# QUASILINEAR DEGENERATE ELLIPTIC UNILATERAL PROBLEMS

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We will be concerned with the existence result of a degenerate elliptic unilateral problem of the form $Au + H(x,u,\nabla u) = f$, where $A$ is a Leray-Lions operator from $W^{1,p}(\Omega,w)$ into its dual. On the nonlinear lower-order term $H(x,u,\nabla u)$, we assume that it is a Carathéodory function having natural growth with respect to $|\nabla u|$, but without assuming the sign condition. The right-hand side $f$ belongs to $L^1(\Omega)$.

## 1. Introduction

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, let $p$ be a real number such that $1 < p < \infty$, and let $w = \{w_i(x),\ 0 \le i \le N\}$ be a vector of weight functions on $\Omega$, that is, each $w_i(x)$ is a measurable a.e. strictly positive function on $\Omega$, satisfying some integrability conditions. And let $Au = -\operatorname{div}(a(x,u,\nabla u))$ be a Leray-Lions operator defined from the weighted Sobolev space $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$.

The aim of this paper is to study the degenerate unilateral problems associated to a nonlinear operator of the form

$$\begin{gathered}-\operatorname{div}\big(a(x,u,\nabla u)\big) + H(x,u,\nabla u) = f \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{1.1}$$

where $H$ is a nonlinear lower-order term having natural growth with respect to $|\nabla u|$. With respect to $|u|$ we do not assume any growth restrictions (i.e., $H(x,s,\xi) \le \gamma(x) + g(s)\sum_{i=1}^N w_i|\xi_i|^p$). The right-hand side $f$ belongs to $L^1(\Omega)$. More precisely, we prove the existence of solutions for the following nonlinear Dirichlet problems:

$$\begin{gathered}u \ge \psi \quad \text{a.e. in } \Omega,\\ T_k(u) \in W_0^{1,p}(\Omega,w) \quad \forall k > 0,\\ \int_\Omega \big(a(x,u,\nabla u)\big) T_k(u-v) + \int_\Omega H(x,u,\nabla u) T_k(u-v)\\ \le \int_\Omega f\, T_k(u-v) \quad \forall v \in K_\psi \cap L^\infty(\Omega),\end{gathered} \tag{P}$$


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where $K_\psi = \{u \in W_0^{1,p}(\Omega, w),\ u \geq \psi \text{ a.e. in } \Omega\}$ and $T_k$ is the usual truncation operator. Note that the existence result is proved without assuming the sign condition $H(x,s,\xi)s \geq 0$. For that we prove the strong convergence of the truncations $T_k(u_n)$ in $W_0^{1,p}(\Omega, w)$, where $u_n$ is a solution of the approximate problem. If we take $\psi = -\infty$, we obtain the existence result of problem (1.1) in the case of equation.

Recently in [9] Porretta studied problem (1.1) in the classical Sobolev space $W_0^{1,p}(\Omega)$ where the right-hand side is a measure. We point out that another work in this direction can be found in [6] where problem (1.1) is studied with $f \in L^m(\Omega)$, for which the authors have proved that there exists a bounded weak solution for $m > (N/2)$, and an unbounded entropy solution for $(N/2) > m > (2N/(N+2))$. A different approach (without using the sign condition) was introduced also in [5] when $b(x,s,\xi) = \lambda s - |\xi|^2$ with $\lambda > 0$. Our paper can be seen as a generalization of [9] in the weighted case and as a continuation of [1, 2, 3, 4] where in [1] the case of degenerate variational equation is treated and in [2, 3] the variational degenerate inequalities are studied, while in [4] the authors were concerned with problem (1.1) with the right-hand side $f$ assumed to belong either to $W^{-1,p'}(\Omega, w^*)$ or to $L^1(\Omega)$. In [1, 2, 3, 4] we suppose that the lower-order term satisfies the sign condition $H(x,s,\xi)s \geq$ for all $s \in \mathbb{R}$, while in [4] (where $f \in L^1(\Omega)$), we have assumed also the exact natural growth, that is, $|H(x,s,\xi)| \geq \rho \sum_{i=1}^N w_i |\xi_i|^p$.

## 2. Preliminaries

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$. Let $1 < p < \infty$, and let $w = \{w_i(x);\ i = 1,\dots,N\}$, $0 \leq i \leq N$, be a vector of weight functions, that is, every component $w_i(x)$ is a measurable function which is strictly positive a.e. in $\Omega$. Further, we suppose in all our considerations that, for $0 \leq i \leq N$,

$$w_i \in L^1_{\text{loc}}(\Omega), \qquad w_i^{-1/(p-1)} \in L^1_{\text{loc}}(\Omega). \tag{2.1}$$

We define the weighted space with weight $\gamma$ in $\Omega$ as

$$L^p(\Omega, \gamma) = \{u(x),\ u\gamma^{1/p} \in L^p(\Omega)\}, \tag{2.2}$$

which is endowed with the norm

$$\|u\|_{p,\gamma} = \left( \int_\Omega |u(x)|^p \gamma(x)\,dx \right)^{1/p}. \tag{2.3}$$

We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \forall i = 1,\dots,N. \tag{2.4}$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left( \int_\Omega |u(x)|^p w_0\, dx + \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{1/p}. \tag{2.5}$$

To deal with the Dirichlet problem, we use the space

$$X = W_0^{1,p}(\Omega, w), \tag{2.6}$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.5). Note that $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|\cdot\|_{1,p,w})$ is a reflexive Banach space.

We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}$, $i = 1,\dots,N$, and $p'$ is the conjugate of $p$, that is, $p' = p/(p-1)$. For more details we refer the reader to [7].

## 3. Main general results

**3.1. Basic assumptions and statement of result.** We state the following assumptions.

$(H_1)$ The expression

$$|||u|||_X = \left( \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{1/p}, \tag{3.1}$$

is a norm defined on $X$ and is equivalent to the norm (2.5). Note that $(X, \|u\|_X)$ is a uniformly convex (and reflexive) Banach space.

$(H_2)$ There exist a weight function $\sigma$ on $\Omega$ and a parameter $q$, satisfying

$$1 < q < p + p', \tag{3.2}$$

$$\sigma^{1-q'} \in L^1_{\mathrm{loc}}(\Omega), \tag{3.3}$$

with $q' = q/(q-1)$ and such that the Hardy inequality

$$\left( \int_\Omega |u|^q \sigma(x) dx \right)^{1/q} \leq C \left( \sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^p w_i(x) dx \right)^{1/p} \tag{3.4}$$

holds for every $u \in X$ with a constant $C > 0$ independent of $u$. Moreover, the imbedding

$$W_0^{1,p}(\Omega, w) \hookrightarrow \hookrightarrow L^q(\Omega, \sigma) \tag{3.5}$$

determined by inequality (3.4) is compact.

On the other hand, we consider the nonlinear elliptic differential operator in divergence form, defined from $W_0^{1,p}(\Omega,w)$ into its dual $W^{-1,p'}(\Omega,w^*)$ by

$$Au = -\operatorname{div}(a(x,u,\nabla u)), \tag{3.6}$$

where $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ is a Carathéodory function satisfying the following conditions.

For all $s\in\mathbb{R}$, $\xi,\eta\in\mathbb{R}^N$ and for almost every $x\in\Omega$,

$$|a_i(x,s,\xi)| \le \beta w_i^{1/p}(x)\left[k(x)+\sigma^{1/p'}|s|^{q/p'}+\sum_{j=1}^{N} w_j^{1/p'}(x)|\xi_j|^{p-1}\right] \quad \text{for } i=1,\dots,N, \tag{3.7}$$

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)>0 \quad \forall \xi\neq\eta, \tag{3.8}$$

$$a(x,s,\xi)\xi \ge \alpha\sum_{i=1}^{N} w_i(x)|\xi_i|^p, \tag{3.9}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and $\alpha$, $\beta$ are positive constants.

Furthermore, let $H(x,s,\xi):\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, $\xi\in\mathbb{R}^N$, the growth condition

$$|H(x,s,\xi)| \le \gamma(s)+g(s)\sum_{i=1}^{N} w_i(x)|\xi_i|^p \tag{3.10}$$

is satisfied, where $g:\mathbb{R}\to\mathbb{R}^+$ is a continuous positive function that belongs to $L^1(\mathbb{R})$, while $\gamma(x)$ belongs to $L^1(\Omega)$.

Finally, let the convex set

$$K_\psi = \{u\in W_0^{1,p}(\Omega,w),\ u\ge\psi \text{ a.e. in } \Omega\}. \tag{3.11}$$

We will prove the following existence theorem.

Theorem 3.1. *Assume that the assumptions* $(H_1)$, $(H_2)$, *and (3.7)–(3.11) hold and let* $f$ *belong to* $L^1(\Omega)$. *Then, there exists a measurable function* $u$ *which is a solution of the following problem:*

$$\begin{gathered} u>\psi \quad \text{a.e. in } \Omega, \\ T_k(u)\in W_0^{1,p}(\Omega,w), \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx+\int_\Omega H(x,u,\nabla u)T_k(u-v)\,dx \\ \le \int_\Omega f T_k(u-v)\,dx \quad \forall v\in K_\psi\cap L^\infty(\Omega)\ \forall k>0. \end{gathered} \tag{P}$$

*Remark 3.2.* The statement of Theorem 3.1 generalizes in the weighted case the analogous one in [9] with $\mu\in L^1(\Omega)$ (where the case of equation is treated).

*Remark 3.3.* We remark that in the case of $\psi = -\infty$, Theorem 3.1 gives the existence of solution in the case of equation, that is, the problem

$$\begin{gathered} T_k(u) \in W_0^{1,p}(\Omega,w), \quad k>0, \\ \int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)\,dx + \int_\Omega H(x,u,\nabla u)T_k(u-v)\,dx \\ \leq \int_\Omega f\,T_k(u-v)\,dx \quad \forall v \in W_0^{1,p}(\Omega,w)\cap L^\infty(\Omega) \end{gathered} \tag{3.12}$$

has at least one solution.

**3.2. Approximate problem.** Let $\Omega_n$ be a sequence of compact subsets of $\Omega$ such that $\Omega_n$ is increasing to $\Omega$ as $n \to \infty$.

We consider the sequence of approximate problem

$$\begin{gathered} u_n \in K_\psi, \\ \int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n-v)\,dx + \int_\Omega H_n(x,u_n,\nabla u_n)(u_n-v)\,dx \\ \leq \int_\Omega f_n(u_n-v)\,dx \quad \forall v \in K_\psi, \end{gathered} \tag{$\mathcal{P}_n$}$$

where $f_n$ are regular functions which strongly converge to $f$ in $L^1(\Omega)$ and $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$ and

$$H_n(x,s,\xi) = \frac{H(x,s,\xi)}{1+(1/n)\,|H(x,s,\xi)|}\chi_{\Omega_n}, \tag{3.13}$$

where $\chi_{\Omega_n}$ is the characteristic function of $\Omega_n$.

Note that $|H_n(x,s,\xi)| \leq |H(x,s,\xi)|$ and $|H_n(x,s,\xi)| \leq n$.

Proposition 3.4. *Assume that the assumptions* $(H_1)$, $(H_2)$, *and (3.7)–(3.11) hold. Then, the problem* ($\mathcal{P}_n$) *has at least one solution* $u_n \in K_\psi$.

*Proof.* We define the operator $G_n : X \to X^*$ by

$$\langle G_n u, v\rangle = \int_\Omega H_n(x,u,\nabla u)v\,dx. \tag{3.14}$$

Thanks to Hölder's inequality, we have, for all $u \in X$ and all $v \in X$,

$$\begin{aligned} \left|\int_\Omega H_n(x,u,\nabla u)v\,dx\right| &\leq \left(\int_\Omega |H_n(x,u,\nabla u)|^{q'}\sigma^{-q'/q}dx\right)^{1/q'}\left(\int_\Omega |v|^q\sigma\,dx\right)^{1/q} \\ &\leq n\left(\int_{\Omega_n}\sigma^{1-q'}dx\right)^{1/q'}\|v\|_{q,\sigma} \\ &\leq c_n|||v|||. \end{aligned} \tag{3.15}$$

The last inequality is due to (3.3) and (3.5). Consequently, in view of [3, Lemma 4.1], we deduce that the operator $B_n = A + G_n$ is pseudomonotone.

On the other hand, we show that $B_n$ is coercive, in the following sense: there exists $v_0 \in K_\psi$ such that

$$\frac{\langle B_n v, v - v_0\rangle}{|||v|||} \longrightarrow +\infty \quad \text{if } |||v||| \longrightarrow \infty,\ v \in K_\psi. \tag{3.16}$$

Let $v_0 \in K_\psi$. From Hölder's inequality, the growth condition (3.7), and the compact imbedding (3.5), we have

$$\begin{aligned}
\langle Av, v_0\rangle &= \sum_{i=1}^{N} \int_\Omega a_i(x, v, \nabla v)\frac{\partial v_0}{\partial x_i}dx \\
&\le \sum_{i=1}^{N} \left(\int_\Omega \left|a_i(x, v, \nabla v)\right|^{p'} w_i^{-p'/p} dx\right)^{1/p'} \left(\int_\Omega \left|\frac{\partial v_0}{\partial x_i}\right|^p w_i\, dx\right)^{1/p} \\
&\le c_1 |||\, v_0\, |||\left(\int_\Omega k(x)^{p'} + |v|^q\sigma + \sum_{j=1}^{N}\left|\frac{\partial v}{\partial x_j}\right|^p w_j\, dx\right)^{1/p'} \\
&\le c_2\left(c_3 + |||v|||^{q/p'} + |||v|||^{p-1}\right),
\end{aligned} \tag{3.17}$$

where $c_i$ are various constants.

Thanks to (3.9), we obtain

$$\frac{\langle Av, v\rangle}{|||v|||} - \frac{\langle Av, v_0\rangle}{|||v|||} \ge \alpha|||v|||^{p-1} - |||v|||^{p-2} - |||v|||^{(q/p')-1} - \frac{c}{|||v|||}. \tag{3.18}$$

In view of (3.2), we have $p - 1 > (q/p') - 1$, then,

$$\frac{\langle Av, v - v_0\rangle}{|||v|||} \longrightarrow \infty \quad \text{as } |||v||| \longrightarrow \infty. \tag{3.19}$$

On the other hand,

$$\begin{aligned}
\left|\langle G_n v, v\rangle\right| &= \left|\int_{\Omega_n} H_n(x, v, \nabla v) v\, dx\right| \\
&\le \left(\int_{\Omega_n} \left|H_n(x, v, \nabla v)\right|^{q'} \sigma^{-q'/q} dx\right)^{1/q'} \left(\int_{\Omega_n} |v|^q \sigma\, dx\right)^{1/q} \\
&\le n\left(\int_{\Omega_n} \sigma^{-q'/q} dx\right)^{1/q'} |||v||| \\
&\le c_n |||v|||,
\end{aligned} \tag{3.20}$$

hence (since $\langle G_n v, v\rangle / |||v|||$ and $\langle G_n v, v_0\rangle$ are bounded), we have

$$\frac{\langle B_n v, v - v_0\rangle}{|||v|||} = \frac{\langle Av, v - v_0\rangle}{|||v|||} + \frac{\langle G_n v, v\rangle}{|||v|||} - \frac{\langle G_n v, v_0\rangle}{|||v|||} \longrightarrow \infty \quad \text{as } |||v||| \to \infty. \tag{3.21}$$

Finally, $B_n$ is pseudomonotone and coercive, hence by using [8, Chapter 2, Theorem 8.2], the approximate problem $(\mathcal{P}_n)$ has at least one solution. □

### 3.3. A priori estimate

Proposition 3.5. *Assume that the assumptions* $(H_1)$, $(H_2)$, *and (3.7)–(3.11) hold, and let* $u_n$ *be a solution of the approximate problem* $(\mathcal{P}_n)$. *Then, there exists a constant c (which does not depend on the n and k) such that*

$$\int_\Omega \sum_{i=1}^N \left| \frac{\partial T_k(u_n)}{\partial x_i} \right|^p w_i\, dx \le ck, \tag{3.22}$$

*where* $k > 0$.

*Proof.* Let $v = u_n - \eta \exp(G(u_n)) T_k(u_n^+ - \psi^+)$, where $\eta$ is a real positive and $G(s) = \int_0^s (g(t)/\alpha)\, dt$ (note that the function $g$ is the one that appeared in (3.10)). Since $v \in W_0^{1,p}(\Omega, w)$ and for $\eta$ small enough, we have $v \ge \psi$, thus $v$ is an admissible test function in $(\mathcal{P}_n)$, then

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla(\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+))\, dx \\
&\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)(\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+))\, dx \\
&\quad \le \int_\Omega f_n(\eta \exp(G(u_n)) T_k(u_n^+ - \psi^+))\, dx
\end{aligned} \tag{3.23}$$

which implies that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla(\exp(G(u_n)) T_k(u_n^+ - \psi^+))\, dx \\
&\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)\exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx \\
&\quad \le \int_\Omega f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx.
\end{aligned} \tag{3.24}$$

Then,

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha} \exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx \\
&\qquad + \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^+ - \psi^+)\exp(G(u_n))\, dx \\
&\quad \le -\int_\Omega H_n(x,u_n,\nabla u_n)\exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx \\
&\qquad + \int_\Omega f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx \\
&\quad \le \int_\Omega \gamma(x)\exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx \\
&\qquad + \int_\Omega g(u_n)\sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i \exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx \\
&\qquad + \int_\Omega f_n \exp(G(u_n)) T_k(u_n^+ - \psi^+)\, dx.
\end{aligned} \tag{3.25}$$

In view of (3.9), we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^+-\psi^+)\exp(G(u_n))\,dx\\ &\le \int_\Omega \gamma(x)\exp(G(u_n))T_k(u_n^+-\psi^+)\,dx\\ &\quad+\int_\Omega f_n\exp(G(u_n))T_k(u_n^+-\psi^+)\,dx\\ &\le c_1k,\end{aligned}\tag{3.26}$$

where $c_1$ is a positive constant and does not depend on $n$.

Consequently, we have

$$\begin{aligned}\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n^+\exp(G(u_n))\,dx\\ \le \int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n)\nabla\psi^+\exp(G(u_n))\,dx+c_1k.\end{aligned}\tag{3.27}$$

Thanks to (3.9) and Young's inequality, we obtain

$$\int_{\{|u_n^+-\psi^+|\le k\}}\sum_{i=1}^N\left|\frac{\partial u_n^+}{\partial x_i}\right|^p w_i\,dx\le c_2k.\tag{3.28}$$

Since $\{x\in\Omega,\ |u_n^+|\le k\}\subset\{x\in\Omega,\ |u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}$, hence

$$\begin{aligned}&\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k(u_n^+)}{\partial x_i}\right|^p w_i\,dx\\ &\quad=\int_{\{|u_n^+|\le k\}}\sum_{i=1}^N\left|\frac{\partial u_n^+}{\partial x_i}\right|^p w_i\,dx\le\int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}}\sum_{i=1}^N\left|\frac{\partial u_n^+}{\partial x_i}\right|^p w_i\,dx,\end{aligned}\tag{3.29}$$

from which in addition to (3.10) we deduce that

$$\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k(u_n^+)}{\partial x_i}\right|^p w_i\,dx\le c_3k\quad\forall k>0,\tag{3.30}$$

where $c_3$ is a positive constant.

On the other hand, taking $v=u_n+\exp(-G(u_n))T_k(u_n^-)$ as the test function in $(\mathcal{P}_n)$, we obtain

$$\begin{aligned}-\int_\Omega a(x,u_n,\nabla u_n)\nabla(\eta\exp(-G(u_n))T_ku_n^-)\,dx\\ -\int_\Omega H_n(x,u_n,\nabla u_n)(\eta\exp(-G(u_n))T_k(u_n^-))\,dx\\ \le-\int_\Omega f_n(\eta\exp(-G(u_n))T_k(u_n^-))\,dx.\end{aligned}\tag{3.31}$$

Using (3.10), we can write

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(-G(u_n))T_k(u_n^-)\,dx \\
&\quad -\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))\,dx \\
&\le \int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n^-)\,dx \\
&\quad +\int_\Omega g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i\exp(-G(u_n))T_k(u_n^-)\,dx \\
&\quad -\int_\Omega f_n\exp(-G(u_n))T_k(u_n^-)\,dx.
\end{aligned} \tag{3.32}$$

In virtue of (3.9) and since the functions $\gamma$ and $f_n$ lie in $L^1(\Omega)$ with $\|f_n\|_{L^1(\Omega)} \le \|f\|_{L^1(\Omega)}$, we get

$$\begin{aligned}
&-\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n^-)\exp(-G(u_n))\,dx \\
&\quad =\int_{\{u_n\le 0\}} a(x,u_n,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))\,dx \\
&\quad \le c_3k.
\end{aligned} \tag{3.33}$$

By using, again, (3.9), we deduce that

$$\int_{\{u_n\le 0\}}\sum_{i=1}^N\left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p w_i\,dx \le c_3k, \tag{3.34}$$

that is,

$$\int_\Omega\sum_{i=1}^N\left|\frac{\partial T_k(u_n^-)}{\partial x_i}\right|^p w_i\,dx \le c_4k, \tag{3.35}$$

where $c_4$ is a positive constant.

Combining (3.30) and (3.35), we conclude (3.22). □

### 3.4. Strong convergence of truncations

PROPOSITION 3.6. *Let $u_n$ be solutions of the problems $(\mathcal{P}_n)$, then there exists a measurable function $u$ such that*

$$T_k(u_n) \longrightarrow T_k(u) \quad \textit{strongly in } W_0^{1,p}(\Omega,w). \tag{3.36}$$

*Proof.*

*Step 1.* We prove that $u_n$ converges to some function $u$ locally in measure (and therefore we can always assume that the convergence is a.e. in $\Omega$ after passing to a suitable subsequence). We will show that $u_n$ is a Cauchy sequence in measure in any ball $B_R$.

On the one hand, thanks to (3.22), we have

$$\begin{aligned} T_k(u_n) &\rightharpoonup v_k \quad \text{weakly in } W_0^{1,p}(\Omega,w), \\ T_k(u_n) &\longrightarrow v_k \quad \text{strongly in } L^q(\Omega,\sigma) \text{ and a.e. in } \Omega, \end{aligned} \tag{3.37}$$

for some function $v_k \in W_0^{1,p}(\Omega,w)$.

On the other hand, letting $k>0$ large enough, we have

$$\begin{aligned} k\,\text{meas}(\{|u_n|>k\}\cap B_R) &= \int_{\{\{|u_n|>k\}\cap B_R\}} |T_k(u_n)|\,dx \le \int_{B_R} |T_k(u_n)|\,dx \\ &\le \left(\int_\Omega |T_k(u_n)|^q \sigma\,dx\right)^{1/q} \left(\int_{B_R} \sigma^{1-q'}\,dx\right)^{1/q'} \\ &\le c_R \left(\int_\Omega \sum_{i=1}^N \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p w_i(x)dx\right)^{1/p} \quad (\text{due to } (3.5)) \\ &\le ck^{1/p} \quad (\text{due to } (3.22)) \end{aligned} \tag{3.38}$$

which implies that

$$\text{meas}(\{|u_n|>k\}\cap B_R) \le \frac{c}{k^{1-(1/p)}} \quad \forall k>1. \tag{3.39}$$

For every $\delta>0$, we have

$$\begin{aligned} &\text{meas}(\{|u_n-u_m|>\delta\}\cap B_R) \\ &\quad\le \text{meas}(\{|u_n|>k\}\cap B_R) + \text{meas}(\{|u_m|>k\}\cap B_R) \\ &\qquad + \text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}. \end{aligned} \tag{3.40}$$

Let $\varepsilon>0$, combining (3.37), (3.39), and (3.40), we can deduce that there exists some $k(\varepsilon)>0$ such that $\text{meas}(\{|u_n-u_m|>\delta\}\cap B_R)<\varepsilon$ for all $n,m\ge n_0(k(\varepsilon),\delta,R)$. This proves that $(u_n)$ is a Cauchy sequence in measure in $B_R$, thus it converges almost everywhere to some measurable function $u$. Then by using (3.37), we have

$$\begin{aligned} T_k(u_n) &\rightharpoonup T_k(u) \quad \text{weakly in } W_0^{1,p}(\Omega,w), \\ T_k(u_n) &\longrightarrow T_k(u) \quad \text{strongly in } L^q(\Omega,\sigma) \text{ and a.e. in } \Omega. \end{aligned} \tag{3.41}$$

*Step 2.* We claim that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\le|u_n|\le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{3.42}$$

Consider the function $v = u_n - \eta\exp(G(u_n))T_1(u_n-T_j(u_n))^+$.

For $j$ large enough and $\eta$ small enough, we can deduce that $v \geq \psi$, and since $v \in W_0^{1,p}(\Omega, w)$, hence $v$ is a test function in $(\mathcal{P}_n)$. Then, we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla\Big(\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+\Big)dx \\
&\qquad+\int_\Omega H_n(x,u_n,\nabla u_n)\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx \\
&\quad\leq\int_\Omega f_n\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx.
\end{aligned}\tag{3.43}$$

From the growth condition (3.10), we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{g(u_n)}{\alpha}\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx \\
&\qquad+\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1\big(u_n-T_j(u_n)\big)^+\exp\big(G(u_n)\big)dx \\
&\quad\leq\int_\Omega \gamma(x)\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx \\
&\qquad+\int_\Omega g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_i\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx \\
&\qquad+\int_\Omega f_n\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx,
\end{aligned}\tag{3.44}$$

which gives by using (3.9) the following inequality:

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1\big(u_n-T_j(u_n)\big)^+\exp\big(G(u_n)\big)dx \\
&\quad\leq\int_\Omega \gamma(x)\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx \\
&\qquad+\int_\Omega f_n\exp\big(G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^+dx.
\end{aligned}\tag{3.45}$$

By Lebesgue's theorem the right-hand side goes to zero as $n$ and $j$ tend to infinity. Therefore, passing to the limit firstly in $n$ and secondly in $j$, we obtain from (3.45)

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{j\leq u_n\leq j+1\}}a(x,u_n,\nabla u_n)\nabla u_n\,dx=0.\tag{3.46}$$

On the other hand, taking $v=u_n+\exp(-G(u_n))T_1(u_n-T_j(u_n))^-$ in $(\mathcal{P}_n)$ (which is an admissible test function), we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla\big(-\exp\big(-G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^-\big)dx \\
&\qquad+\int_\Omega H_n(x,u_n,\nabla u_n)\big(-\exp\big(-G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^-\big)dx \\
&\quad\leq\int_\Omega f_n\big(-\exp\big(-G(u_n)\big)T_1\big(u_n-T_j(u_n)\big)^-\big)dx,
\end{aligned}\tag{3.47}$$

which implies that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp(-G(u_n))T_1(u_n-T_j(u_n))^- dx \\
&\quad -\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^- \exp(-G(u_n))\,dx \\
&\leq \int_\Omega H_n(x,u_n,\nabla u_n)\exp(-G(u_n))T_1(u_n-T_j(u_n))^- dx \\
&\quad -\int_\Omega f_n \exp(-G(u_n))T_1(u_n-T_j(u_n))^- dx.
\end{aligned} \tag{3.48}$$

In virtue of (3.9) and (3.10), it is possible to conclude that

$$\begin{aligned}
&-\int_\Omega a(x,u_n,\nabla u_n)\nabla T_1(u_n-T_j(u_n))^- \exp(-G(u_n))\,dx \\
&\quad\leq \int_\Omega \gamma(x)\exp(-G(u_n))T_1(u_n-T_j(u_n))^- dx \\
&\qquad -\int_\Omega f_n \exp(-G(u_n))T_1(u_n-T_j(u_n))^- dx.
\end{aligned} \tag{3.49}$$

The second term in the right-hand side of the previous inequality can be neglected since it is nonnegative, and by Lebesgue's theorem the first term goes to zero as $n$ and $j$ tend to infinity. Then (3.49) becomes

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{-j-1\leq u_n\leq -j\}} a(x,u_n,\nabla u_n)\nabla u_n\,dx = 0. \tag{3.50}$$

Finally, (3.42) follows from (3.46) and (3.50).

*Step 3.* We will show that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx = 0, \tag{3.51}$$

where $h_j$ is a real variable function defined as

$$\begin{aligned}
&h_j(s) = 1 \quad \text{if } |s|\leq j, \\
&h_j(s) = 0 \quad \text{if } |s|\geq j+1, \\
&h_j(s) = j+1-s \quad \text{if } j\leq s\leq j+1, \\
&h_j(s) = s+j+1 \quad \text{if } -j-1\leq s\leq -j,
\end{aligned} \tag{3.52}$$

with $j$ a nonnegative real parameter.

Let $v = u_n + \exp(-G(u_n))T_k(u_n)^-(1 - h_j(u_n))$, $v$ a test function in $(\mathcal{P}_n)$. Then we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n)))\,dx\\&\qquad+\int_\Omega H_n(x,u_n,\nabla u_n)(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n)))\,dx\\&\quad\le\int_\Omega f_n(-\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n)))\,dx.\end{aligned}\tag{3.53}$$

By using (3.10), we have

$$\begin{aligned}&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\frac{g(u_n)}{\alpha}\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx\\&\qquad-\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)^-\exp(-G(u_n))(1-h_j(u_n))\,dx\\&\qquad+\int_\Omega a(x,u_n,\nabla u_n)\nabla h_j(u_n)\exp(-G(u_n))T_k(u_n)^-\,dx\\&\qquad-\int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx\\&\qquad-\int_\Omega \exp(-G(u_n))g(u_n)\sum_{i=1}^N\left|\frac{\partial u_n}{\partial x_i}\right|^p w_iT_k(u_n)^-(1-h_j(u_n))\,dx\\&\quad\le-\int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx.\end{aligned}\tag{3.54}$$

Thanks to (3.9), we can deduce that

$$\begin{aligned}&-\int_{\{u_n\le 0\}} a(x,u_n,\nabla u_n)\nabla T_k(u_n)\exp(-G(u_n))(1-h_j(u_n))\,dx\\&\qquad-\int_{\{-j-1\le u_n\le -j\}} a(x,u_n,\nabla u_n)\nabla u_n\exp(-G(u_n))T_k(u_n)^-\,dx\\&\qquad+\int_\Omega \gamma(x)\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx\\&\quad\ge\int_\Omega f_n\exp(-G(u_n))T_k(u_n)^-(1-h_j(u_n))\,dx.\end{aligned}\tag{3.55}$$

In view of (3.35), the second integral tends to zero as $n$ and $j$ go to infinity. And by Lebesgue's theorem, it is possible to conclude that the third and fourth integrals converge to zero as $n$ and $j$ go to infinity. Then (3.41) implies that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{u_n\le 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx=0.\tag{3.56}$$

On the other hand, take $v = u_n - \eta \exp(G(u_n))T_k(u_n^+ - \psi^+)(1 - h_j(u_n))$ which is an admissible test function in $(\mathcal{P}_n)$, then, we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla\left(\eta \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)\right)dx \\
&\qquad + \int_\Omega H_n(x,u_n,\nabla u_n)\left(\eta \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)\right)dx \\
&\quad \le \int_\Omega f_n\left(\eta \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)\right)dx,
\end{aligned} \tag{3.57}$$

which in addition to (3.10) implies that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\qquad + \int_\Omega a(x,u_n,\nabla u_n)\nabla T_k\left(u_n^+ - \psi^+\right)\exp\left(G(u_n)\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\qquad + \int_{\{j \le u_n \le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)dx \\
&\quad \le \int_\Omega g(u_n)\sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\qquad + \int_\Omega f_n \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\qquad + \int_\Omega \gamma(x)\exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)dx,
\end{aligned} \tag{3.58}$$

which takes, by using (3.9), the form

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k\left(u_n^+ - \psi^+\right)\exp\left(G(u_n)\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\quad \le -\int_{\{j \le u_n \le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)dx \\
&\qquad + \int_\Omega \gamma(x)\exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\qquad + \int_\Omega f_n \exp\left(G(u_n)\right)T_k\left(u_n^+ - \psi^+\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\quad = \varepsilon_1(j,n).
\end{aligned} \tag{3.59}$$

In virtue of (3.42) and Lebesgue's theorem, we can conclude that $\varepsilon_1(j,n)$ converges to zero as $n$ and $j$ go to infinity.

From (3.59), we have

$$\begin{aligned}
&\int_{\{|u_n^+ - \psi^+| \le k\}} a(x,u_n,\nabla u_n)\nabla u_n^+ \exp\left(G(u_n)\right)\left(1 - h_j\left(u_n\right)\right)dx \\
&\quad \le \int_{\{|u_n^+ - \psi^+| \le k\}} a(x,u_n,\nabla u_n)\nabla \psi^+ \exp\left(G(u_n)\right)\left(1 - h_j\left(u_n\right)\right)dx + \varepsilon_1(j,n).
\end{aligned} \tag{3.60}$$

Thanks to the growth condition (3.7) and Young's inequality, it is possible to conclude that

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n^+ \exp\left(G(u_n)\right)\left(1-h_j(u_n)\right)dx \le \varepsilon_2(j,n), \tag{3.61}$$

where $\varepsilon_2(j,n)$ tends to 0 as $n$ and $j$ go to infinity.

Since $\exp(G(u_n))$ is bounded, then

$$\int_{\{|u_n^+-\psi^+|\le k\}} a(x,u_n,\nabla u_n)\nabla u_n^+ \exp\left(G(u_n)\right)\left(1-h_j(u_n)\right)dx \le \varepsilon_3(j,n). \tag{3.62}$$

Since $\{x\in\Omega,\ |u_n^+|\le h\}\subset\{x\in\Omega,\ |u_n^+-\psi^+|\le h+\|\psi^+\|_\infty\}$, hence

$$\begin{aligned}\int_{\{|u_n^+|\le k\}} & a(x,u_n,\nabla u_n)\nabla u_n\left(1-h_j(u_n)\right)dx\\ &\le \int_{\{|u_n^+-\psi^+|\le k+\|\psi^+\|_\infty\}} a(x,u_n,\nabla u_n)\nabla u_n\left(1-h_j(u_n)\right)dx\\ &\le \varepsilon_3(j,n),\end{aligned} \tag{3.63}$$

which yields, for all $k>0$,

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{u_n\ge 0\}} a\left(x,T_k(u_n),\nabla T_k(u_n)\right)\nabla T_k(u_n)\left(1-h_j(u_n)\right)dx = 0. \tag{3.64}$$

Using (3.56) and (3.64), we conclude (3.51).

*Step 4.* We prove that

$$\begin{aligned}\lim_{j\to\infty}\lim_{n\to\infty}\int_\Omega & \left(a\left(x,T_k(u_n),\nabla T_k(u_n)\right)-a\left(x,T_k(u_n),\nabla T_k(u)\right)\right)\\ &\times\left(\nabla T_k(u_n)-\nabla T_k(u)\right)h_j(u_n)\,dx = 0.\end{aligned} \tag{3.65}$$

On the one hand, let $v = u_n - \eta\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)$ with $h_j$ defined as in (3.52) and $\eta$ small enough such that $v\in K_\psi$, then, we take $v$ as the test function in $(\mathcal{P}_n)$ and we obtain

$$\begin{aligned}\int_\Omega & a(x,u_n,\nabla u_n)\nabla\left(\eta\exp\left(G(u_n)\right)\left(T_k(u_n)-T_k(u)\right)^+h_j(u_n)\right)dx\\ &+\int_\Omega H_n(x,u_n,\nabla u_n)\left(\eta\exp\left(G(u_n)\right)\left(T_k(u_n)-T_k(u)\right)^+h_j(u_n)\right)dx\\ \le & \int_\Omega f_n\left(\eta\exp\left(G(u_n)\right)\left(T_k(u_n)-T_k(u)\right)^+h_j(u_n)\right)dx.\end{aligned} \tag{3.66}$$

Similarly, using (3.9) and (3.10), we obtain

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla (T_k(u_n)-T_k(u))^+\exp(G(u_n))h_j(u_n)\,dx\\
&\qquad-\int_{\{j\le u_n\le j+1\}} a(x,u_n,\nabla u_n)\nabla u_n\exp(G(u_n))(T_k(u_n)-T_k(u))^+\,dx\\
&\quad\le\int_\Omega \gamma(x)\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx\\
&\qquad+\int_\Omega f_n\exp(G(u_n))(T_k(u_n)-T_k(u))^+h_j(u_n)\,dx,
\end{aligned}\tag{3.67}$$

that is,

$$\int_{\{T_k(u_n)-T_k(u)\ge 0\}} a(x,u_n,\nabla u_n)\nabla (T_k(u_n)-T_k(u))\exp(G(u_n))h_j(u_n)\,dx\le\varepsilon_4(j,n).\tag{3.68}$$

Applying again (3.42) and Lebesgue's theorem, we deduce that $\varepsilon_4(j,n)$ goes to zero as $n$ and $j$ tend to infinity. Moreover, (3.68) becomes

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(u)\ge 0\}} a(x,T_k(u_n),\nabla T_k(u_n))\nabla (T_k(u_n)-T_k(u))\exp(G(u_n))h_j(u_n)\,dx\\
&\qquad+\int_{\{T_k(u_n)-T_k(u)\ge 0,\ |u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)\exp(G(u_n))h_j(u_n)\,dx\\
&\quad\le\varepsilon_4(j,n).
\end{aligned}\tag{3.69}$$

Since $h_j(u_n)=0$ if $|u_n|>j+1$, hence, we obtain

$$\begin{aligned}
&\int_{\{T_k(u_n)-T_k(u)\ge 0,\ |u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(u)\exp(G(u_n))h_j(u_n)\,dx\\
&\quad=\int_{\{T_k(u_n)-T_k(u)\ge 0,\ |u_n|>k\}} a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))\nabla T_k(u)\\
&\qquad\times\exp(G(u_n))h_j(u_n)\,dx\\
&\quad\le\varepsilon_5(j,n)
\end{aligned}\tag{3.70}$$

which gives

$$\begin{aligned}
\int_{\{T_k(u_n)-T_k(u)\ge 0\}} &a(x,T_k(u_n),\nabla T_k(u_n))\nabla (T_k(u_n)-T_k(u))\\
&\times\exp(G(u_n))h_j(u_n)\,dx\\
&\quad\le\varepsilon_6(j,n),
\end{aligned}\tag{3.71}$$

where $\varepsilon_6(j,n)=c(\int_{\{|u_n|>k\}}|a(x,T_{j+1}(u_n),\nabla T_{j+1}(u_n))||\nabla T_k(u)|\exp(G(u_n))h_j(u_n)\,dx+\varepsilon_5(j,n))$ which goes to zero as $n$ and $j$ tend to infinity.

On the other hand, take $v = u_n + \exp(-G(u_n))(T_k(u_n) - T_k(u))^- h_j(u_n)$ as the test function in $(\mathcal{P}_n)$ and, reasoning as in (3.71), it is possible to conclude that

$$\lim_{j\to\infty}\lim_{n\to\infty}\int_{\{T_k(u_n)-T_k(u)\leq 0\}} (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))) \times(\nabla T_k(u_n) - \nabla T_k(u))h_j(u_n)\,dx = 0. \tag{3.72}$$

Combining (3.71) and (3.72), we deduce (3.65).

*Step 5.* We show that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega,w). \tag{3.73}$$

Firstly, we have

$$\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))\,dx \\
&\quad= \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))) \\
&\qquad\times(\nabla T_k(u_n) - \nabla T_k(u))h_j(u_n)\,dx \\
&\qquad+ \int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))) \\
&\qquad\times(\nabla T_k(u_n) - \nabla T_k(u))(1-h_j(u_n))\,dx.
\end{aligned} \tag{3.74}$$

Thanks to (3.65) the first integral of the right-hand side converges to zero as $n$ and $j$ tend to infinity.

For the second term, we have

$$\begin{aligned}
&\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)))(\nabla T_k(u_n) - \nabla T_k(u))(1-h_j(u_n))\,dx \\
&\quad= \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)(1-h_j(u_n))\,dx \\
&\qquad- \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)(1-h_j(u_n))\,dx \\
&\qquad- \int_\Omega a(x,T_k(u_n),\nabla T_k(u))(\nabla T_k(u_n) - \nabla T_k(u))(1-h_j(u_n))\,dx.
\end{aligned} \tag{3.75}$$

By (3.51) the first integral of the right-hand side goes to zero as $n$ and $j$ tend to infinity, and since $(a(x,T_k(u_n),\nabla T_k(u_n)))$ is bounded in $\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$ uniformly on $n$ while $\nabla T_k(u)(1-h_j(u_n))$ converges to zero, hence, the second integral converges to zero. For the third integral, it converges to zero because $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $\prod_{i=1}^N L^p(\Omega,w_i)$.

Finally, we conclude that

$$\lim_{n\to+\infty}\int_\Omega (a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))) \times(\nabla T_k(u_n) - \nabla T_k(u))\,dx = 0. \tag{3.76}$$

Then [1, Lemma 3.1] implies that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega, w). \tag{3.77}$$

This completes the proof of Proposition 3.6. □

### 3.5. Proof of Theorem 3.1

Let $\varphi \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega)$ and taking $v = T_k(u_n - \varphi)$ as test function in $(\mathcal{P}_n)$, we have

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi)\, dx \\ &\qquad + \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi)\, dx \\ &\quad \leq \int_\Omega f_n T_k(u_n - \varphi)\, dx, \end{aligned} \tag{3.78}$$

which implies that

$$\begin{aligned} &\int_\Omega a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \nabla T_k(u_n - \varphi)\, dx \\ &\qquad + \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi)\, dx \\ &\quad \leq \int_\Omega f_n T_k(u_n - \varphi)\, dx. \end{aligned} \tag{3.79}$$

In view of Proposition 3.6 and the growth condition (3.7), we deduce that

$$\begin{aligned} &a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \\ &\quad \longrightarrow a(x, T_{k+\|\varphi\|_\infty}(u), \nabla T_{k+\|\varphi\|_\infty}(u)) \quad \text{strongly in } \prod_{i=1}^N L^{p'}(\Omega, w_i). \end{aligned} \tag{3.80}$$

Moreover, using, again, Proposition 3.6, we have

$$T_k(u_n - \varphi) \longrightarrow T_k(u - \varphi) \quad \text{strongly in } W_0^{1,p}(\Omega, w). \tag{3.81}$$

Then,

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) T_k(u_n - \varphi)\, dx \\ &\quad = \int_\Omega a(x, T_{k+\|\varphi\|_\infty}(u), \nabla T_{k+\|\varphi\|_\infty}(u)) T_k(u - \varphi) dx. \end{aligned} \tag{3.82}$$

On the other hand, we claim that

$$H_n(x, u_n, \nabla u_n) \longrightarrow H(x, u, \nabla u) \quad \text{strongly in } L^1(\Omega). \tag{3.83}$$

Let $v = u_n + \exp(-G(u_n))\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds$. Since $v \in W_0^{1,p}(\Omega, w)$ and $v \geq \psi$, hence $v$ is an admissible test function in $(\mathcal{P}_n)$. Then,

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla\big(-\exp\big(-G(u_n)\big)\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\big)dx \\
&\quad + \int_\Omega H_n(x,u_n,\nabla u_n)\big(-\exp\big(-G(u_n)\big)\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\big)dx \\
&\leq \int_\Omega f_n\big(-\exp\big(-G(u_n)\big)\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\big)dx
\end{aligned} \tag{3.84}$$

which implies that

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \frac{g(u_n)}{\alpha}\exp\big(-G(u_n)\big)\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\,dx \\
&\quad + \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \exp\big(-G(u_n)\big)g(u_n)\chi_{\{u_n<-h\}}dx \\
&\leq \int_\Omega \gamma(x)\exp\big(-G(u_n)\big)\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\,dx \\
&\quad + \int_\Omega g(u_n)\sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i \exp\big(-G(u_n)\big)\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\,dx \\
&\quad - \int_\Omega f_n \exp\big(-G(u_n)\big)\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds\,dx.
\end{aligned} \tag{3.85}$$

By (3.9) and since $\int_{u_n}^0 g(s)\chi_{\{s<-h\}}ds \leq \int_{-\infty}^{-h} g(s)ds$, we have

$$\begin{aligned}
&\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \exp\big(-G(u_n)\big)g(u_n)\chi_{\{u_n<-h\}}dx \\
&\quad \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\int_{-\infty}^{-h} g(s)ds\big(\|\gamma\|_{L^1(\Omega)} + \|f_n\|_{L^1(\Omega)}\big) \\
&\quad \leq \exp\left(\frac{\|g\|_{L^1(\mathbb{R})}}{\alpha}\right)\int_{-\infty}^{-h} g(s)ds\big(\|\gamma\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega)}\big).
\end{aligned} \tag{3.86}$$

Using again (3.9), we obtain

$$\int_{\{u_n<-h\}} g(u_n)\sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i\,dx \leq c\int_{-\infty}^{-h} g(s)ds \tag{3.87}$$

and since $g \in L^1(\mathbb{R})$, we deduce that

$$\lim_{h\to+\infty}\ \sup_{n\in\mathbb{N}}\int_{\{u_n<-h\}} g(u_n)\sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^p w_i\,dx = 0. \tag{3.88}$$

On the other hand, letting $M = \exp(-G(u_n))\int_0^{+\infty} g(s)ds$ and $h \geq M + \|\psi^+\|_{L^\infty(\Omega)}$, we consider $v = u_n - \exp(G(u_n))\int_0^{u_n} g(s)\chi_{\{s>h\}}ds$. Since $v \in W_0^{1,p}(\Omega, w)$ and $v \geq \psi$, $v$ is an admissible test function in $(\mathcal{P}_n)$. Then, similarly as in (3.88), we deduce that

$$\lim_{h\to+\infty} \sup_{n\in\mathbb{N}} \int_{\{u_n>h\}} g(u_n) \sum_{i=1}^{N} \left| \frac{\partial u_n}{\partial x_i} \right|^p w_i \, dx = 0. \tag{3.89}$$

Combining (3.88), (3.89), Proposition 3.6, and Vitali's theorem, we conclude (3.60).

On the other hand, letting $\varphi \in K_\psi \cap L^\infty(\Omega)$ and taking $v = u_n - T_k(u_n - \varphi)$ as test function in $(\mathcal{P}_n)$, we get

$$\begin{gathered} u_n \in K_\psi \quad \forall k > 0, \\ \int_\Omega a(x, u_n, \nabla u_n) \nabla T_k(u_n - \varphi)\, dx + \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi)\, dx \\ \leq \int_\Omega f_n T_k(u_n - \varphi)\, dx \quad \forall \varphi \in K_\psi \cap L^\infty(\Omega), \end{gathered} \tag{3.90}$$

which implies that

$$\begin{gathered} u_n \in K_\psi \quad \forall k > 0, \\ \int_\Omega a(x, T_{k+\|\varphi\|_\infty}(u_n), \nabla T_{k+\|\varphi\|_\infty}(u_n)) \nabla T_k(u_n - \varphi)\, dx + \int_\Omega H_n(x, u_n, \nabla u_n) T_k(u_n - \varphi)\, dx \\ \leq \int_\Omega f_n T_k(u_n - \varphi)\, dx \quad \forall \varphi \in K_\psi \cap L^\infty(\Omega). \end{gathered} \tag{3.91}$$

Finally, from (3.82) and (3.83), we can pass to the limit in (3.91). This completes the proof of Theorem 3.1.

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