SUBDOMINANT POSITIVE SOLUTIONS OF THE DISCRETE EQUATION $\Delta u(k+n) = -p(k)u(k)$

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A delayed discrete equation $\Delta u(k+n) = -p(k)u(k)$ with positive coefficient p is considered. Sufficient conditions with respect to p are formulated in order to guarantee the existence of positive solutions if $k \to \infty$. As a tool of the proof of corresponding result, the method described in the author's previous papers is used. Except for the fact of the existence of positive solutions, their upper estimation is given. The analysis shows that every positive solution of the indicated family of positive solutions tends to zero (if $k \to \infty$) with the speed not smaller than the speed characterized by the function $\sqrt{k} \cdot (n/(n+1))^k$. A comparison with the known results is given and some open questions are discussed.

1. Introduction and motivation

In this contribution, the delayed scalar linear discrete equation

$$\Delta u(k+n) = -p(k)u(k) \tag{1.1}$$

with fixed $n \in \mathbb{N} \setminus \{0\}$, $\mathbb{N} := \{0,1,\ldots\}$, and variable $k \in N(a)$, $N(a) := \{a, a+1, \ldots\}$, $a \in \mathbb{N}$, is considered. The function $p : N(a) \to \mathbb{R}$ is supposed to be positive. We are interested in the existence of positive solutions of (1.1). As a tool of the proof, the method described in [2, 5] is used.

Equation (1.1) can be considered as a discrete analogue of the delayed linear differential equation of the form

$$\dot{x}(t) = -c(t)x(t-\tau) \tag{1.2}$$

with positive coefficient c on $I = [t_0, \infty)$, $t_0 \in \mathbb{R}$, which was considered in many works. We mention at least the books by Győri and Ladas [14] and by Erbe et al. [12] and the papers by Domshlak and Stavroulakis [9], by Elbert and Stavroulakis [11], by Győri and Pituk [16], and by Jaroš and Stavroulakis [18]. Note that close problems were investigated, for example, by Castillo [3], Čermák [4], Kalas and Baráková [19], and Slyusarchuk [22].

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In [6], it was investigated that if (1.2) admits a positive solution $\tilde{x}$ on an interval I, then it admits on I *two positive solutions* x_1 and x_2, satisfying

$$\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 0. \tag{1.3}$$

Moreover, every solution x of (1.2) on I is represented by the formula

$$x(t) = Kx_1(t) + O(x_2(t)), \tag{1.4}$$

where $K \in \mathbb{R}$ depends on x and O is the Landau order symbol. In this formula, the solutions x_1, x_2 can be changed to any couple of positive on I solutions $\tilde{x}_1, \tilde{x}_2$ of (1.2) satisfying the property

$$\lim_{t\to\infty} \frac{\tilde{x}_2(t)}{\tilde{x}_1(t)} = 0 \tag{1.5}$$

(see [6, pages 638-639]). This invariance property led to the following terminology: if (x_1, x_2) is a fixed couple of positive solutions (having the above-indicated properties) of (1.2), then the solution x_1 is called a *dominant solution* and the solution x_2 is called a *subdominant solution.* Subdominant solutions can serve as an analogy to "small solutions" as they are used, for example, in the book by Hale and Verduyn Lunel [17], and dominant solutions express an analogy to the notion of "special solution" which is used in many investigations (see, e.g., Rjabov [20]).

In the present contribution, we will give sufficient conditions for the existence of positive solutions of (1.1). We will discuss known sufficient conditions too, and we will show that our conditions have a more general character than the previous ones. Otherwise the method of the proof of corresponding result permits to express an estimation of the considered positive solution. Taking into account the fact that this solution tends to zero (if $k \to \infty$) with speed not smaller than the speed characterized by the function $\sqrt{k} \cdot (n/(n+1))^k$, we can conclude that this solution is an analogy to the notion of subdominant solution introduced above, in the case of scalar delayed linear differential equations. Moreover, the supporting motivation for the terminology used is the fact that our result does not hold for nondelayed equations of type (1.1), that is, it does not hold if $n = 0$. This is in full accordance with differential equations again, since obviously the subdominant solution does not appear if $\tau = 0$, in (1.2), that is, it does not appear in the case of ordinary differential equations.

2. Preliminary

We consider the scalar discrete equation

$$\Delta u(k+\tilde{n}) = f(k, u(k), u(k+1), \dots, u(k+\tilde{n})), \tag{2.1}$$

where $f(k, u_0, u_1, \dots, u_{\tilde{n}})$ is defined on $N(a) \times \mathbb{R}^{\tilde{n}+1}$, with values in $\mathbb{R}$, $a \in \mathbb{N}$, and $\tilde{n} \in \mathbb{N}$.

Together with the discrete equation (2.1), we consider an initial problem. It is posed as follows: for a given $s \in \mathbb{N}$, we are seeking the solution of (2.1) satisfying $\tilde{n}+1$ initial conditions

$$u(a+s+m) = u^{s+m} \in \mathbb{R}, \quad m = 0,1,\dots,\tilde{n}, \tag{2.2}$$

with prescribed constants u^{s+m}.

We recall that the solution of the initial problem (2.1), (2.2) is defined as an infinite sequence of numbers

$$\begin{aligned}&\{u(a+s) = u^s,\ u(a+s+1) = u^{s+1},\dots,\\ &\ u(a+s+\tilde{n}) = u^{s+\tilde{n}}, u(a+s+\tilde{n}+1), u(a+s+\tilde{n}+2),\dots\}\end{aligned} \tag{2.3}$$

such that, for any $k \in N(a+s)$, equality (2.1) holds.

The existence and uniqueness of the solution of the initial problem (2.1), (2.2) are obvious for every $k \in N(a+s)$. If the function f satisfies the Lipschitz condition with respect to u-arguments, then the initial problem (2.1), (2.2) depends continuously on the initial data [1].

We define, for every $k \in N(a)$, a set $\omega(k)$ as

$$\omega(k) := \{u \in \mathbb{R} : b(k) < u < c(k)\}, \tag{2.4}$$

where $b(k)$, $c(k)$, $b(k) < c(k)$ are real functions defined on $N(a)$.

The following theorem is taken from the investigation in [2].

Theorem 2.1. *Suppose that $f(k,u_0,u_1,\dots,u_{\tilde{n}})$ is defined on $N(a)\times\mathbb{R}^{\tilde{n}+1}$ with values in $\mathbb{R}$ and for all $(k,u_0,u_1,\dots,u_{\tilde{n}}),(k,v_0,v_1,\dots,v_{\tilde{n}}) \in N(a)\times\mathbb{R}^{\tilde{n}+1}$:*

$$\left| f(k,u_0,u_1,\dots,u_{\tilde{n}}) - f(k,v_0,v_1,\dots,v_{\tilde{n}}) \right| \le \lambda(k)\sum_{i=0}^{\tilde{n}} |u_i - v_i|, \tag{2.5}$$

where $\lambda(k)$ is a nonnegative function defined on $N(a)$. If, moreover, the inequalities

$$f(k,u_0,u_1,\dots,u_{\tilde{n}-1},b(k+\tilde{n})) - b(k+\tilde{n}+1) + b(k+\tilde{n}) < 0, \tag{2.6}$$

$$f(k,u_0,u_1,\dots,u_{\tilde{n}-1},c(k+\tilde{n})) - c(k+\tilde{n}+1) + c(k+\tilde{n}) > 0 \tag{2.7}$$

hold for every $k \in N(a)$, every $u_0 \in \omega(k)$, and $u_1 \in \omega(k+1),\dots,u_{\tilde{n}-1} \in \omega(k+\tilde{n}-1)$, then there exists an initial problem

$$u^*(a+m) = u_m^* \in \mathbb{R}, \quad m = 0,1,\dots,\tilde{n}, \tag{2.8}$$

with

$$u_0^* \in \omega(a), u_1^* \in \omega(a+1),\dots,u_n^* \in \omega(a+\tilde{n}) \tag{2.9}$$

such that the corresponding solution $u = u^*(k)$ of (2.1) satisfies the inequalities

$$b(k) < u^*(k) < c(k), \tag{2.10}$$

for every $k \in N(a)$.

3. Existence of subdominant positive solutions

In this section, we prove the existence of a positive solution of (1.1). In the proof of the corresponding theorem (see Theorem 3.2 below), the following elementary lemma concerning asymptotic expansion of the indicated function is necessary. The proof is omitted since it can be done easily with the aid of binomial formula.

Lemma 3.1. *For $k \to \infty$ and fixed $\sigma, d \in \mathbb{R}$, the following asymptotic representation holds:*

$$\left(1+\frac{d}{k}\right)^{\sigma} = 1+\frac{\sigma d}{k}+\frac{\sigma(\sigma-1)d^2}{2k^2}+\frac{\sigma(\sigma-1)(\sigma-2)d^3}{6k^3}+O\left(\frac{1}{k^4}\right). \tag{3.1}$$

Theorem 3.2 (subdominant positive solution). *Let $a \in \mathbb{N}$ and $n \in \mathbb{N}\setminus\{0\}$ be fixed. Suppose that there exists a constant $\theta \in [0,1)$ such that the function $p : N(a) \to \mathbb{R}$ satisfies the inequalities*

$$0 < p(k) \le \left(\frac{n}{n+1}\right)^{n} \cdot \left(\frac{1}{n+1}+\frac{\theta n}{8k^2}\right), \tag{3.2}$$

for every $k \in N(a)$. Then there exist a positive integer $a_1 \ge a$ and a solution $u = u(k)$, $k \in N(a_1)$, of (1.1) such that the inequalities

$$0 < u(k) < \sqrt{k}\cdot\left(\frac{n}{n+1}\right)^{k} \tag{3.3}$$

hold for every $k \in N(a_1)$.

Proof. In the proof, Theorem 2.1 with $\tilde{n} = n$ is used. We define

$$\begin{gathered} f\left(k,u(k),u(k+1),\dots,u(k+n)\right) := -p(k)u(k), \\ b(k) := 0, \qquad c(k) := \sqrt{k}\cdot\left(\frac{n}{n+1}\right)^{k}, \end{gathered} \tag{3.4}$$

for every $k \in N(a)$. In this case (see (2.4)),

$$\omega(k) := \{u \in \mathbb{R} : b(k) < u < c(k)\} \equiv \left\{u \in \mathbb{R} : 0 < u < \sqrt{k}\cdot\left(\frac{n}{n+1}\right)^{k}\right\}. \tag{3.5}$$

Due to the linearity of equation (1.1), the Lipschitz-type condition (2.5) is obviously satisfied with $\lambda(k) \equiv p(k)$. We verify that the inequality of type (2.6) holds. It is easy to see that, for every $k \in N(a)$, $\tilde{n} = n$,

$$f\left(k,u_0,u_1,\dots,u_{n-1},b(k+n)\right) - b(k+n+1)+b(k+n) = -p(k)u_0 < 0 \tag{3.6}$$

since the function p is, by (3.2), positive and u_0 is a positive term too since $u_0 \in \omega(k)$.

We start the verification of inequality (2.7). We get, for sufficiently large $k \in N(a)$ and for $\tilde{n} = n$,

$$\begin{aligned} & f(k, u_0, u_1, \dots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\ & \quad = -p(k)u_0 - \sqrt{k+n+1} \cdot \left(\frac{n}{n+1}\right)^{k+n+1} + \sqrt{k+n} \cdot \left(\frac{n}{n+1}\right)^{k+n}. \end{aligned} \tag{3.7}$$

Since $u_0 \in \omega(k)$, that is,

$$-u_0 > -\sqrt{k} \cdot n^k/(n+1)^k, \quad k \in N(a), \tag{3.8}$$

we get

$$\begin{aligned} & f(k, u_0, u_1, \dots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) \\ & \quad > -p(k)\sqrt{k} \cdot \left(\frac{n}{n+1}\right)^k - \left(\frac{n}{n+1}\right)^k \cdot \left(\frac{n}{n+1}\right)^{n+1} \sqrt{k+n+1} \\ & \quad\quad + \left(\frac{n}{n+1}\right)^k \cdot \left(\frac{n}{n+1}\right)^n \sqrt{k+n} = \mathcal{H}_1 \end{aligned} \tag{3.9}$$

with

$$\mathcal{H}_1 := \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \left[-p(k) - \left(\frac{n}{n+1}\right)^{n+1} \cdot \sqrt{1 + \frac{n+1}{k}} + \left(\frac{n}{n+1}\right)^n \cdot \sqrt{1 + \frac{n}{k}} \right]. \tag{3.10}$$

Now applying formula (3.1) twice, with $\sigma = 1/2$, $d = n+1$, to the expression

$$\sqrt{1 + \frac{n+1}{k}} \tag{3.11}$$

and, with $\sigma = 1/2$, $d = n$, to the expression

$$\sqrt{1 + \frac{n}{k}}, \tag{3.12}$$

we obtain

$$\begin{aligned} \mathcal{H}_1 = & \left(\frac{n}{n+1}\right)^k \sqrt{k} \\ & \times \left[-p(k) - \left(\frac{n}{n+1}\right)^{n+1} \cdot \left(1 + \frac{n+1}{2k} - \frac{(n+1)^2}{8k^2} + \frac{(n+1)^3}{16k^3} + O\left(\frac{1}{k^4}\right)\right) \right. \\ & \quad \left. + \left(\frac{n}{n+1}\right)^n \cdot \left(1 + \frac{n}{2k} - \frac{n^2}{8k^2} + \frac{n^3}{16k^3} + O\left(\frac{1}{k^4}\right)\right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \left[-p(k) - \left(\frac{n}{n+1}\right)^{n+1} + \left(\frac{n}{n+1}\right)^n \right.\\
&\qquad + \frac{1}{k}\left(\frac{-n^{n+1}}{2(n+1)^n} + \frac{n^{n+1}}{2(n+1)^n}\right) + \frac{1}{k^2}\left(\frac{n^{n+1}}{8(n+1)^{n-1}} - \frac{n^{n+2}}{8(n+1)^n}\right)\\
&\qquad \left. + \frac{1}{k^3}\left(\frac{-n^{n+1}}{16(n+1)^{n-2}} + \frac{n^{n+3}}{16(n+1)^n}\right) + O\left(\frac{1}{k^4}\right)\right]\\
&= \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \left[-p(k) + \left(\frac{n}{n+1}\right)^n \frac{-n+n+1}{n+1} + \frac{1}{k^2}\frac{n^{n+1}(n+1) - n^{n+2}}{8(n+1)^n} \right.\\
&\qquad \left. + \frac{1}{k^3}\frac{-n^{n+1}(n+1)^2 + n^{n+3}}{16(n+1)^n} + O\left(\frac{1}{k^4}\right)\right] = \mathcal{H}_2
\end{aligned}
\tag{3.13}
$$

with

$$
\begin{aligned}
\mathcal{H}_2 := \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \left[-p(k) + \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} + \frac{1}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n \right.\\
\left. + \frac{1}{16k^3}\frac{-2n^{n+2} - n^{n+1}}{(n+1)^n} + O\left(\frac{1}{k^4}\right)\right].
\end{aligned}
\tag{3.14}
$$

Due to inequality (3.2), we obtain that

$$
\begin{aligned}
\mathcal{H}_2 &\geq \left(\frac{n}{n+1}\right)^k \sqrt{k}\\
&\quad \cdot \left[-\left(\frac{n}{n+1}\right)^n \cdot \left(\frac{1}{n+1} + \frac{\theta n}{8k^2}\right) + \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} \right.\\
&\quad \left. + \frac{1}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n + \frac{1}{16k^3}\frac{-2n^{n+2} - n^{n+1}}{(n+1)^n} + O\left(\frac{1}{k^4}\right)\right]\\
&= \left(\frac{n}{n+1}\right)^k \sqrt{k} \cdot \mathcal{H}_3
\end{aligned}
\tag{3.15}
$$

with

$$
\mathcal{H}_3 := \frac{1-\theta}{8k^2}\left(\frac{n}{n+1}\right)^n \cdot n - \frac{1}{16k^3}\frac{n^{n+1}(1+2n)}{(n+1)^n} + O\left(\frac{1}{k^4}\right). \tag{3.16}
$$

Now, it is obvious that there exists an integer $a_1 \geq a$ such that the inequality $\mathcal{H}_3 > 0$ holds for every $k \in N(a_1)$. Consequently,

$$
f(k, u_0, u_1, \ldots, u_{n-1}, c(k+n)) - c(k+n+1) + c(k+n) > 0, \tag{3.17}
$$

that is, inequality (2.7) holds for every $k \in N(a_1)$. So, all the suppositions of Theorem 2.1 are met with $a := a_1$, $\tilde{n} = n$. Then, following its affirmation, there exists an initial problem

$$u^*(a_1+m) = u_m^* \in \mathbb{R}, \quad m = 0,1,\dots,n, \tag{3.18}$$

with

$$u_0^* \in \omega(a_1), u_1^* \in \omega(a_1+1),\dots,u_n^* \in \omega(a_1+n) \tag{3.19}$$

such that the corresponding solution $u = u^*(k)$ of (1.1) satisfies the inequalities

$$b(k) = 0 < u^*(k) < c(k) = \sqrt{k}\cdot\left(\frac{n}{n+1}\right)^k, \tag{3.20}$$

for every $k \in N(a_1)$, that is, (3.3) holds. The theorem is proved. □

4. Comparisons and concluding remarks

We remark that analogous (in a sense) problems are discussed, for example, in [10, 13, 14, 15, 21]. The following known result (see [14, page 192]) will be formulated with a notation adapted with respect to our notation.

Theorem 4.1. *Assume $n \in \mathbb{N}\setminus\{0\}$, $p(k) > 0$ for $k \geq 0$, and*

$$p(k) \leq \frac{n^n}{(n+1)^{n+1}}. \tag{4.1}$$

Then the difference equation (1.1), where $k = 0,1,2,\dots$, has a positive solution

$$\{u(0),u(1),u(2),\dots\}. \tag{4.2}$$

Comparing this result with the result given by Theorem 3.2, we conclude that inequality (3.2) is a substantial improvement over (4.1) since the choice $\theta = 0$ in (3.2) gives inequality (4.1). Moreover, inequality (3.2), unlike inequality (4.1), involves the variable k on the right-hand side. As noted in [14, page 179], for $p(k) \equiv p = \text{const}$, inequality (4.1) is sharp in a sense, since in this case the necessary and sufficient condition for the oscillation of all solutions of (1.1) is the inequality

$$p > \frac{n^n}{(n+1)^{n+1}}. \tag{4.3}$$

Inequality (3.2) can be considered as a discrete analogy of the inequality

$$c(t) \leq \frac{1}{e} + \frac{1}{8et^2} \tag{4.4}$$

(t is supposed to be sufficiently large) used in [11, Theorem 3], in order to give a guarantee of the existence of a positive solution of (1.2).

5. Open questions

We indicate problems, still unsolved , whose solution will lead to progress in the considered theory.

Open Question 5.1. Does the affirmation of Theorem 2.1 remain valid if $\theta = 1$? In other words, can inequality (3.2) be replaced by a weaker one

$$0 < p(k) \leq \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{1}{n+1} + \frac{n}{8k^2}\right)? \tag{5.1}$$

Open Question 5.2. As a motivation for the following problem, we state this known fact: equation (1.1) with "limiting" value of coefficient (corresponding to $\theta = 0$), that is, the equation

$$\Delta u(k+n) = -\frac{n^n}{(n+1)^{n+1}} \cdot u(k), \tag{5.2}$$

admits two positive and asymptotically noncomparable solutions: a dominant one (we use a similar terminology as involved in Section 1)

$$u_1(k) = k \cdot \left(\frac{n}{n+1}\right)^k \tag{5.3}$$

and a subdominant one

$$u_2(k) = \left(\frac{n}{n+1}\right)^k, \tag{5.4}$$

since

$$\lim_{k\to\infty} \frac{u_2(k)}{u_1(k)} = \lim_{k\to\infty} \frac{1}{k} = 0. \tag{5.5}$$

In this connection, the next problem arises: is it possible to prove (under the same conditions as indicated in Theorem 3.2) the existence of the second solution $u^\star(k)$ of the equation

$$\lim_{k\to\infty} \frac{u(k)}{u^\star(k)} = 0? \tag{5.6}$$

In other words, is the couple of solutions $u^\star(k)$ and $u(k)$ a couple of dominant and subdominant solutions?

Open Question 5.3. Together with the investigation of linear discrete problems, the development of methods for the investigation of nonlinear discrete problems is a very important problem too. Is it, for example, possible (based on the similarity of continuous and discrete methods) to obtain analogies of the results of the investigation of singular problems for ordinary differential equations performed in [7, 8] in the discrete case?

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References

[1] R. P. Agarwal, *Difference equations and inequalities. Theory, methods, and applications*, 2nd ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 228, Marcel Dekker, New York, 2000.

[2] J. Baštinec, J. Diblík, and B. Zhang, *Existence of bounded solutions of discrete delayed equations*, Proceedings of the International Conference on Computers and Mathematics (Augsburg, 2001), CRC Press, in press.

[3] S. Castillo, *Asymptotic formulae for solutions of linear functional-differential systems*, Funct. Differ. Equ. **6** (1999), no. 1-2, 55–68.

[4] J. Čermák, *The asymptotic bounds of solutions of linear delay systems*, J. Math. Anal. Appl. **225** (1998), no. 2, 373–388.

[5] J. Diblík, *Discrete retract principle for systems of discrete equations*, Comput. Math. Appl. **42** (2001), no. 3–5, 515–528.

[6] J. Diblík and N. Koksch, *Positive solutions of the equation $\dot{x}(t) = -c(t)x(t-\tau)$ in the critical case*, J. Math. Anal. Appl. **250** (2000), no. 2, 635–659.

[7] J. Diblík and M. Růžičková, *Existence of positive solutions of a singular initial problem for nonlinear system of differential equations*, to appear in Rocky Mountain J. Math.

[8] ———, *Existence of positive solutions of n-dimensional system of nonlinear differential equations entering into a singular point*, Arch. Math. (Brno) **36** (2000), no. suppl., 435–446.

[9] Y. Domshlak and I. P. Stavroulakis, *Oscillations of first-order delay differential equations in a critical state*, Appl. Anal. **61** (1996), no. 3-4, 359–371.

[10] S. N. Elaydi, *An Introduction to Difference Equations*, 2nd ed., Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1999.

[11] Á. Elbert and I. P. Stavroulakis, *Oscillation and nonoscillation criteria for delay differential equations*, Proc. Amer. Math. Soc. **123** (1995), no. 5, 1503–1510.

[12] L. H. Erbe, Q. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 190, Marcel Dekker, New York, 1995.

[13] L. H. Erbe and B. G. Zhang, *Oscillation of discrete analogues of delay equations*, Differential Integral Equations **2** (1989), no. 3, 300–309.

[14] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991.

[15] I. Győri and M. Pituk, *Asymptotic formulae for the solutions of a linear delay difference equation*, J. Math. Anal. Appl. **195** (1995), no. 2, 376–392.

[16] ———, *L^2-perturbation of a linear delay differential equation*, J. Math. Anal. Appl. **195** (1995), no. 2, 415–427.

[17] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, Applied Mathematical Sciences, vol. 99, Springer-Verlag, New York, 1993.

[18] J. Jaroš and I. P. Stavroulakis, *Oscillation tests for delay equations*, Rocky Mountain J. Math. **29** (1999), no. 1, 197–207.

[19] J. Kalas and L. Baráková, *Stability and asymptotic behaviour of a two-dimensional differential system with delay*, J. Math. Anal. Appl. **269** (2002), no. 1, 278–300.

[20] Ju. A. Rjabov, *Certain asymptotic properties of linear systems with small time-lag*, Trudy Sem. Teor. Differencial. Uravneniĭ s Otklon. Argumentom Univ. Družby Narodov Patrisa Lumumby **3** (1965), 153–164 (Russian).
[21] J. Shen and I. P. Stavroulakis, *Oscillation criteria for delay difference equations*, Electron. J. Differential Equations **2001** (2001), no. 10, 1–15.
[22] V. E. Slyusarchuk, *Necessary and sufficient conditions for the oscillation of solutions of nonlinear differential equations with pulse influence in the Banach space*, Ukrainian Math. J. **51** (1999), no. 1, 107–118.

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