$. Then condition (2.25) is satisfied and $b_1(t)$ is given by

$$b_1(t) = -\frac{1-\lambda}{1-\lambda e^{-2(1-\lambda)t}}. \tag{2.26}$$

This example is a particular case of Example 4.5. In fact, (2.26) is the same with (4.11) for $\beta = 0$ and $\kappa = 1-\lambda > 0$.

3. The inverse spectral problem

It is well known that any probability measure μ on the real line with finite moments and infinite support determines uniquely a pair of real sequences α_n, b_n with $\alpha_n > 0$ and a class of orthonormal polynomials $P_n(x)$ ($\int_{-\infty}^{\infty} P_n(x)P_m(x)d\mu(x) = \delta_{nm}$), which satisfy the relation

$$\begin{gathered} \alpha_n P_{n+1}(x) + \alpha_{n-1}P_{n-1}(x) + b_n P_n(x) = xP_n(x), \\ P_0(x) = 0, \qquad P_1(x) = 1. \end{gathered} \tag{3.1}$$

Conversely, for any pair of real sequences α_n, b_n with $\alpha_n > 0$, there exists at least one probability measure μ such that the polynomials (3.1) are orthonormal. The measure μ is unique if and only if the tridiagonal operator $L(0)$, defined on finite linear combination of an orthonormal basis e_n, $n = 1,2,\ldots$, of a Hilbert space H:

$$\begin{aligned} L(0)e_n &= \alpha_n e_{n+1} + \alpha_{n-1}e_{n-1} + b_n e_n, \\ L(0)e_1 &= \alpha_1 e_2 + b_1 e_1, \end{aligned} \tag{3.2}$$

is (essentially) selfadjoint (see [1, 8, 11] for these subjects and their relationships). If $L(0)$ is selfadjoint, then there exists a one parameter family E_t, $-\infty < t < \infty$, of orthogonal projections on H such that for every x, $\|x\| = 1$, the function $F(t) = (E_t x, x)$, where $(\cdot,\cdot)$ means scalar product, is a distribution function, that is, a nondecreasing function which is continuous on the right and satisfies $F(-\infty) = 0$, $F(\infty) = 1$. In particular, for $x = e_1$, the distribution function

$$F(t) = (E_t e_1, e_1) \tag{3.3}$$

is the distribution function which corresponds to the unique probability measure μ, that is, μ and F are connected by (see Theorem 3.1)

$$\mu((-\infty, t]) = F(t). \tag{3.4}$$

The direct problem of orthogonal polynomials is the following.

Given the real sequences $\alpha_n > 0$ and b_n, find the measure of orthogonality of the polynomials which are defined by (3.1). This problem has a long history. Only the problem of finding conditions on α_n and b_n, such that the above problem has a unique solution (note that at least one solution always exists), is connected with many important problems in analysis, for instance, the moment problem, the problem of selfadjoint extensions of an unbounded symmetric operator, and others [1, 8, 11] (see also [5]). Note that the measure of orthogonality is a probability measure on the real line with finite moments and support consisting of infinitely many points.

The inverse problem of orthogonal polynomials is the following.

Given a probability measure μ on the real line with finite moments and infinite support, find the coefficients α_n and b_n which define the polynomials P_n in (3.1). Sometimes we say the inverse problem of the operator $L(0)$ or the inverse problem of μ instead of the inverse problem of the polynomials (3.1).

There exists a standard procedure, which determines uniquely the sequences α_n and b_n but this procedure involves many and arduous calculations, and exact solutions are very difficult to be found. In fact, multiplication by $P_n(x)$ in (3.1) and integration gives

$$b_n = \int_{-\infty}^{\infty} xP_n^2(x)d\mu(x). \tag{3.5}$$

First, we find from (3.5) that $b_1 = \int_{-\infty}^{\infty} x\,d\mu(x)$. Consequently, taking $n = 1$ in (3.1) and multiplying the relation $\alpha_1 P_2(x) + b_1 = x$ by $P_2(x)$, we obtain $\alpha_1^2 = \int_{-\infty}^{\infty} x^2 d\mu(x) - b_1^2$. After this, knowing the polynomial $P_2(x)$, we determine b_2 from (3.5). Then we find α_2, the polynomial $P_3(x)$, and so on. The inverse problem of a selfadjoint operator is the problem of finding the operator when some of its properties are given, for instance, its spectrum. It is well known that the spectrum is not always enough for the solution of this problem. In the present case, the spectrum of $L(0)$ is not enough to determine uniquely the sequences α_n, b_n. However, the knowledge of the distribution function $(E_t e_1, e_1)$ is enough, because of the following well-known theorem, which we prove for completeness.

Theorem 3.1. *Let $L(0)$, defined by (3.2), be selfadjoint and let E_t, $-\infty < t < \infty$ be its spectral family. Then the measure which corresponds to the distribution function $(E_t e_1, e_1)$ is the unique measure of orthogonality of the polynomials P_n defined by (3.1).*

Proof. Let $L(0) = T$. Then T can be written as

$$T = \int_{-\infty}^{\infty} t\,dE_t, \tag{3.6}$$

in the sense that

$$(Tx, y) = \int_{-\infty}^{\infty} t\,d(E_t x, y) \tag{3.7}$$

for every x, y in the definition domain of T. Then the operator $P_m(T)P_n(T)$ can be written as follows:

$$\begin{aligned} P_m(T)P_n(T) &= \int_{-\infty}^{\infty} P_m(t)P_n(t)dE_t, \\ (P_m(T)P_n(T)e_1, e_1) &= \int_{-\infty}^{\infty} P_m(t)P_n(t)d(E_t e_1, e_1). \end{aligned} \tag{3.8}$$

The operator $P_n(T)$ acting on the element e_1 produces the vector e_n, that is,

$$P_n(T)e_1 = e_n, \quad n = 1, 2, \ldots. \tag{3.9}$$

Relation (3.9) is obvious for $n = 1$ and for $n \geq 2$ it follows from (3.2) and the relation $\alpha_n P_{n+1}(T) + \alpha_{n-1}P_{n-1}(T) + b_n P_n(T) = TP_n(T)$, by induction. Thus,

$$\begin{aligned}\int_{-\infty}^{\infty} P_m(t)P_n(t)d(E_t e_1, e_1) &= (P_m(T)P_n(T)e_1, e_1) \\ &= (P_n(T)e_1, P_m(T)e_1) \\ &= (e_n, e_m) = \delta_{n,m}.\end{aligned} \tag{3.10}$$

□

The measure which corresponds to the distribution function (3.3) is called spectral measure of the tridiagonal operator $L(0)$.

Another theorem that we will need is the following one.

Theorem 3.2. *Assume that the operator*

$$L(0) : L(0)e_n = \alpha_n e_{n+1} + \alpha_{n-1}e_{n-1} + b_n e_n, \quad n = 1, 2, \ldots, \tag{3.11}$$

is essentially selfadjoint with spectral measure μ. Then the operator

$$L_1(0) : L_1(0)e_n = \alpha_n e_{n+1} + \alpha_{n-1}e_{n-1} - b_n e_n \tag{3.12}$$

is also essentially selfadjoint with spectral measure $\mu\tau^{-1}$, where $\tau(x) = -x$.

Proof. By Theorem 3.1 and by the equivalence of the properties "essential selfadjointness" of $L(0)$ and "uniqueness of the measure of orthogonality" of the corresponding polynomials, it is enough to prove that $\mu\tau^{-1}$ is a measure of orthogonality of the polynomials

$$\begin{gathered}\alpha_n R_{n+1}(x) + \alpha_{n-1}R_{n-1}(x) - b_n R_n(x) = xR_n(x), \\ R_0(x) = 0, \qquad R_1(x) = 1,\end{gathered} \tag{3.13}$$

provided that μ is a measure of orthogonality of the polynomials (3.1). It is easy to see that the polynomials $R_n(x)$ and $P_n(x)$ are related by

$$R_n(x) = (-1)^n P_n(-x). \tag{3.14}$$

Assume that

$$\int_{-\infty}^{\infty} P_n(x)P_m(x)d\mu = \delta_{n,m}. \tag{3.15}$$

Then by a well-known property in measure theory, we have

$$\begin{aligned}\int_{-\infty}^{\infty} R_n(x)R_m(x)d\mu\tau^{-1} &= \int_{-\infty}^{\infty} R_n(\tau(x))R_m(\tau(x))d\mu \\ &= \int_{-\infty}^{\infty} R_n(-x)R_m(-x)d\mu \\ &= (-1)^{n+m}\int_{-\infty}^{\infty} P_n(x)P_m(x)d\mu = \delta_{n,m}.\end{aligned} \tag{3.16}$$

□

Given the initial conditions α_n and b_n of the Toda lattice, we assume that the operator $L(0)$ is (essentially) selfadjoint which means that the measure μ of orthogonality of the polynomials (3.1) is unique.

The method of the inverse spectral problem works as follows: system (1.1), (1.2) is equivalent to the equation

$$\frac{dL(t)}{dt} = M(t)L(t) - L(t)M(t), \tag{3.17}$$

where $L(t)$ and $M(t)$ are the tridiagonal operators

$$\begin{aligned} L(t)e_n &= \alpha_n(t)e_{n+1} + \alpha_{n-1}(t)e_{n-1} + b_n(t)e_n, \\ M(t)e_n &= \alpha_n(t)e_{n+1} - \alpha_{n-1}(t)e_{n-1}. \end{aligned} \tag{3.18}$$

Under suitable assumptions on α_n, b_n, one finds the spectral measure $\mu^{(t)}$ of the operator $L(t)$ [7]. Note that the spectral measure μ can be found by solving the direct problem of the operator $L(0)$. For the Toda lattice, as it is written in (1.1), (1.2), $\mu^{(t)}$ has the form

$$d\mu^{(t)}(x) = \frac{e^{2xt}d\mu(x)}{\int_{-\infty}^{\infty} e^{2xt}d\mu(x)}, \quad t \geq 0. \tag{3.19}$$

The solution of the Toda lattice is obtained by solving the inverse problem of $L(t)$. In fact, starting from the spectrum measure $\mu^{(t)}$, we consider the linearly independent elements $1, x, x^2, \dots$ of the space $L_2(\mu^{(t)})$, the orthogonalization of which by the use of the Gram-Schmidt method gives the orthogonal polynomials $P_n(t,x)$ which satisfy

$$\begin{gathered} \alpha_n(t)P_{n+1}(t,x) + \alpha_{n-1}(t)P_{n-1}(t,x) + b_n(t)P_n(t,x) = xP_n(t,x), \\ P_0(t,x) = 0, \qquad P_1(t,x) = 1, \end{gathered} \tag{3.20}$$

with $\alpha_n(t) > 0$ and $b_n(t)$ real. By a well-known procedure, the sequences $\alpha_n(t)$, $b_n(t)$ can be found from the above recurrence relation. Moreover, they satisfy system (1.1), (1.2). In our case, it is enough to determine exactly the function $b_1(t)$, which is given by

$$b_1(t) = \frac{\int_{-\infty}^{\infty} xe^{2xt}d\mu(x)}{\int_{-\infty}^{\infty} e^{2xt}d\mu(x)}, \quad t \geq 0. \tag{3.21}$$

What we need for the solution of the Toda lattice is the spectral measure μ of $L(0)$. If the spectrum of $L(0)$ is discrete with eigenvalues λ_n and normalized eigenvectors x_n, then

$$\mu(\{\lambda_n\}) = |(e_1, x_n)|^2 = \sigma_n^2 \tag{3.22}$$

and $b_1(t)$ is given by

$$b_1(t) = \frac{\sum_{n=1}^{\infty} \lambda_n e^{2\lambda_n t}\sigma_n^2}{\sum_{n=1}^{\infty} e^{2\lambda_n t}\sigma_n^2}. \tag{3.23}$$

In Section 4, we begin with a probability measure μ without knowing the initial conditions $\alpha_n(0)$, $b_n(0)$. Then we determine the element $b_1(t)$ from (3.21) or (3.23) and examine the validity of the relation (1.4). After this, the solution of the Toda lattice is given by (1.5). The solution of the inverse problem of μ is given by

$$\begin{aligned} \alpha_n^2(0) &= \left(\frac{n(n-1)}{2}\delta + n\right)\alpha_1^2(0), \\ b_n(0) &= [(n-1)\delta + 1]b_1(0) + (n-1)c. \end{aligned} \tag{3.24}$$

Remark 3.3. The usefulness of Theorem 3.2 is that if the support of the measure μ lies in the interval $[\alpha, \infty)$, $\alpha \in \mathbb{R}$, and if it is not a bounded set, then the integrals in (3.21) may not be finite. In this case, we find the solution $\alpha_n(0)$, $b_n(0)$ of the inverse spectral problem of the measure $\mu\tau^{-1}$, $\tau(x) = -x$, whose support is bounded from above. Then the solution of the inverse spectral problem of μ, due to Theorem 3.2, is $\alpha_n(0)$, $-b_n(0)$, $n = 1, 2, \ldots$.

4. Examples

In all the following examples, we begin with a probability measure μ on the real line with finite moments $\mu_n = \int_{-\infty}^{\infty} x^n d\mu(x)$, $\mu_0 = 1$, and infinite support.

Example 4.1. Consider the probability measure μ whose distribution function is given by

$$F(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^x \xi^\alpha e^{-\xi} d\xi, \quad \alpha > -1, \tag{4.1}$$

where Γ is the gamma function. The moments

$$\mu_k = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^k x^\alpha e^{-x} dx = \frac{\Gamma(\alpha+k+1)}{\Gamma(\alpha+1)}, \quad k = 0, 1, 2, \ldots, \tag{4.2}$$

are finite. Moreover, we can see that there exists a positive number r such that the series

$$\sum_{m=1}^{\infty} \frac{\mu_m r^m}{m!} \tag{4.3}$$

converges and by a well-known criterion (see, e.g., [2, Theorem 30.1]), the moment problem is determined or equivalently the operator $L(0)$ is selfadjoint (see [1]).

Since the integrals of (3.21) do not exist for this measure, we consider the measure $\mu\tau^{-1}$, $\tau(x) = -x$ instead of μ. We find from (3.21)

$$\begin{aligned} b_1(t) &= -\frac{\alpha+1}{2t+1}, & \alpha_1(t) &= \frac{\sqrt{\alpha+1}}{2t+1}, \\ \frac{\dot\alpha_1(t)}{\alpha_1(t)} &= \frac{2}{\alpha+1} b_1(t) & & \left(\delta = \frac{2}{\alpha+1},\ c = 0\right). \end{aligned} \tag{4.4}$$

Thus, from (1.5), we obtain

$$\begin{aligned} \alpha_n(t) &= \frac{\sqrt{n(n+\alpha)}}{2t+1}, \\ b_n(t) &= -\frac{2n+\alpha-1}{2t+1}, \quad t \geq 0,\ n = 1,2,\dots. \end{aligned} \tag{4.5}$$

Conclusion 4.2. The solution of the Toda lattice with initial conditions

$$\begin{aligned} \alpha_n(0) &= \sqrt{n(n+\alpha)}, \\ b_n(0) &= -(2n+\alpha-1) \end{aligned} \tag{4.6}$$

is given by (4.5). The solution of the inverse problem of $\mu\tau^{-1}$ is given by (4.6). Due to Theorem 3.2, the solution of the inverse problem of the measure μ is

$$\begin{aligned} \alpha_n(0) &= \sqrt{n(n+\alpha)}, \\ b_n(0) &= (2n+\alpha-1). \end{aligned} \tag{4.7}$$

Remark 4.3. The solution of the inverse problem that we studied in this example is well known in the theory of Laguerre polynomials defined by

$$\begin{gathered} \sqrt{n(n+\alpha)}P_{n+1}(x) + \sqrt{(n-1)(n-1+\alpha)}P_{n-1}(x) + (2n+\alpha-1)P_n(x) = xP_n(x), \\ P_0(x) = 0, \qquad P_1(x) = 1. \end{gathered} \tag{4.8}$$

In fact, it is well known that the measure of orthogonality of $P_n(x)$ is unique and its distribution function is given by (4.1) (see [3]).

Remark 4.4. In the following examples, we have a difficulty to establish the convergence of the series (4.3). We avoid this difficulty as follows: suppose that we start with a measure of the form (3.19) and we have found exactly a solution of the Toda lattice $\alpha_n(t)$, $b_n(t)$ for $t \geq 0$. This means that we have found exactly the coefficients of the polynomials $P_n(t,x)$ which are orthonormal with respect to the measure $\mu^{(t)}$. In all the examples that we will give, we can see from the coefficients $\alpha_n(t)$, $b_n(t)$ that $\mu^{(t)}$ is the unique measure of orthogonality for every $t \geq 0$. In fact, the well-known criterion of Carleman [1] can easily be applied. For $t = 0$, we find the coefficients of the polynomials whose measure of orthogonality is the measure μ. In this way, we solve both the inverse and the direct spectral problem of μ.

Example 4.5. Consider the discrete probability measure

$$\mu(\{-\kappa n+\beta\}) = (1-\lambda)\lambda^{n-1}, \quad 0<\lambda<1,\ n = 1,2,\dots,\ \kappa > 0. \tag{4.9}$$

Obviously, its support is infinite consisting of the points $-\kappa+\beta$, $-2\kappa+\beta$, $-3\kappa+\beta,\dots$, and the moments $\mu_k = (1-\lambda)\sum_{n=1}^{\infty}(-\kappa n+\beta)^k\lambda^{n-1}$, $k=0,1,2,\dots$, are finite. We have

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2xt}\,d\mu(x) &= (1-\lambda)\sum_{n=1}^{\infty} e^{-2\kappa nt+2\beta t}\lambda^{n-1}\\ &= \frac{(1-\lambda)e^{2\beta t}}{\lambda}\sum_{n=1}^{\infty}\left(\frac{\lambda}{e^{2\kappa t}}\right)^n = \frac{(1-\lambda)e^{2\beta t}}{e^{2\kappa t}-\lambda},\\ \int_{-\infty}^{\infty} xe^{2xt}\,d\mu(x) &= \sum_{n=1}^{\infty}(-\kappa n+\beta)e^{-2\kappa nt}e^{2\beta t}(1-\lambda)\lambda^{n-1}\\ &= \frac{\beta(1-\lambda)e^{2\beta t}}{e^{2\kappa t}-\lambda} - \frac{\kappa(1-\lambda)e^{2\beta t}}{e^{2\kappa t}(1-\mu)^2},\quad \mu = \frac{\lambda}{e^{2\kappa t}}.\end{aligned} \tag{4.10}$$

Thus, from (3.21), we obtain

$$b_1(t) = \beta - \frac{\kappa}{1-\lambda e^{-2\kappa t}}. \tag{4.11}$$

From (1.2), we find

$$\frac{\dot b_1(t)}{2} = \alpha_1^2(t) = \frac{\lambda\kappa^2 e^{-2\kappa t}}{\left(1-\lambda e^{-2\kappa t}\right)^2},\qquad \alpha_1(t) = \frac{\kappa\sqrt{\lambda}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}} \tag{4.12}$$

and after some manipulation, we obtain

$$\frac{\dot\alpha_1(t)}{\alpha_1(t)} = 2b_1(t)+\kappa-2\beta. \tag{4.13}$$

Now, from (1.5), we find the solution of the Toda lattice which is

$$\begin{aligned}\alpha_n(t) &= \frac{n\kappa\sqrt{\lambda}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}},\\ b_n(t) &= (2n-1)\left(\beta - \frac{\kappa}{1-\lambda e^{-2\kappa t}}\right)+\kappa(n-1)-2\beta(n-1).\end{aligned} \tag{4.14}$$

The inverse problem of $L(0)$ can be solved by setting $t=0$ in (4.14), that is,

$$\begin{aligned}\alpha_n(0) &= \frac{n\kappa\sqrt{\lambda}}{1-\lambda},\\ b_n(0) &= -\frac{(1+\lambda)\kappa n}{1-\lambda}+\beta+\frac{\kappa}{1-\lambda}-\kappa.\end{aligned} \tag{4.15}$$

Due to Theorem 3.2, the solution of the inverse problem of the measure

$$\mu(\{\kappa n-\beta\}) = (1-\lambda)\lambda^{n-1},\quad 0<\lambda<1,\ n=1,2,\dots,\ \kappa>0 \tag{4.16}$$

is

$$\alpha_n(0) = \frac{n\kappa\sqrt{\lambda}}{1-\lambda},\qquad b_n(0) = \frac{(1+\lambda)\kappa n}{1-\lambda}-\beta-\frac{\kappa}{1-\lambda}+\kappa. \tag{4.17}$$

Remark 4.6. For $\beta = 0$, $\kappa = 1$, the inverse problem of the measure in (4.16) has been solved in [4] by a different method. Also for $\beta = 0$, $\kappa = 1 - \lambda$, the inverse problem of the measure in (4.16) can be solved by using a result of Stieltjes in [9, 10]. In fact, Stieltjes considered the continued fraction

$$F(z,\lambda) = \cfrac{1}{z + \cfrac{1}{1 + \cfrac{\lambda}{z + \cfrac{2}{1 + \cfrac{2\lambda}{z + \cfrac{3}{1 + \cfrac{3\lambda}{z + \ddots}}}}}}} \tag{4.18}$$

This fraction, by the identity

$$z + c_1 - \frac{c_1 c_2}{c_2 + k_1} = z + \frac{c_1}{1 + c_2/k_1}, \tag{4.19}$$

where $\alpha_n^2 = c_{2n-1}c_{2n}$ and $b_n = c_{2n-2}c_{2n-1}$, $b_1 = c_1$, can be transformed into the fraction

$$F(z,\lambda) = \cfrac{1}{z + b_1 - \cfrac{\alpha_1^2}{z + b_2 - \cfrac{\alpha_2^2}{z + b_3 - \ddots}}} = \int_{-\infty}^{\infty} \frac{d\mu(x)}{z + x} \tag{4.20}$$

Stieltjes gives in [9, 10] what we call nowadays the Stieltjes transform of the measure (4.16):

$$\int_{-\infty}^{\infty} \frac{d\mu(x)}{z + x} = \sum_{n=1}^{\infty} \frac{(1-\lambda)\lambda^{n-1}}{z + n(1-\lambda)}. \tag{4.21}$$

From the analytic theory of continued fractions [6], it follows that the coefficients α_n, b_n of the orthogonal polynomials corresponding to μ are given by $\alpha_n = \sqrt{\lambda} n$ and $b_n = n(1+\lambda) - \lambda$.

In fact, it is well known that if the tridiagonal operator $L(0)$ is selfadjoint with spectral measure μ, then the Stieltjes transform of μ is given by

$$\int_{-\infty}^{\infty} \frac{d\mu(x)}{z + x} = \cfrac{1}{z + b_1 - \cfrac{\alpha_1^2}{z + b_2 - \cfrac{\alpha_2^2}{z + b_3 - \ddots}}} \tag{4.22}$$

In case the measure is discrete with mass points $\lambda_1, \lambda_2, \ldots$, the Stieltjes transform is given by

$$\int_{-\infty}^{\infty} \frac{d\mu(x)}{z+x} = \sum_{k=1}^{\infty} \frac{\mu(\{\lambda_k\})}{z+\lambda_k}. \tag{4.23}$$

Conversely, from the solution of the inverse problem, the Stieltjes transform of the measure (4.16) follows. This transform was presented by Stieltjes without proof in [9, 10] (see also [12, page 367]).

Example 4.5 is a particular case of the following example.

Example 4.7. Consider the probability measure

$$\mu(\{\beta-\kappa n\}) = \frac{(\alpha)_{n-1}(1-\lambda)^\alpha \lambda^{n-1}}{(n-1)!}, \quad \alpha > 0,\ 0 < \lambda < 1,\ n = 1,2,\ldots, \tag{4.24}$$

where $(\alpha)_{n-1} = \alpha(\alpha+1)\cdots(\alpha+n-2)$. The support of this measure is infinite and the moments

$$\mu_\kappa = \sum_{n=1}^{\infty} \frac{(-\kappa n+\beta)^\kappa \alpha(\alpha+1)\cdots(\alpha+n-2)(1-\lambda)^\alpha \lambda^{n-1}}{(n-1)!} \tag{4.25}$$

are finite.

From (4.24), we obtain

$$\begin{gathered}
\int_{-\infty}^{\infty} e^{2xt} d\mu = \sum_{n=1}^{\infty} \frac{e^{2t(-\kappa n+\beta)}(\alpha)_{n-1}\lambda^{n-1}(1-\lambda)^\alpha}{(n-1)!} = \frac{(1-\lambda)^\alpha e^{2\beta t - 2\kappa t}}{\left(1-\lambda e^{-2\kappa t}\right)^\alpha}, \\
\int_{-\infty}^{\infty} x e^{2xt} d\mu = (1-\lambda)^\alpha e^{2\beta t-2\kappa t} \frac{\left[(\kappa\lambda - \kappa\alpha\lambda - \beta\lambda)e^{-2\kappa t} + \beta - \kappa\right]}{\left(1-\lambda e^{-2\kappa t}\right)^{\alpha+1}}.
\end{gathered} \tag{4.26}$$

Thus,

$$\begin{aligned}
b_1(t) &= \beta + \frac{\lambda\kappa e^{-2\kappa t}(1-\alpha) - \kappa}{1-\lambda e^{-2\kappa t}}, \\
\dot{b}_1(t) &= \frac{2\lambda\kappa^2 \alpha e^{-2\kappa t}}{\left(1-\lambda e^{-2\kappa t}\right)^2} = 2\alpha_1^2(t), \\
\alpha_1(t) &= \frac{\kappa\sqrt{\lambda\alpha}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}}, \\
\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} &= \frac{2}{\alpha} b_1(t) + \frac{2\kappa}{\alpha} - \frac{2\beta}{\alpha} - \kappa.
\end{aligned} \tag{4.27}$$

From this relation, we see that in this case we have $\delta = 2/\alpha$ and $c = 2\kappa/\alpha - 2\beta/\alpha - \kappa$. Then from (1.5), it follows that

$$\begin{aligned} \alpha_n(t) &= \frac{\kappa\sqrt{n(n+\alpha-1)\lambda}e^{-\kappa t}}{1-\lambda e^{-2\kappa t}}, \\ b_n(t) &= \left(\frac{-\kappa(1+\lambda e^{-2\kappa t})}{1-\lambda e^{-2\kappa t}}\right)n + \left(1-\frac{2}{a}\right)b_1(t) - \frac{2\kappa}{\alpha} + \frac{2\beta}{\alpha} + \kappa. \end{aligned} \tag{4.28}$$

The solution of the inverse problem is given by

$$\begin{aligned} \alpha_n(0) &= \frac{\kappa\sqrt{n(n+\alpha-1)\lambda}}{1-\lambda}, \\ b_n(0) &= \frac{-\kappa(1+\lambda)n + \lambda\kappa(2-\alpha) + \beta(1-\lambda)}{1-\lambda}. \end{aligned} \tag{4.29}$$

We note that this inverse spectral problem is well known in the theory of the measure of orthogonality of the Meixner polynomials (see [3, page 175]), which we find here by an alternative method.

Using Theorem 3.2, we can derive the solution of the inverse problem of the measure (4.24). This solution is given by

$$\begin{aligned} \alpha_n(0) &= \frac{\kappa\sqrt{n(n+\alpha-1)\lambda}}{1-\lambda}, \\ b_n(0) &= \frac{\kappa(1+\lambda)n - \lambda\kappa(2-\alpha) - \beta(1-\lambda)}{1-\lambda}. \end{aligned} \tag{4.30}$$

Example 4.8. Consider the probability measure

$$\mu(\{\beta-\gamma n\}) = \frac{e^{-\alpha}\alpha^{n-1}}{(n-1)!}, \quad \alpha > 0,\ \gamma > 0,\ n = 1,2,\dots. \tag{4.31}$$

From (4.31), we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} e^{2xt}d\mu &= e^{-\alpha+2\beta t-2\gamma t+\alpha e^{-2\gamma t}}, \\ \int_{-\infty}^{\infty} xe^{2xt}d\mu &= e^{-\alpha+2\beta t-2\gamma t+\alpha e^{-2\gamma t}}\left(\beta-\gamma-\alpha\gamma e^{-2\gamma t}\right). \end{aligned} \tag{4.32}$$

Thus,

$$b_1(t) = \beta - \gamma - \alpha\gamma e^{-2\gamma t}. \tag{4.33}$$

From (4.33),

$$\dot{b}_1(t) = 2\alpha\gamma^2 e^{-2\gamma t} = 2\alpha_1^2(t), \tag{4.34}$$

and from (1.2), we find

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = -\gamma. \tag{4.35}$$

From (4.35), we see that in this case we have $\delta = 0$ and $c = -\gamma$. Then from (1.5), it follows that

$$\alpha_n(t) = \gamma\sqrt{n\alpha}e^{-\gamma t}, \qquad b_n(t) = \beta - \gamma - \alpha\gamma e^{-2\gamma t} - (n-1)\gamma. \tag{4.36}$$

The solution of the inverse problem is

$$\alpha_n(0) = \gamma\sqrt{n\alpha}, \qquad b_n(0) = \beta - \alpha\gamma - n\gamma \tag{4.37}$$

for $\beta = 1/\gamma$, $\alpha = 1/\gamma^2$. Thus, we have obtained the example studied in [7]. Here we note that we obtain an alternative derivation of the measure of orthogonality of the Charlier polynomials (see appendix in [7]).

In this example, we solved the inverse problem of the measure (4.31). Due to Theorem 3.2, we can see that the solution of the inverse problem of the measure

$$\mu(\{\gamma n - \beta\}) = \frac{e^{-\alpha}\alpha^{n-1}}{(n-1)!}, \quad \alpha > 0,\ \gamma > 0,\ n = 1,2,\ldots, \tag{4.38}$$

is

$$\alpha_n(0) = \gamma\sqrt{n\alpha}, \qquad b_n(0) = -\beta + \alpha\gamma + n\gamma. \tag{4.39}$$

Example 4.9. Consider the probability measure μ whose distribution function is given by

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} e^{-(\xi-m)^2/2\sigma^2}\,d\xi, \quad m,\sigma > 0. \tag{4.40}$$

The moments $\mu_k = (1/\sigma\sqrt{2\pi})\int_{-\infty}^{\infty} x^k e^{-(x-m)^2/2\sigma^2}\,dx$ are finite.

We calculate the integrals

$$\begin{aligned}\int_{-\infty}^{\infty} e^{2xt}\,d\mu &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2xt-(x-m)^2/2\sigma^2}\,dx = e^{2t(m+\sigma^2 t)},\\ \int_{-\infty}^{\infty} xe^{2xt}\,d\mu &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{2xt-(x-m)^2/2\sigma^2}\,dx = (m+2\sigma^2 t)e^{2t(m+\sigma^2 t)}.\end{aligned} \tag{4.41}$$

Thus relation (3.21) gives

$$b_1(t) = m + 2\sigma^2 t \tag{4.42}$$

and so

$$\frac{\dot{\alpha}_1(t)}{\alpha_1(t)} = 0 \quad (\delta = c = 0). \tag{4.43}$$

Then from (1.5), we obtain

$$\alpha_n(t) = \sigma\sqrt{n}, \quad b_n(t) = b_1(t) = m + 2\sigma^2 t, \quad n = 1,2,\ldots. \tag{4.44}$$

As a consequence, the solution of the inverse problem of μ is

$$\alpha_n(0) = \sigma\sqrt{n}, \quad b_n(0) = m, \quad n = 1,2,\ldots. \tag{4.45}$$

Due to Theorem 3.2, the solution of the inverse problem of the measure $\mu\tau^{-1}$ is

$$\alpha_n(0) = \sigma\sqrt{n}, \quad b_n(0) = -m, \quad n = 1,2,\ldots. \tag{4.46}$$

Remark 4.10. In [7], we proved that the Toda lattice has a unique solution $\alpha_n(t)$, $b_n(t)$ provided that the tridiagonal operator $L(0)$ is (essentially) selfadjoint and bounded from above. This means that the support of the measure μ is a set bounded from above. Examples showed that there exist spectral measures μ with support not bounded from above such that the integrals in (3.21) do not exist (see, e.g., the measure μ in Example 4.1 or the measure $\mu\tau^{-1}$ in Examples 4.5, 4.7, and 4.8). What can be said for the integrability of these systems? From (1.6), we see that in these cases the solution $\alpha_n(t)$, $b_n(t)$ has poles for some $t > 0$.

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E. K. Ifantis: Department of Mathematics, University of Patras, 26500 Patras, Greece
E-mail address: ifantis@math.upatras.gr

K. N. Vlachou: Department of Mathematics, University of Patras, 26500 Patras, Greece