ON A CLASS OF SEMILINEAR ELLIPTIC EQUATIONS WITH BOUNDARY CONDITIONS AND POTENTIALS WHICH CHANGE SIGN

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We study the existence of nontrivial solutions for the problem $\Delta u = u$, in a bounded smooth domain $\Omega \subset \mathbb{R}^{\mathbb{N}}$, with a semilinear boundary condition given by $\partial u/\partial \nu = \lambda u - W(x)g(u)$, on the boundary of the domain, where W is a potential changing sign, g has a superlinear growth condition, and the parameter $\lambda \in]0,\lambda_1]$; λ_1 is the first eigenvalue of the Steklov problem. The proofs are based on the variational and min-max methods.

1. Introduction

In this paper, we study the existence of nontrivial solutions of the following problem:

(P_λ)

$$\begin{aligned} \Delta u &= u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} &= \lambda u - W(x)g(u) \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain set of $\mathbb{R}^{\mathbb{N}}$, $\mathbb{N} \geq 3$ with smooth boundary $\partial\Omega$, $\Delta u = \nabla \cdot (\nabla u)$ is the Laplacian and $\partial/\partial\nu$ is the outer normal derivative; the parameter $\lambda \in]0,\lambda_1]$, where λ_1 is the first eigenvalue of the Steklov problem (see [5]), $W \in C(\overline{\Omega})$ different from zero almost everywhere and changes sign, while $g(u)$ is a continuous and superlinear function (see $(G1)$, $(G2)$, $(G3)$) below.

In the case of $W \equiv 0$, (P_λ) becomes a linear eigenvalue problem and it is known as the Steklov problem studied in [5], which proved the existence, the simplicity, and the isolation of the first eigenvalue λ_1.

The study of the similar problem when the nonlinear term is placed in the equation, that is, when one considers problem of the form $-\Delta u = \lambda u + W(x)g(u)$ with Dirichlet boundary condition, there is more work; hence, in the case where g behaves as a power near 0 and infinity, Alama and Tarantello in [2] showed the existence of a positive solution, provided that f is odd, and found that a necessary and sufficient condition to obtain

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such a solution is

$$\int_{\Omega} W(x)e_1^p dx < 0, \tag{1.2}$$

where e_1 denotes a positive eigenfunction of Laplacian related to the first eigenvalue, with $p \in]2,2^*[$, $2^* = 2\mathbb{N}/(\mathbb{N}-2)$ if $\mathbb{N} > 2$, $2^* = +\infty$ if $\mathbb{N} = 2$. Also, in [3], it was proved that (1.2) is a necessary and sufficient condition to obtain a positive solution; recently, Margone in [14], proved some results of existence in case that $0 < \lambda \leq \lambda_1$, close to λ_1; by using mountain pass lemma (see [4]) and linking-type theorem (see [15]). Finally, in [1], Alama and Delpino proved under some restriction on the sign of $W(x)$ the existence of nontrivial solution, by using two different approach: one involving min-max methods, the other Morse theory methods.

However, nonlinear boundary conditions have only been considered in recent years, for the Laplacian with boundary conditions, see, for example [6, 7, 8, 12, 13, 16], where the authors discussed mountain pass theorem on an order interval with Dirichlet boundary condition. For elliptic systems with nonlinear boundary conditions, see [9, 10].

The main purpose of this work is to study one problem of Neumman boundary value, in the case $\lambda = \lambda_1$ because if $\lambda < \lambda_1$, it is easy to prove that the functional Φ_λ has a condition of mountain pass structure. We show two results of existence obtained as critical points of the functional related at (P_λ), by using mountain pass lemma introduced in [4] and linking-type theorem introduced in [15].

The rest of this paper is organized as follows: in Section 2, we cite the main results and in Section 3, we prove the main results.

2. Main results

In the sequel, we consider the following functional:

$$G(u) = \int_0^u g(t)dt. \tag{2.1}$$

Then, we show the following existence results for (P_λ).

Theorem 2.1. *Let g be a continuous real-valued function on $\mathbb{R}$ such that the following assumptions hold:*

(G1) $g(u)u \geq 0$ *for all* $u \in \mathbb{R}$,
(G2) $|g(u)| \leq C|u|^{r-1}$ *for all* $u \in \mathbb{R}$, *and for some* $r \in]2,2(\mathbb{N}-1)/(\mathbb{N}-2)[$,
(G3) $g(u)u \geq (s+1)G(u)$ *for* $u > R$, *R sufficiently large, and for some* $s \in]1,\mathbb{N}/(\mathbb{N}-2)[$,
(G4) $\lim_{u\to 0}(g(u)/|u|^{r-2}u) = a > 0$,
(G5) $g(u)u \geq c|u|^{s+1}$ *for* $|u| > R$, *and R sufficiently large,*
(G6) $W^-(g(u)u - (s+1)G(u)) \leq \gamma|u|^2$, $|u| > R$, *for some*

$$\gamma \in \left]0, \left(\frac{s+1}{2} - 1\right)(\lambda_2 - \lambda_1)\right[, \tag{2.2}$$

where λ_2 is the second eigenvalue of the Steklov problem, and $W^-(x) = -\min\{W(x),0\}$, $W^- = \max_{x\in\partial\Omega} W^-(x)$; moreover, let

(W_0) $W^+(x) = \max\{W(x),0\}, \text{meas}(\{x \in \partial\Omega : W(x) = 0\}) = 0$,
(W_1) $\int_{\partial\Omega} W(x)e_1^r d\sigma < 0$, *where* e_1 *is a positive eigenfunction related to* λ_1,

then (P_λ) *has a positive solution* u_λ *for any* $\lambda \in (0,\lambda_1]$.

Remarks 2.2. (i) Condition $(G6)$ was introduced by Girardi and Matzeu (see [11]) and plays a crucial role in the proof of Palais-Smale condition.

(ii) Condition (W_1) is necessary and sufficient to obtain such a solution and was introduced by Alama and Tarantello, (see [3]), for semilinear elliptic equations with Dirichlet boundary conditions.

Theorem 2.3. *Let g satisfy conditions* $(G1)$–$(G3)$, $(G5)$, $(G6)$, *and* (W_0). *If W verifies the further assumptions,*

(W_2) $\int_{\partial\Omega} W(x)G(te_1)d\sigma > 0$, *for all* $t \in \mathbb{R}\backslash\{0\}$,
(W_3) $\int_D W(x)G(te_1)d\sigma > c$, *for all* $t \in \mathbb{R}$ *and for some* $c \in \mathbb{R}$, *where D is a nonempty open subset in* $\partial\Omega$ *such that* $\operatorname{supp} W^- \subset D$,

then (P_{λ_1}) *has a nontrivial solution.*

Remark 2.4. Note that the solution found in Theorem 2.3 is surely not always positive because (W_1) does not hold. Moreover, condition (W_2), which appears in Theorem 2.3, is in some sense complementary to (W_1) if g is a power.

3. Proof of the main results

It is well known that the solutions of (P_λ) are critical points of the functional

$$\Phi_\lambda(u) = \frac{1}{2}\left(\|\nabla u\|_2^2 + \|u\|_2^2 - \lambda\int_{\partial\Omega}|u|^2 d\sigma\right) - \int_{\partial\Omega} W(x)G(u)d\sigma, \quad u \in H^1(\Omega). \tag{3.1}$$

In order to prove the main results, we apply the mountain pass theorem (see [4]) and a suitable version of the linking-type theorem (see [15]) to the functional Φ_λ.

The following lemma is the key for proving our theorems, in which we consider $\lambda = \lambda_1$ because if $\lambda < \lambda_1$, the argument is the same.

Lemma 3.1. *Under assumptions* (W_0), $(G2)$, $(G3)$, $(G5)$, $(G6)$, *the functional* $\Phi_\lambda(u)$ *satisfies the Palais-Smale condition on* $H^1(\Omega)$. *That is, any sequence* $(u_n)_n$ *in* $H^1(\Omega)$, *such that*

$$(\Phi_\lambda(u_n))_n \text{ is bounded and } \Phi'_\lambda(u_n) \longrightarrow 0 \tag{3.2}$$

possesses a converging subsequence.

Proof. Let $(u_n)_n \subset H^1(\Omega)$ be a Palais-Smale sequence, namely, there exist c_1 and c_2 such that

$$c_1 \le \frac{1}{2}\left(\|\nabla u_n\|_2^2 + \|u_n\|_2^2 - \lambda_1\int_{\partial\Omega}|u_n|^2 d\sigma\right) - \int_{\partial\Omega} W(x)G(u_n)d\sigma \le c_2, \tag{3.3}$$

$$\begin{aligned}\sup_{\{\phi\in H^1(\Omega),\|\phi\|_{1,2}=1\}} &\left\{\int_\Omega (\nabla u_n\nabla\phi + u_n\phi)dx - \lambda_1\int_{\partial\Omega} u_n\phi\, d\sigma\right.\\ &\left.- \int_{\partial\Omega} W(x)g(u_n)\phi\, d\sigma\right\} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty.\end{aligned} \tag{3.4}$$

We are going to show that $(u_n)_n$ is bounded in $H^1(\Omega)$. By assumptions $(G3)$ and $(G6)$, and from (3.3) and (3.4), we get for some constant $c_R > 0$ depending on the number R of $(G3)$,

$$\begin{aligned}
\int_\Omega \left(|\nabla u_n|^2 + u_n^2 \right) dx &= \lambda_1 \int_{\partial\Omega} u_n^2 d\sigma - \int_{\partial\Omega} W(x) g(u_n) u_n d\sigma + \epsilon_n ||u_n||_{1,2} \\
&\geq \lambda_1 \int_{\partial\Omega} u_n^2 d\sigma + \int_{\partial\Omega} W^+(x) g(u_n) u_n d\sigma \\
&\quad - \int_{\partial\Omega} W^-(x) g(u_n) u_n d\sigma + \epsilon_n ||u_n||_{1,2} \\
&\geq \lambda_1 \int_{\partial\Omega} u_n^2 d\sigma + (s+1) \int_{\partial\Omega} W^+(x) G(u_n) d\sigma - \gamma \int_{\partial\Omega \cap \{|u|>R\}} |u_n|^2 d\sigma \\
&\quad - (s+1) \int_{\partial\Omega \cap \{|u|>R\}} W^-(x) G(u_n) d\sigma + c_R + \epsilon_n ||u_n||_{1,2} \\
&\geq \lambda_1 \int_{\partial\Omega} u_n^2 d\sigma + (s+1) \left[\frac{1}{2} ||u_n||_{1,2}^2 - \frac{\lambda_1}{2} \int_{\partial\Omega} u_n^2 d\sigma - c_2 \right] \\
&\quad - \gamma \int_{\partial\Omega} u_n^2 d\sigma + c_R + \epsilon_n ||u_n||_{1,2}.
\end{aligned} \tag{3.5}$$

Set $X_1 = \text{vect}(e_1)$, then, there exist $k_n \in \mathbb{R}$ such that $u_n = k_n e_1 + v_n$, where $v_n \in X_1^\perp$.

Using the variational characterization of λ_2, (3.5) becomes

$$\left(\frac{s+1}{2} - 1 \right) \left(1 - \frac{\lambda_1}{\lambda_2} \right) ||v_n||_{1,2}^2 + \epsilon_n ||v_n||_{1,2} \leq \gamma \int_{\partial\Omega} (k_n e_1 + v_n)^2 d\sigma + c, \tag{3.6}$$

where ϵ_n is an infinitesimal sequence of positive numbers.

On the other hand, using variational characterization of λ_1, it follows that

$$\left[\left(\frac{s+1}{2} - 1 \right) \left(1 - \frac{\lambda_1}{\lambda_2} \right) - \frac{\gamma}{\lambda_2} \right] ||v_n||_{1,2}^2 + \epsilon_n ||v_n||_{1,2} \leq c + \gamma k_n^2 \int_{\partial\Omega} e_1^2 d\sigma. \tag{3.7}$$

On the other side, by (2.2) and taking into acount that $\epsilon_n \to 0$, we deduce that

$$||v_n||_{1,2}^2 \leq \text{const} \left(1 + k_n^2 \right), \tag{3.8}$$

hence, it suffices to prove that $(|k_n|)_n$ is bounded. So, if $|k_n| \to +\infty$ (at least a subsequence), therefore $(v_n/|k_n|)_n$ is bounded in $H^1(\Omega)$, so a subsequence, also called $(v_n/|k_n|)_n$, weakly converges in $H^1(\Omega)$ at some f and that

$$f(x) + e_1(x) \neq 0 \quad \text{a.e. in } \overline{\Omega}. \tag{3.9}$$

Indeed, if (3.9) is false, taking into acount that

$$\int_\Omega \left(\nabla \left(\frac{v_n}{|k_n|} \right) \nabla e_1 + \frac{v_n}{|k_n|} e_1 \right) dx = 0 \quad \forall n \in \mathbb{N} \tag{3.10}$$

as $n \to +\infty$, we obtain $\|e_1\|_{1,2}^2 = \lambda_1 \int_{\partial\Omega} e_1^2 = 0$, which is an absurdum as we know that e_1 is the principal eigenvector related with λ_1.

From (3.4), we obtain

$$\int_\Omega (\nabla u_n \nabla \phi + u_n \phi)\,dx - \lambda_1 \int_{\partial\Omega} u_n \phi\, d\sigma - \int_{\partial\Omega} W(x) g(u_n)\phi\, d\sigma = \eta_n \tag{3.11}$$

with $\lim_{n\to+\infty} \eta_n = 0$ in $\mathbb{R}$.

Let $\phi_n = (k_n e_1 + v_n)|k_n|^{-1}\phi$, where ϕ is a regular function with support compact in $\overline{\Omega}$ and $\operatorname{meas}(\operatorname{supp}\phi \cap \partial\Omega) \neq 0$; then

$$\begin{aligned}&\int_\Omega (\nabla(k_n e_1 + v_n)\nabla\phi_n + (k_n e_1 + v_n)\phi_n)\,dx \\ &\qquad - \lambda_1 \int_{\partial\Omega} (k_n e_1 + v_n)\phi_n d\sigma - \int_{\partial\Omega} W(x) g(k_n e_1 + v_n)\phi_n d\sigma = \eta_n,\end{aligned} \tag{3.12}$$

hence

$$\begin{aligned}\frac{1}{|k_n|}\int_\Omega [\nabla v_n \nabla \phi_n + v_n\phi_n]\,dx - \frac{\lambda_1}{|k_n|}\int_{\partial\Omega} v_n \phi_n d\sigma \\ = \frac{1}{|k_n|}\int_{\partial\Omega} W(x) g(k_n e_1 + v_n)\phi_n d\sigma + o(1)\end{aligned} \tag{3.13}$$

for n large enough.

So, Hölder inequality and (3.8) imply that $(1/|k_n|)\int_\Omega (\nabla v_n \nabla\phi_n + v_n\phi_n)dx$ and $(\lambda_1/|k_n|)\int_{\partial\Omega} v_n\phi_n d\sigma$ are bounded.

On the other side, combining (W_0) and (3.9), it follows that either

$$\int_{\operatorname{Supp} W^+} |h(x) + e_1(x)|^{s+1} d\sigma > 0 \qquad \text{or} \qquad \int_{\operatorname{Supp} W^-} |h(x) + e_1(x)|^{s+1} d\sigma > 0. \tag{3.14}$$

In the first case, we take ϕ regular nonnegative function with $\operatorname{meas}(\operatorname{supp}\phi \cap \operatorname{supp} W^+) \neq 0$ such that

$$\int_{\operatorname{Supp} W^+} W^+(x)\phi(x)|h(x) + e_1(x)|^{s+1} d\sigma > 0, \tag{3.15}$$

then, by $(G6)$ and (3.15), we get for some positive constant c,

$$\begin{aligned}\frac{1}{|k_n|}\int_{\partial\Omega} W(x) g(k_n e_1 + v_n)\phi_n d\sigma &\geq \frac{c}{|k_n|^2}\int_{\operatorname{supp} W^+} W^+(x)|k_n e_1 + v_n|^{s+1}\phi\, d\sigma - c \\ &\geq c k_n^{s-1}\int_{\operatorname{supp} W^+} W^+(x)\left|e_1 + \frac{v_n}{k_n}\right|^{s+1}\phi\, d\sigma - c \longrightarrow +\infty.\end{aligned} \tag{3.16}$$

This and formula (3.13) contradict the bound of $(1/|k_n|)\int_\Omega (\nabla v_n \nabla\phi_n + v_n\phi_n)d\sigma$ and $(\lambda_1/|k_n|)\int_{\partial\Omega} v_n\phi_n d\sigma$.

For the second case, it suffices to take ϕ nonnegative function with meas(supp $\phi \cap$ supp $W^-) \neq 0$ such that

$$\int_{\text{Supp}\,W^-} W^-(x)\phi(x)\,|h(x)+e_1(x)|^{s+1}d\sigma > 0. \tag{3.17}$$

Finally, we have proved that $(u_n)_n$ is bounded, this implies the existence of a subsequence weakly converging in $H^1(\Omega)$. On the other side, thanks to $(G2)$ and the compact embedding $H^1(\Omega) \hookrightarrow L^r(\partial\Omega)$ for $r \in]2, 2(N-1)/(\mathbb{N}-2)[$, we have the strong convergence. □

Lemma 3.2. *The origin is a strict locale minimizer of* Φ_λ.

Proof. First, remark that each $u \in H^1(\Omega)$ can be written as $u = te_1 + v$, where $t \in \mathbb{R}$, and $v \in X_1^\perp$, then

$$\int_\Omega (|\nabla u|^2 + |u|^2)\,dx = t^2\lambda_1 \int_{\partial\Omega} e_1^2 d\sigma + \|v\|_{1,2}^2. \tag{3.18}$$

Choosing e_1 such that $\int_{\partial\Omega} e_1^2 d\sigma = 1/\lambda_1$, one gets, for all u satisfying $\|u\|_{1,2} \le 1/2\|e_1\|_\infty$,

$$t^2 < \|u\|_{1,2}^2 < \frac{1}{4\|e_1\|_\infty^2}. \tag{3.19}$$

Hence, by variational characterization of the eigenvalues of the Laplacian with boundary conditions and for a suitable function $F(t,v)$, we obtain

$$\begin{aligned}\Phi_{\lambda_1}(u) &\ge \frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\|v\|_{1,2}^2 - \int_{\partial\Omega} W(x)G(te_1+v)\,d\sigma \\ &\ge \frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\|v\|_{1,2}^2 - |t|^r\int_{\partial\Omega} W(x)e_1^r d\sigma + F(t,v),\end{aligned} \tag{3.20}$$

where by $(G4)$,

$$\begin{aligned}F(t,v) &= \int_{\partial\Omega} W(x)[\,|te_1|^r - G(te_1)]\,d\sigma + \int_{\partial\Omega} W(x)[G(te_1) - G(te_1+v)]\,d\sigma \\ &= \int_{\partial\Omega} W(x)[G(te_1) - G(te_1+v)]\,d\sigma + o(|t|^r).\end{aligned} \tag{3.21}$$

On the other hand, using arrangement-finite theorem, there exists a function $0 < \theta \equiv \theta(x,t,v) < 1$ such that

$$|G(te_1+v) - G(te_1)| = |g(te_1+\theta v(x))v(x)| \tag{3.22}$$

In case that $|te_1 + \theta v(x)| \geq 1$, by (3.19), we deduce

$$|\theta v(x)| \geq 2|t|\,\|e_1\|_\infty - |t|\,\|e_1\|_\infty \geq |t|\,\|e_1\|_\infty, \tag{3.23}$$

so by $(G2)$,

$$\begin{aligned} |g(te_1 + \theta v(x))v(x)| &\leq C|te_1 + \theta v(x)|^{r-1}|v(x)| \\ &\leq 2^{r-2}C|\theta v(x)|^{r-1}|v(x)| \leq 2^{r-1}C|v(x)|^r, \end{aligned} \tag{3.24}$$

while, if $|te_1 + \theta v(x)| \leq 1$, using again $(G2)$, one obtains

$$\begin{aligned} |W(x)|\,|g(te_1 + \theta v(x))v(x)| &\leq C|te_1 + \theta v(x)|^{r-1}v(x) \\ &\leq C\Big[|te_1|^{r-1} + |v(x)|^r\Big] \leq \epsilon|te_1|^r + C_\epsilon|v(x)|^r, \end{aligned} \tag{3.25}$$

where ϵ, C_ϵ are two positive constants.

Set $A = -\int_{\partial\Omega} W(x)e_1^r d\sigma > 0$. Combining (3.21), (3.24), and (3.25), and using (W_1), (3.20) becomes

$$\begin{aligned} \Phi_{\lambda_1}(u) &\geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|v\|_{1,2}^2 - t^r\int_{\partial\Omega} W(x)e_1^r - |F(t,v)| \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|v\|_{1,2}^2 + t^r A - 2^{r-1}C\int_{\partial\Omega\cap\{|u|>1\}}|W(x)|\,|v(x)|^r d\sigma \\ &\quad - \int_{\partial\Omega\cap\{|u|\leq 1\}}\Big[\epsilon|te_1|^r + C_\epsilon|v(x)|^r\Big] + \theta(|t|^r) \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|v\|_{1,2}^2 + t^r(A - C_1\epsilon) - C_2\|v\|_r^r + o(|t|^r), \end{aligned} \tag{3.26}$$

where C_1, C_2 are two positive constants.

Hence, using Sobolev trace embedding, for $\epsilon < A/C_1$, we deduce

$$\Phi_{\lambda_1}(u) \geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|v\|_{1,2}^2 + C_3 t^r - C_4\|v\|_{1,2}^r + o(|t|^r). \tag{3.27}$$

For $r > 2$, the least expression is strictly positive as $\|v\|_{1,2}$ is close to 0. □

Proof of Theorem 2.1. We will study only the case $\lambda = \lambda_1$ because if $\lambda < \lambda_1$, it is easily proved that the functional Φ_λ has a condition of mountain pass structure.

Now, it suffices to prove that there exist $\overline{u} \in H^1(\Omega)$ such that $\|\overline{u}\|_{1,2} > \rho$, ρ large enough satisfying $\Phi_\lambda(\overline{u}) < 0$ which completes the proof of Theorem 2.3.

Let $t \in \mathbb{R}$ and $\phi \in C_0^\infty(\operatorname{supp} W^+)$, where $W^+(x) = \max(W(x), 0)$ (note that ϕ is well defined, thanks to (W_0)).

Using $(G4)$, we obtain

$$\begin{aligned} \Phi_{\lambda_1}(t\phi) &= \frac{t^2}{2}\left(\|\phi\|_{1,2}^2 - \lambda_1\int_{\partial\Omega}\phi^2 d\sigma\right) - \int_{\partial\Omega} W(x)G(t\phi)d\sigma \\ &\leq \frac{t^2}{2}\|\phi\|_{1,2}^2 - Ct^r\int_{\operatorname{supp} W^+} W^+(x)|\phi|^r d\sigma \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty. \end{aligned} \tag{3.28}$$

Then, there exists $t_0 > 0$ large enough, such that $\overline{u} = t_0\phi$. Hence, using mountain pass lemma, there exists a critical point u of Φ_{λ_1} at the level

$$c = \inf_{\gamma\in\Gamma} \max_{v\in\gamma([0,1])} \Phi_{\lambda_1}(v) > 0, \tag{3.29}$$

where $\Gamma = \{\gamma \in C([0,1], H^1(\Omega)) : \gamma(0) = 0,\ \gamma(\overline{u}) = 1\}$ is the class of the path joining the origin to $\overline{u}$.

The positivity of u can be checked by a standard argument based on (3.29) (which yields the nonnegativity of u) and by the strong maximum principle of Vazquez [17] (which yields the strict positivity of u). □

The proof of Theorem 2.3 is based on Lemma 3.1 and the following version of the linking theorem, see [15].

Proposition 3.3. *Let E be a real Banach space with $E = X_1 \oplus X_2$, where X_1 is finite dimensional. Suppose $J \in C^1(E,\mathbb{R})$ satisfies the Palais-Smale condition and*

($J1$) *there are two constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$, for all $u \in X_2$: $\|u\|_E = \rho$,*

($J2$) *there exists $\overline{x} \in X_2$ with $\|\overline{x}\| = 1$ and $R > \rho$ such that, if*

$$Q = \{u \in E : u = w + \delta\overline{x} \text{ with } w \in X_1,\ \|w\| \leq R,\ \delta \in (0,R)\}, \tag{3.30}$$

then $J_{|\partial Q} \leq 0$.

Then J possesses a critical value $c \geq \alpha$.

Proof of Theorem 2.3. Set $E = H^1(\Omega)$ and $J = \Phi_\lambda$ in Proposition 3.3.

First, thanks to Lemma 3.1, Φ_λ satisfies Palais-Smale condition.

We take $X_1 = \{te_1/t \in \mathbb{R}\}$, then $X_2 = \{v \in H^1(\Omega)/\int_\Omega ve_1\,dx = 0\}$ and let $v \in X_2$, $\|v\|_{1,2} = \rho$, then

$$\begin{aligned}
\Phi_{\lambda_1}(v) &= \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + |v|^2\right)dx - \frac{\lambda_1}{2}\int_{\partial\Omega} v^2\,d\sigma - \int_{\partial\Omega} W(x)G(u)\,d\sigma \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)\|v\|_{1,2}^2 - C\sup_{\partial\Omega} W(x)\int_{\partial\Omega} |v|^r\,d\sigma \\
&\geq \frac{1}{2}\left(1 - \frac{\lambda_1}{\lambda_2}\right)\rho^2 - C\rho^r.
\end{aligned} \tag{3.31}$$

Then, for ρ small enough, we have $\Phi_{\lambda_1}(v) \geq \alpha$, so ($J1$) is verified.

As for the proof of ($J2$), first of all, we note that, as also observed in [15], it is enough to prove the following two properties:

(a) $\Phi_{\lambda_1}(te_1) \leq 0$ for all $t \in \mathbb{R}$;

(b) there exist $\overline{v} \in X_2\setminus\{0\}$ and $\rho_0 > \rho$ such that $\Phi_{\lambda_1}(u) \leq 0$ for all $u \in X_1 \oplus [\overline{v}]$ and $|u| \geq \rho_0$.

For (a), we have

$$\Phi_{\lambda_1}(te_1) = -\int_{\partial\Omega} W(x)G(te_1) \tag{3.32}$$

which is not positive by (W_2), and (a) follows.

On the other side, let $\overline{v}$ be a sufficiently regular function in $X_2\backslash\{0\}$ such that $\operatorname{supp}\overline{v} \subset \overline{\Omega}\backslash D$ and $\operatorname{meas}(\operatorname{supp}\overline{v} \cap \partial\Omega) \neq 0$, Hence, for $u \in X_1 \oplus [\overline{v}] = \{te_1 + \delta\overline{v},\ (t,\delta) \in \mathbb{R}^2\}$, we obtain

$$\begin{aligned}\Phi_{\lambda_1}(u) &= \frac{\delta^2}{2}\left[\int_\Omega \left(|\nabla\overline{v}|^2 + |\overline{v}|^2\right)dx - \lambda_1\int_{\partial\Omega}|\overline{v}|^2 d\sigma\right] - \int_{\partial\Omega} W(x)G(te_1 + \delta\overline{v})\,d\sigma \\ &\leq \frac{\delta^2}{2}\int_\Omega \left(|\nabla\overline{v}|^2 + |\overline{v}|^2\right)dx - \int_{\partial\Omega\backslash D} W^+(x)G(te_1 + \delta\overline{v})\,d\sigma - \int_D W(x)G(te_1)\,d\sigma + c,\end{aligned} \tag{3.33}$$

therefore, by (W_3), one gets

$$\Phi_{\lambda_1}(te_1 + \delta\overline{v}) \leq c(t^2 + \delta^2) - c\int_{\partial\Omega\backslash D} W^+(x)\left|te_1 + \delta\overline{v}\right|^{s+1} d\sigma + c. \tag{3.34}$$

We observe now that the map

$$te_1 + \delta\overline{v} \in X_1 \oplus [\overline{v}] \longrightarrow (t,\delta) \in \mathbb{R}^2 \tag{3.35}$$

is an isomorphism and that

$$te_1 + \delta\overline{v} \longrightarrow \left(\int_{\partial\Omega\backslash D} W^+(x)\left|te_1 + \delta\overline{v}\right|^{s+1} d\sigma\right)^{1/(s+1)} \tag{3.36}$$

yields a norm from $X_1 \oplus [\overline{v}]$ as it easily can be deduced from the fact that $-te_1(x) \neq \delta\overline{v}(x)$ in $\overline{\Omega}\backslash D$ if $\delta^2 + t^2 \neq 0$ (indeed $e_1(x) > 0$ everywhere on $\overline{\Omega}$, while $\overline{v}$ has a compact support in $\overline{\Omega}\backslash D$) therefore, as all the norms are equivalents in a finite dimensional space, we get, for some positive constant c,

$$\Phi_{\lambda_1}(te_1 + \delta\overline{v}) \leq c(t^2 + \delta^2) - c(t^{s+1} + \delta^{s+1}) + c \tag{3.37}$$

then,

$$\lim_{t^2+\delta^2 \to +\infty} \Phi_{\lambda_1}(te_1 + \delta\overline{v}) = -\infty, \tag{3.38}$$

hence, Φ_λ satisfies the assumptions of Proposition 3.3, which completes the proof of Theorem 2.3. □

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