LIPSCHITZ FUNCTIONS WITH UNEXPECTEDLY LARGE SETS OF NONDIFFERENTIABILITY POINTS

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It is known that every G_δ subset E of the plane containing a dense set of lines, even if it has measure zero, has the property that every real-valued Lipschitz function on $\mathbb{R}^2$ has a point of differentiability in E. Here we show that the set of points of differentiability of Lipschitz functions inside such sets may be surprisingly tiny: we construct a G_δ set $E \subset \mathbb{R}^2$ containing a dense set of lines for which there is a pair of real-valued Lipschitz functions on $\mathbb{R}^2$ having no common point of differentiability in E, and there is a real-valued Lipschitz function on $\mathbb{R}^2$ whose set of points of differentiability in E is uniformly purely unrectifiable.

1. Introduction and results

One of the important results of Lebesgue tells us that Lipschitz functions on the real line are differentiable almost everywhere. This result is remarkably sharp: it is not difficult to see that for every Lebesgue null set E on the real line there is a real-valued Lipschitz function which is nondifferentiable at any point of E. The higher-dimensional extension of Lebesgue's result, due to Rademacher, says that Lipschitz functions on $\mathbb{R}^n$ are also differentiable almost everywhere. Here, however, the sharpness of Lebesgue's theorem seems to be lost, as there are null sets in $\mathbb{R}^2$ in which every real-valued Lipschitz function has a point of differentiability. A plethora of such examples may be constructed using the following statement of [6], where it is proved not only in the plane, but in every Banach space with a smooth norm. Recall that a set is G_δ if it is an intersection of a sequence of open sets.

Theorem 1.1. *Suppose that E is a G_δ subset of $\mathbb{R}^2$ having the property that for any two points $u, v \in \mathbb{R}^2$ and for any $\varepsilon > 0$ there is a Lipschitz $\gamma : [0,1] \mapsto \mathbb{R}^2$ such that $\|\gamma(0) - u\| < \varepsilon$, $\|\gamma(1) - v\| < \varepsilon$, $\int_0^1 \|\gamma'(t) - (v - u)\| < \varepsilon$, and $\mu\{t \in [0,1] : \gamma(t) \notin E\} < \varepsilon$. Then every real-valued Lipschitz function defined on a nonempty open subset of the plane is differentiable at some point of E.*

The most well-known examples of sets E satisfying the condition of Theorem 1.1 are constructed by requiring that the curves γ be lines and that the Lebesgue measure

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$\mu\{t \in [0,1] : \gamma(t) \notin E\}$ be not only small, but the set is in fact empty. They are given by the formula

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B(L_k, \varrho_k), \tag{1.1}$$

where $B(S,\varrho)$ denotes the set $\{z : \operatorname{dist}(z,S) < \varrho\}$ and L_k is a sequence of lines in $\mathbb{R}^2$ which is dense in the space of lines; the latter condition means that for any $u,v \in \mathbb{R}^2$ and $\varepsilon > 0$ there is k such that both u and v are within distance ε of L_k. The set E has measure zero if $\sum_{k=1}^{\infty} \varrho_k < \infty$ and the set of lines contained in E is always dense in the space of lines. This may be seen by noting that the sets $\{(u,v) \in \mathbb{R}^2 \times \mathbb{R}^2 : u \neq v, [u+n(u-v), v+n(v-u)] \subset \bigcup_{k=n}^{\infty} B(L_k,\varrho_k)\}$ are open and dense in $\mathbb{R}^4$ and for any (u,v) in their intersection (which is dense in $\mathbb{R}^4$ by the Baire category theorem) the line passing through u,v lies in E.

Here we show that the set of points of differentiability of real-valued Lipschitz functions inside a particular set E of the form described in (1.1), although nonempty by Theorem 1.1, may still be extremely small.

Our first example will give a pair of real-valued Lipschitz functions on $\mathbb{R}^2$ with no common points of differentiability in E; in other words, we construct a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable at no point of E. The example will even provide a function which is "uniformly nondifferentiable on E" in the sense that the quantity

$$\varepsilon^*(f,z) = \limsup_{r\to 0^+} \frac{\sup\{\|f(u)+f(v)-2f((u+v)/2)\| : u,v \in B(z,r)\}}{r} \tag{1.2}$$

is, on E, bounded away from zero. In this connection, recall that the only known analogues of Theorem 1.1 for vector-valued functions do not show differentiability, but the so-called ε-differentiability. (See [3, 4] where the emphasis is on the infinite-dimensional case and [2] for a considerably more precise result in the finite-dimensional case. Here we ignore the results of [5] because they are purely infinite dimensional.) The concept of ε-differentiability measures the nondifferentiability of $f : \mathbb{R}^m \to \mathbb{R}^n$ by the quantity

$$\varepsilon(f,z) = \inf_M \limsup_{r\to 0^+} \frac{\sup\{\|f(u)-f(z)-M(u-z)\| : u \in B(z,r)\}}{r}, \tag{1.3}$$

where the infimum is over the set of $n \times m$ matrices. An ε-differentiability result for a set E and a function f would say that E contain points with $\varepsilon(f,z)$ arbitrarily small; this is (considerably) stronger than requiring that the set E contain points with $\varepsilon^*(f,z)$ arbitrarily small. Our example therefore shows that ε-differentiability results for vector-valued functions cannot be extended to all sets for which we have full differentiability results for real-valued functions.

Our second example will provide a real-valued Lipschitz function on $\mathbb{R}^2$ whose set of differentiability points inside E is small in the sense of rectifiability. Recall that a subset

N of $\mathbb{R}^2$ is called purely unrectifiable if it meets every rectifiable curve in a set of one-dimensional measure zero. A somewhat stronger notion of uniform pure unrectifiability is defined by requiring the existence of an $\eta > 0$ such that for every segment I of the unit circle of length η and for every $\varepsilon > 0$ there is an open set G containing N with the property that $\mu(\gamma^{-1}(G)) < \varepsilon$ for every Lipschitz $\gamma : [0,1] \to \mathbb{R}^2$ such that $\gamma'(t) \in I$ for almost every t. Although these are basic concepts, not much appears to be known about them. In particular, it is not known whether for G_δ sets the notions of pure and uniform pure unrectifiability coincide or not. Some information will eventually be found in [1]: an equivalent definition of uniform pure unrectifiability is obtained by fixing the η as any number less than π, and for us the most relevant point is that uniform pure unrectifiability characterises the sets N for which there is a real-valued Lipschitz function having no directional derivative at any point of N. Using this result, we could have easily obtained our first example from the second; we have not done it partly because the second example is considerably harder but mainly because in this way we would not obtain a uniform estimate of nondifferentiability of the pair of functions. We explain the reasoning behind this after stating our result.

Theorem 1.2. *There is a G_δ subset E of $\mathbb{R}^2$ containing a dense set of lines for which we can construct*

(i) *a Lipschitz function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable at no point of E, and which even satisfies that, for a fixed $\varepsilon > 0$, f is not ε-differentiable at any point of E,*

(ii) *a real-valued Lipschitz function on $\mathbb{R}^2$ whose set of points of differentiability in E is uniformly purely unrectifiable.*

As we have already pointed out, if we take the function, say h, from (ii) and use the result from [1] to find a real-valued Lipschitz function g on $\mathbb{R}^2$ which is nondifferentiable at every point of the uniformly purely unrectifiable set N of the points of differentiability of h in E, the pair (g,h) will provide an example satisfying the first part of (i). However, this would not easily provide an example of an $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is not ε-differentiable on E, since for every $\varepsilon > 0$ the set of points $z \in E$ at which $\varepsilon(h,z) < \varepsilon$ must be of positive measure on some lines lying in E. (This is explained in [6] and is behind the ε-differentiability results alluded to above.) As we do not have any control of the behaviour of g at most of these points, the proof of ε-nondifferentiability of (g,h) would require further arguments.

Yet another curious difference between the one- and two-dimensional situation arises in this connection. To explain it, recall (a special case of) the result of Zahorski [7] that for every G_δ set $N \subset \mathbb{R}$ of measure zero there is $\psi : \mathbb{R} \to \mathbb{R}$ with $\mathrm{Lip}(\psi) \le 1$, which is differentiable at every point of $\mathbb{R} \setminus N$, and at the points of N it satisfies

$$\limsup_{y\to x} \frac{\psi(y)-\psi(x)}{y-x} = 1, \qquad \liminf_{y\to x} \frac{\psi(y)-\psi(x)}{y-x} = -1. \tag{1.4}$$

This result may be used to show that the set of points of differentiability of a real-valued Lipschitz function h that lie in a set E satisfying the assumptions of Theorem 1.1 cannot be too small: its Hausdorff (one-dimensional) measure must be positive, since otherwise it would project to a null set on the x-axis and a suitable linear combination of h and Zahorski's function ψ would provide a Lipschitz function differentiable at no points of E. (A stronger version of Zahorski's results is used in [6] to show that the one-dimensional projections of the set of points of differentiability of a real-valued Lipschitz function that lie in a set E satisfying the assumptions of Theorem 1.1 have a null complement.) Now, a seemingly plausible version of Zahorski's result in the plane may say that for every uniformly purely unrectifiable G_δ set $N \subset \mathbb{R}^2$ there is a Lipschitz $\psi : \mathbb{R}^2 \to \mathbb{R}$ that is differentiable at every point of $\mathbb{R}^2 \setminus N$ and satisfies $\varepsilon(\psi, z) \geq \varepsilon > 0$, for all $z \in N$. But this is false whenever N contains the set of points of the set E from Theorem 1.2 at which the function h from (ii) is differentiable, because then a suitable linear combination of h and ψ would be differentiable at no points of E. Notice that there are such uniformly purely unrectifiable G_δ sets N since every uniformly purely unrectifiable set is obviously contained in a uniformly purely unrectifiable G_δ set.

2. Constructions

We first describe the method of the choice of the lines $L_1, L_2, \ldots$ and the half-widths $\varrho_k > 0$ of the strips $B(L_k, \varrho_k)$ which is common to both examples. In addition to L_k and ϱ_k, we will also construct functions $g_k : \mathbb{R}^2 \to \mathbb{R}^2$ in the first example or $\varphi_k : \mathbb{R}^2 \to \mathbb{R}$ in the second example, and a finite set of lines which we wish to avoid in the future choices of lines; we denote by T_k the union of these "prohibited" lines. The function f for the first example will be obtained as a composition of the g_k, and the function h for the second example as a sum of multiples of the φ_k by suitable functions.

The recursive construction will run as follows. We order a countable dense subset of $\mathbb{R}^4$ into a sequence (u_k, v_k) and start the induction by choosing L_0 and ϱ_0 arbitrarily and letting $T_0 = \partial B(L_0, \varrho_0)$. Whenever L_j, ϱ_j, g_j or φ_j, and T_j have been defined for $j < k$, we choose a line L_k not lying in T_{k-1} which passes within $1/k$ of both u_k and v_k (and satisfying another simple condition in the first example). Then we define ϱ_k by requirements that make it small compared to the data we have so far and continue by defining the functions g_k or φ_k. These functions will be piecewise affine, and we choose a finite union of lines $T_k \supset T_{k-1} \cup \partial B(L_k, \varrho_k)$ so that they are affine on every component of $\mathbb{R}^2 \setminus T_k$; in the first example, we also require that several other functions obtained by composition of g_j, $j \leq k$, be affine on every component of $\mathbb{R}^2 \setminus T_k$. Although the particular requirements on the various choices will be somewhat different in the two constructions; it is clear that we can satisfy both of them at the same time and so get the same set E (which is, of course, defined by (1.1)).

The notation we use is either mostly standard or easy to understand, such as $\langle u, v\rangle$ for the scalar product of the vectors u and v. On two occasions, we find it convenient to use the less standard notation for the cutoff function, which is defined by $\operatorname{cutoff}(x, y) = \min(\max(x, -y), y)$ for $x \in \mathbb{R}$ and $y \geq 0$.

2.1. Proof of Theorem 1.2(i). For this example, we additionally require that the line L_k do not pass through any meeting point of two different lines of T_{k-1}, and that it is not perpendicular to any line of T_{k-1}. The choice of ϱ_k is subject to the conditions that $\varrho_k \leq \varrho_{k-1}/12$ and that, for any $z \in L_k$, $B(z, \varrho_k)$ meets no more than one of the lines of which T_{k-1} consists. The function $g_k : \mathbb{R}^2 \to \mathbb{R}^2$ will be defined by

$$g_k(z) = z - 2\,\text{cutoff}\,(\langle z, v_k\rangle - \alpha_k, \varrho_k)v_k, \tag{2.1}$$

where v_k is a unit vector perpendicular to L_k and $\alpha_k = \langle u, v_k\rangle$ for $u \in L_k$. Geometrically, this definition says that, in the strip $B(L_k, \varrho_k)$, g_k is the reflection about L_k, and each of the remaining half-planes is shifted perpendicularly to L_k so that each of the two lines forming the boundary $\partial B(L_k, \varrho_k)$ of the strip is mapped onto the other one. Finally, $T_k \supset T_{k-1} \cup \partial B(L_k, \varrho_k)$ is chosen so that all compositions $g_j \circ g_{j+1} \circ \cdots \circ g_k$, where $j \le k$, are affine on every component of $\mathbb{R}^2 \setminus T_k$.

For $j \le k$, we let

$$f_{j,k} = g_j \circ g_{j+1} \circ \cdots \circ g_{k-1}, \tag{2.2}$$

with the usual convention that the composition of an empty sequence of functions is the identity. Noting that g_k is an (affine) isometry on each of the three regions into which the plane is divided by $\partial B(L_k, \varrho_k)$, we see that $f_{j,k+1}$ is an affine isometry on each component of $\mathbb{R}^2 \setminus T_k$.

Since $\|g_j(z) - z\| \le 2\varrho_j$ for every $z \in \mathbb{R}^2$, we have, for $j \le k \le l$ and $u \in \mathbb{R}^2$,

$$\begin{gathered} \|f_{k,l}(u) - u\| \le \sum_{i=k}^{l-1} \|g_i(f_{i+1,l}(u)) - f_{i+1,l}(u)\| \le \sum_{i=k}^{l-1} 2\varrho_i \le 3\varrho_k, \\ \|f_{j,k}(u) - f_{j,l}(u)\| \le \|f_{k,l}(u) - u\| \le 3\varrho_k. \end{gathered} \tag{2.3}$$

So the limits

$$f_j = \lim_{k\to\infty} f_{j,k} \tag{2.4}$$

exist and, since $\text{Lip}(g_i) \le 1$ for each i, we have $\text{Lip}(f_j) \le 1$. Moreover, for each $j \le k$,

$$f_j = f_{j,k} \circ f_k = f_{j,k} \circ g_k \circ f_{k+1}. \tag{2.5}$$

We show that $f = f_1$ is the required function. For this, assume that $z \in E$ and consider any k such that $z \in B(L_k, \varrho_k)$. Let $u \in L_k$ and $v_1, v_2 \in \partial B(L_k, \varrho_k)$, $v_1 \ne v_2$, lie on the line through z perpendicular to L_k. By the choice of ϱ_k, $[v_1, v_2]$ may meet at most one line of T_{k-1}, hence the interior of one of the segments $[u, v_1]$, $[u, v_2]$ does not cross any line of T_{k-1}. Choose the notation so that it is $[u, v_1]$ and define $v = u + 2(v_2 - u)$. Then $f_{1,k}$ is an affine isometry on $g_k([u, v]) = [u, v_1]$ and hence by (2.3) and (2.5),

$$\begin{aligned} &\left\| f(u) + f(v) - 2f\left(\frac{(u+v)}{2}\right)\right\| \\ &\quad \ge \left\| f_{1,k}(g_k(u)) + f_{1,k}(g_k(v)) - 2f_{1,k}\left(g_k\left(\frac{(u+v)}{2}\right)\right)\right\| - 12\varrho_{k+1} \\ &\quad = \left\| g_k(u) + g_k(v) - 2g_k\left(\frac{(u+v)}{2}\right)\right\| - 12\varrho_{k+1} \\ &\quad = 2\varrho_k - 12\varrho_{k+1} \ge \varrho_k. \end{aligned} \tag{2.6}$$

Since the distance of the points u,v from z is not more than $3\varrho_k$, this means that $\varepsilon^*(f,z) \geq 1/3$.

2.2. Proof of Theorem 1.2(ii). Here we do not need any further conditions on the choice of L_k, $k \geq 1$. Before choosing ϱ_k, we let $S_k = L_k \cap T_{k-1}$, denote by s_k the number of elements of S_k and choose $0 < \delta_k < 2^{-k-3}/s_k$. We also choose a unit vector e_k parallel to L_k and denote $\alpha_k = \langle z, e_k^\perp \rangle$ where $z \in L_k$; we use the notation $u^\perp = (-u_2, u_1)$ for $u = (u_1, u_2)$. We subject ϱ_k to the conditions $\varrho_k < 16^{-k-3}\sin(\pi/36)$, $\varrho_k \leq \varrho_{k-1}/32$, and $\varrho_k < 2^{-k-1}\operatorname{dist}(z, T_{k-1})$ for $z \in B(L_k, \varrho_k) \setminus B(S_k, \delta_k)$. The last assumption implies

$$B(z, 4\varrho_k) \cap T_{k-1} = \varnothing \quad \text{for } z \in B(L_k, \varrho_k) \setminus B(S_k, \delta_k). \tag{2.7}$$

Finally, we define $T_k \supset T_{k-1} \cup \partial B(L_k, \varrho_k)$ so that the function

$$\varphi_k(z) = \operatorname{cutoff}\left(\langle z, e_k^\perp \rangle - \alpha_k, \min\left(\varrho_k, 2^{-k}\operatorname{dist}(z, T_{k-1})\right)\right) \tag{2.8}$$

is affine on each component of $\mathbb{R}^2 \setminus T_k$.

We let

$$C_k = \sum_{j=0}^{k-1} 2^{-j}(4j + 24); \tag{2.9}$$

these constants will be used to control the Lipschitz constant of a sequence of functions approximating the desired function h. We list here the inequalities involving δ_k and ϱ_k in a form that will be actually used:

$$\sum_{j=k}^{\infty}\left(3\delta_j s_j + 2\varrho_j \csc\left(\frac{\pi}{36}\right)\right) < 2^{-k}, \qquad \sum_{j=k+1}^{\infty} 4\varrho_j < \frac{\varrho_k}{4}, \qquad \sum_{j=k}^{\infty} 3^{j-1} 6\varrho_j < 4^{-k}. \tag{2.10}$$

We start our construction by defining four sequences of functions that describe various aspects of the geometry of the strips $B(L_k, \varrho_k)$. Each of them will have the property that the kth function is constant on each component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^{k} \partial B(L_j, \varrho_j)$.

(1) Let $k_0(z) = 0$ and $k_p(z) = \min\{k > k_{p-1}(z) : z \in B(L_k, \varrho_k)\}$; this formula is understood to imply that $k_p(z) = \infty$ if $z \notin \bigcup_{k > k_{p-1}(z)} B(L_k, \varrho_k)$.

(2) Put $\sigma_j(z) = (-1)^p$ if $k_p(z) \leq j < k_{p+1}(z)$.

(3) Choose $W \subset \{z \in \mathbb{R}^2 : \|z\| = 1\}$ having five elements so that for every line L there is $w \in W$ whose angle with L is no more than $\pi/9$. We also pick $w_0 \in W$ and let $w_0(z) = w_0$. If U is a component of $B(L_k, \varrho_k) \setminus \bigcup_{j=1}^{k-1} \partial B(L_j, \varrho_j)$ on which the angle between $w_{k-1}(z)$ and L_k is bigger than $2\pi/9$ (notice that this angle does not depend on $z \in U$, since w_{k-1} is constant on U), then we choose $w \in W$ whose angle with L_k is no more than $\pi/9$ and let $w_k(z) = w$ for $z \in U$. In all other cases, we let $w_k(z) = w_{k-1}(z)$.

(4) Put $\zeta_k(z) = 1/\langle e_{k+1}, w_k(z) \rangle$ if $|\langle e_{k+1}, w_k(z) \rangle| \geq 1/2$ and $\zeta_k(z) = 0$ otherwise.

The functions h_k approximating h will be defined as a combination of the functions φ_k defined in (2.8). Notice that φ_k is continuous on $\mathbb{R}^2$, affine on each component of $\mathbb{R}^2 \setminus T_k$, $|\varphi_k(z)| \le \varrho_k$, $\|\varphi_k'(z)\| \le 1$, and $\|\varphi_k'(z)\| \le 2^{-k}$ for $z \notin B(L_k, \varrho_k)$. Note also that φ_k is zero on T_{k-1}, on the components of the complement of which both σ_{k-1}, and ζ_{k-1} are constant.

The coefficients of the required combination of the φ_k will depend on yet another sequence m_k of integer-valued functions on $\mathbb{R}^2$; these functions will be constant on the components of $\mathbb{R}^2 \setminus T_k$ and, similarly to the φ_k, the functions h_k approximating h will be continuous on $\mathbb{R}^2$ and affine on each such component. These functions are defined by requiring that

(i) $m_0(z) = 0$ and $h_0(z) = 0$ for all $z \in \mathbb{R}^2$;

(ii) $h_k(z) = h_{k-1}(z) + 2^{-m_{k-1}(z)}\sigma_{k-1}(z)\zeta_{k-1}(z)\varphi_k(z)$;

(iii) $m_k(z) = m_{k-1}(z) + 1$ if $z \notin T_k$ and $\|h_k'(z)\| > C_{m_{k-1}(z)}$;

(iv) $m_k(z) = m_{k-1}(z)$ in all other cases.

The function with a small set of points of differentiability is defined by

$$h(z) = \sum_{k=1}^{\infty} 2^{-m_{k-1}(z)}\sigma_{k-1}(z)\zeta_{k-1}(z)\varphi_k(z) = \lim_{k\to\infty} h_k(z); \tag{2.11}$$

the series converges since $|\zeta_{k-1}(z)| \le 2$ and so its terms are bounded by $2\varrho_k$, where $\sum_k \varrho_k$ converges.

Notice that m_{k-1} is constant on each component of $\mathbb{R}^2 \setminus T_{k-1}$ and that φ_k is zero on T_{k-1}, so h_k is continuous on $\mathbb{R}^2$ and affine on each component of $\mathbb{R}^2 \setminus T_k$. In particular, the functions h_k are Lipschitz. To show that h is Lipschitz as well, we show that

$$||h_k'(z)|| \le C_{m_k} \quad \text{for every } z \notin T_k. \tag{2.12}$$

This clearly holds for $k = 0$ and, if it holds for $k-1$, then either $\|h_k'(z)\| \le C_{m_{k-1}} \le C_{m_k}$ or m_k was defined in (iii), so $m_k = m_{k-1} + 1$ and $\|h_k'(z)\| \le C_{m_{k-1}} + 2^{-m_{k-1}+1} \le C_{m_{k-1}+1} = C_{m_k}$.

Since the sequence C_j is bounded, (2.12) implies that the Lipschitz constants of h_k are bounded by a constant independent of k and hence h is Lipschitz.

We need to show that the set of the points of differentiability of h in E is uniformly purely unrectifiable. We choose $\eta = \pi/18$ in the definition of uniform pure unrectifiability, and let I be an arc of the unit circle of length $\pi/18$. Denote by I_1 and I_2 the arcs of the unit circle concentric with I of length $\pi/9$ and $5\pi/9$, respectively. These angles fit with the definition of w_k: they are chosen so that the angle between any vector $e \in I_1$ and $w \in I_2$ is no more than $\pi/3$ and if the angle between some $e \in I_1$ and w does not exceed $\pi/9$, then $w \in I_2$ and the angle between w and any $e \in I_1$ does not exceed $2\pi/9$.

For $n = 1, 2, \ldots$, denote

$$\begin{aligned} G_n &= \bigcup_{k \ge n, \pm e_k \notin I_1} B(L_k, \varrho_k) \cup \bigcup_{k \ge n} B(S_k, \delta_k), \\ H_n &= \Big\{ z : \sup_k m_k(z) > n + 1 \Big\}. \end{aligned} \tag{2.13}$$

These sets are open: for G_n this is obvious and for H_n it follows by observing that the functions m_k are lower semicontinuous. It is our intention to show that the sets $G_n \cup H_n$ form the required open covers of the set of points of differentiability of h in E. For this purpose, we fix n and start with proving the following statement.

Claim 2.1. Let $z \in \mathbb{R}^2 \setminus G_n$ and simplify the notation by writing k_p for $k_p(z)$ and w_k for $w_k(z)$. Then for any p such that $k_p \geq n$,
(i) $e_{k_q} \in \pm I_1$ for $q \geq p$,
(ii) $\varphi_{k_q}(z) = \langle z, e_{k_q}^{\perp}\rangle - \alpha_{k_q}$ for $q \geq p$,
(iii) $w_k \in \pm I_2$ for all $k \geq k_p$,
and there is $r \geq p$ such that
(iv) $w_k = w_{k_p}$ for $k_p \leq k < k_r$, and $w_k = w_{k_r}$ for $k \geq k_r$,
(v) $\zeta_{k_q-1}(z) = 1/\langle e_{k_q}, w_{k_p}\rangle$ for $p < q < r$, and $\zeta_{k_q-1}(z) = 1/\langle e_{k_q}, w_{k_r}\rangle$ for $q > r$.

The statement (i) follows immediately from $z \in B(L_{k_q}, \varrho_{k_q})$ and $z \notin G_n$, and the statement (ii) follows from $z \in B(L_{k_q}, \varrho_{k_q}) \setminus B(S_{k_q}, \delta_{k_q})$ since for such z we have $\varrho_{k_q} < 2^{-k_q} \operatorname{dist}(z, T_{k_q-1})$. For the remaining statements, first notice that w_k stays constant for $k_{q-1} \leq k < k_q$ and that the angle between w_{k_q} and L_{k_q} never exceeds $2\pi/9$. Hence, by (i) and the definition of I_2, $w_{k_q} \in \pm I_2$ for $q \geq p$, and so $w_k \in \pm I_2$ for all $k \geq k_p$ as claimed in (iii). The statement (iv) is obvious by letting $r = p$ if $w_k = w_{k_p}$ for all $k \geq k_p$. If this is not the case, take the least index after k_p, which must necessarily be of the form k_r, for which $w_{k_r} \neq w_{k_p}$. Then $w_k = w_{k_p}$ for $k_p \leq k < k_r$, and the definition of w_{k_r} gives that the angle between w_{k_r} and L_{k_r} does not exceed $\pi/9$. Since by (i) $e_{k_q} \in \pm I_1$, the angle between w_{k_r} and any e_{k_q}, $q \geq r$, never exceeds $2\pi/9$. Hence, $w_{k_q} = w_{k_r}$ for $q \geq r$ and (iv) follows. From (i) and (iii), we infer that the angle between e_{k_q} and $w_{k_q-1} = w_{k_p}$ did not exceed $\pi/3$, and (v) follows from (iv).

We now show that h is nondifferentiable at any point $z \in E \setminus (G_n \cup H_n)$. Indeed, since $z \in E$, $k_p(z) < \infty$ for all p. So, since $z \notin H_n$, there is an index p such that $k_p \geq n$ and $m := m_{k_p}(z) = m_j(z)$ for all $j \geq k_p$. By Claim 2.1, $w_k(z) \in \pm I_2$ for all $k \geq k_p(z)$, and $e_{k_q(z)} \in \pm I_1$ for $q \geq p$. Consider any $q > p$ and denote $k = k_q(z)$. Since the angle between $w_{k-1}(z)$ and L_k does not exceed $\pi/3$, $|\zeta_{k-1}(z)| \geq 1$ and there are $u \in L_k$ and $v \in \partial B(L_k, 2\varrho_k)$ so that $v - u$ is a multiple of $w_{k-1}(z)^{\perp}$ and z lies on the line segment $[u,v]$; moreover, $\|v - u\| \leq 4\varrho_k$. So, deducing from (2.7) that h_{k-1} is affine on $B(z, 4\varrho_k)$ and that $\varphi_k(u) = 0$ and $\varphi_k(v) = \varphi_k((u+v)/2)$ and they are either both ϱ_k or both $-\varrho_k$, we use that $\sum_{j=k+1}^{\infty} |\varphi_j(u) + \varphi_j(v) - 2\varphi_j((u+v)/2)| \leq \sum_{j=k+1}^{\infty} 4\varrho_j \leq \varrho_k/4$ to estimate $|h(u) + h(v) - 2h((u+v)/2)| \geq 2^{-m}(|\varphi_k(u) + \varphi_k(v) - 2\varphi_k((u+v)/2)| - \varrho_k/2) = 2^{-m-1}\varrho_k$, which means that $\varepsilon^*(h,z) \geq 2^{-m-3} > 0$.

It follows that the proof will be finished once we find $\varepsilon_n \to 0$ (independent of γ) so that $\mu(\gamma^{-1}(G_n \cup H_n)) \leq \varepsilon_n$. Since $G_n \cup H_n$ is open, it suffices to verify this inequality for a dense set of γ (in the topology of uniform convergence), so we may and will assume that γ intersects each T_k in at most finitely many points and so all h_j are differentiable at $\gamma(t)$, for almost every $t \in [0,1]$.

The estimate of the measure of $\gamma^{-1}(G_n)$ is straightforward. Since I has length $\pi/18$, and $2\delta \sec(\pi/36) < 2\delta \sec(\pi/4) < 3\delta$, the γ-preimage of any disk of radius δ is contained in an interval of length at most 3δ and, if $e_k \notin \pm I_1$, the γ-preimage of $B(L_k, \varrho_k)$ is contained in an interval of length at most $2\varrho_k \csc(\pi/36)$. Hence,

$$\begin{aligned}\mu(\gamma^{-1}(G_n)) &\le \sum_{k\ge n}^{\infty}\sum_{z\in S_k}\mu(\gamma^{-1}(B(z,\delta_k))) + \sum_{k\ge n, e_k\notin \pm I_1}^{\infty}\mu(\gamma^{-1}(B(L_k,\varrho_k))) \\ &\le \sum_{k=n}^{\infty}\left(3\delta_k s_k + 2\varrho_k \csc\left(\frac{\pi}{36}\right)\right) < 2^{-n}.\end{aligned} \tag{2.14}$$

To estimate $\mu(\gamma^{-1}(H_n \setminus G_n))$, we have to work a little bit more. Let Σ_p be the least σ-algebra of subsets of $[0,1]$ with respect to which the functions $k_q \circ \gamma$, $0 \le q \le p$ are measurable. Then the conditional expectations $\beta_p = \mathbb{E}(\gamma' \mid \Sigma_p)$ form an $\mathbb{R}^2$-valued martingale such that $\|\beta_p\|_\infty \le 1$.

For any k, the set $B(L_k,\varrho_k) \setminus \bigcup_{j<k}\partial B(L_j,\varrho_j)$ has at most 3^{k-1} components. Let P denote one of these components. Then there is an index p so that $k = k_p(z)$ for all $z \in P$. We show that

$$\int_{\gamma^{-1}(P)} |\langle \beta_p(t), e_k^\perp\rangle|\,dt = \left|\int_{\gamma^{-1}(P)}\langle \gamma'(t), e_k^\perp\rangle dt\right| \le 6\varrho_k. \tag{2.15}$$

Since all $k_q \circ \gamma$, $0 \le q \le p$ are constant on $\gamma^{-1}(P)$, so is β_p. Hence,

$$\int_{\gamma^{-1}(P)} |\langle \beta_p(t), e_k^\perp\rangle|\,dt = \left|\int_{\gamma^{-1}(P)}\langle \beta_p(t), e_k^\perp\rangle dt\right|, \tag{2.16}$$

and the equality follows from the definition of conditional expectations. The inequality is obvious if P does not meet γ or if the angle between L_k and all vectors from I is at least $\pi/6$, since then $\gamma^{-1}(P)$ is contained in an interval of length at most $4\varrho_k$. When the angle between L_k and some vector from I is less than $\pi/6$, the function $t \to \langle \gamma(t), e_k\rangle$ is strictly monotonic. Let $a = \inf\{\langle z, e_k\rangle : z \in P\}$ and $b = \sup\{\langle z, e_k\rangle : z \in P\}$. Since P is an open convex set, there are functions ψ^- and ψ^+ on (a,b) such that ψ^- is convex, ψ^+ is concave, $\psi^- < \psi^+$, and $\partial P \cap \{z : a < \langle z, e_k\rangle < b\}$ is the union of the graphs of ψ^- and ψ^+ (in the coordinate system $e_k, e_k^\perp$). By our assumption on γ, ∂P meets γ only in a finite set, hence $\gamma^{-1}(P)$ is the union of finitely many intervals, say $(a_1,a_2),(a_3,a_4),\ldots,(a_{2d-1},a_{2d})$, where $\langle \gamma(a_1), e_k\rangle, \langle \gamma(a_2), e_k\rangle, \ldots$ is strictly monotonic and for each $1 \le i \le d-1$ both points $\gamma(a_{2i})$ and $\gamma(a_{2i+1})$ lie either on the graph of ψ^- or on the graph of ψ^+. Since ψ^- is convex and oscillates between $\alpha_k - \varrho_k$ and $\alpha_k + \varrho_k$, the sum of $\langle \gamma(a_{2i+1}) - \gamma(a_{2i}), e_k^\perp\rangle = \psi^-(\langle \gamma(a_{2i+1}), e_k\rangle) - \psi^-(\langle \gamma(a_{2i}), e_k\rangle)$ over those i for which the first case occurs is at most $2\varrho_k$. Similarly, we obtain the same estimate of the sum of $\langle \gamma(a_{2i+1}) - \gamma(a_{2i}), e_k^\perp\rangle$ over those i for which the second case occurs. Hence,

$$\begin{aligned}&\left|\sum_{i=1}^{d}\langle \gamma(a_{2i}) - \gamma(a_{2i-1}), e_k^\perp\rangle\right| \\ &\qquad \le |\langle \gamma(a_{2d}) - \gamma(a_1), e_k^\perp\rangle| + \left|\sum_{i=1}^{d-1}\langle \gamma(a_{2i+1}) - \gamma(a_{2i}), e_k^\perp\rangle\right| \le 6\varrho_k,\end{aligned} \tag{2.17}$$

and (2.15) is proved.

For any fixed p, by summing (2.15) first over those components P of $B(L_k,\varrho_k)\setminus\bigcup_{j<k}\partial B(L_j,\varrho_j)$ for which $k_p(z)=k$ on P, which gives no more than 3^{k-1} terms, and then over k, which starts only from p, we get that

$$\int_{A_p} \left|\langle \beta_p(t), e^{\perp}_{k_p(\gamma(t))}\rangle\right| dt \le \sum_{k=p}^{\infty} 3^{k-1}6\varrho_k < 4^{-p}, \tag{2.18}$$

where $A_p=\{t : k_p(\gamma(t))<\infty\}$.

Hence, letting

$$D_p := \{t : k_p(\gamma(t))<\infty \text{ and } \left|\langle \beta_p(t), e^{\perp}_{k_p(\gamma(t))}\rangle\right| > 2^{-p}\}, \tag{2.19}$$

we conclude from the Markov inequality that

$$\mu(D_p) < 2^{-p}. \tag{2.20}$$

For each $\nu\in I_2$, we infer from $\gamma'(t)\in I\subset I_1$ that $1/2\le\langle\gamma'(t),\nu\rangle\le 1$. Hence,

$$\mu^{\nu}(A) := \frac{\int_A \langle\gamma',\nu\rangle\,dt}{\int_0^1\langle\gamma',\nu\rangle\,dt} \tag{2.21}$$

is a well-defined probability measure on $[0,1]$. Since $\mathbb{E}(\langle\gamma',\nu\rangle|\Sigma_p)=\langle\beta_p,\nu\rangle$ and $\mathbb{E}(\langle\gamma',\nu^{\perp}\rangle|\Sigma_p)=\langle\beta_p,\nu^{\perp}\rangle$,

$$\begin{aligned}\mathbb{E}\left(\frac{\langle\beta_p,\nu^{\perp}\rangle}{\langle\beta_p,\nu\rangle}\cdot\langle\gamma',\nu\rangle\,\middle|\,\Sigma_p\right) &= \frac{\langle\beta_p,\nu^{\perp}\rangle\cdot\mathbb{E}(\langle\gamma',\nu\rangle\,|\,\Sigma_p)}{\langle\beta_p,\nu\rangle} = \langle\beta_p,\nu^{\perp}\rangle\\ &= \mathbb{E}(\langle\gamma',\nu^{\perp}\rangle\,|\,\Sigma_p) = \mathbb{E}\left(\frac{\langle\gamma',\nu^{\perp}\rangle}{\langle\gamma',\nu\rangle}\cdot\langle\gamma',\nu\rangle\,\middle|\,\Sigma_p\right).\end{aligned} \tag{2.22}$$

Therefore, $\langle\beta_p,\nu^{\perp}\rangle/\langle\beta_p,\nu\rangle$ is a real-valued martingale with respect to the measure μ^{ν} and filtration Σ_p. Since both $\langle\beta_p,\nu^{\perp}\rangle$ and $\langle\beta_p,\nu\rangle$ are in the interval $[1/2,1]$, the martingale is bounded by 2. From this, it follows that the $L^2(\mu^{\nu})$ norm of the martingale is bounded by 2, moreover,

$$\left\|\frac{\langle\beta_0,\nu^{\perp}\rangle}{\langle\beta_0,\nu\rangle}\right\|^2_{L^2(\mu^{\nu})} + \sum_{p=1}^{\infty}\left\| -\frac{\langle\beta_{2p-1},\nu^{\perp}\rangle}{\langle\beta_{2p-1},\nu\rangle} + \frac{\langle\beta_{2p},\nu^{\perp}\rangle}{\langle\beta_{2p},\nu\rangle}\right\|^2_{L^2(\mu^{\nu})} \le 4. \tag{2.23}$$

Let

$$\beta^{\nu}_p = \sum_{q=0}^{p}(-1)^q\frac{\langle\beta_q,\nu^{\perp}\rangle}{\langle\beta_q,\nu\rangle}. \tag{2.24}$$

Then β^{ν}_{2p-1} is a μ^{ν} martingale with respect to the σ-algebras Σ_{2p-1} with $L^2(\mu^{\nu})$-norm bounded by 2. By Kolmogorov's martingale inequality, $\mu^{\nu}\{t : \sup_p|\beta^{\nu}_{2p-1}|>n\}<4/n^2$. Since the terms of the series defining β^{ν}_p are bounded by 2, we conclude that $\sup_p|\beta^{\nu}_p|\le\sup_q|\beta^{\nu}_{2q-1}|+2$ and so $\mu^{\nu}\{t : \sup_p|\beta^{\nu}_p|>n+2\}\le 4/n^2$ whenever $\nu\in I_2$. Since $\mu\le 2\mu^{\nu}$, the Lebesgue measure of these sets is at most $8/n^2$. The same estimate holds also for $v \in -I_2$, since $\beta_p^{-v} = \beta_p^v$. Hence, denoting

$$B = \left\{ t : \sup_p \left| \sum_{q=0}^{p} (-1)^q \frac{\langle \beta_q, v^\perp \rangle}{\langle \beta_q, v \rangle} \right| > n+2 \text{ for some } v \in W \cap \pm I_2 \right\}, \tag{2.25}$$

we have

$$\mu(B) \le \frac{40}{n^2}. \tag{2.26}$$

We show that

$$\mu\left(\gamma^{-1}(H_n \setminus G_n) \setminus \left(B \cup \bigcup_{p=n}^{\infty} D_p \right) \right) = 0. \tag{2.27}$$

By (2.20) and (2.26), this will give $\mu(\gamma^{-1}(H_n \setminus G_n)) < 2^{-n+1} + 40/n^2$, and so finish the proof.

To establish (2.27), suppose that $t \in (0,1) \setminus (B \cup \bigcup_{p=n}^{\infty} D_p)$ is such that $z = \gamma(t) \in H_n \setminus G_n$ and all h_j are differentiable at z and simplify the notation by denoting $m_k(z) = m_k$, $w_k(z) = w_k$, and $k_p(z) = k_p$. We will need an estimate, for any $k < l$, of

$$\|h_l'(z) - h_k'(z)\| = \left\| \sum_{j=k+1}^{l} 2^{-m_{j-1}} \sigma_{j-1}(z) \zeta_{j-1}(z) \varphi_j'(z) \right\|. \tag{2.28}$$

Let p be the least index such that $k_p > k$ and let q be the largest index such that $k_q \le l$. Recall that $|\sigma_{j-1}(z)| = 1$, $|\zeta_{j-1}(z)| \le 2$, $\|\varphi_j'(z)\| \le 1$, and $m_{j-1} \ge m_k$ for all $k+1 \le j \le l$. Hence, the norm of each term of the series is trivially estimated by 2^{-m_k+1}. If $z \notin B(L_j, \varrho_j)$, we also have $\|\varphi_j'(z)\| \le 2^{-j}$, and so the contribution of the terms for which $z \notin B(L_j, \varrho_j)$ is at most

$$\sum_{j=k+1}^{l} 2^{-m_k} |\zeta_{j-1}(z)| \|\varphi_j'(z)\| \le 2^{-m_k} \sum_{j=k+1}^{l} 2^{-j+1} \le 2^{-m_k+1}. \tag{2.29}$$

Using this, the trivial estimate for $j = k_p$ and $j = k_q$, the simple fact that $\sigma_{k_s-1}(z) = (-1)^{s-1}$, and noting that the untreated indices j are of the form $j = k_s$, where $p < s < q$, we get

$$\begin{aligned} \|h_l'(z) - h_k'(z)\| &\le 6 \cdot 2^{-m_k} + \left\| \sum_{p<s<q} 2^{-m_{k_s-1}} (-1)^{s-1} \zeta_{k_s-1}(z) \varphi_{k_s}'(z) \right\| \\ &\le 2^{-m_k+3} + \left\| \sum_{p<s<q} 2^{-m_{k_s-1}} (-1)^{s-1} \zeta_{k_s-1}(z) \varphi_{k_s}'(z) \right\|. \end{aligned} \tag{2.30}$$

A simple corollary of this is that $m_{k_r} \le r$ for all r. Indeed, since $\|h_{k_r}'(z)\| \le C_{m_{k_r}}$ for all r by (2.12), we get from (2.30) with $k = k_r$ and $l \le k_{r+1}$ that $\|h_l'(z)\| \le C_{m_{k_r}} + 2^{-m_{k_r}+3} \le C_{m_{k_r}+1}$

for all $k_r < l \le k_{r+1}$. By the definition of m_l, this gives $m_l \le m_{k_r} + 1$ for all $k_r < l \le k_{r+1}$; in particular, $m_{k_{r+1}} \le m_{k_r} + 1$. Since this holds for all r, $m_{k_r} \le r$.

We now turn our attention to the estimate of the sum in (2.30) under the special assumptions that for all $p < s < q$, $w_{k_s} = w_{k_p}$ and $m_{k_s} = m_{k_p} \ge n$. Since $k_p \ge m_{k_p} \ge n$, Claim 2.1 shows that $\varphi'_{k_s}(z) = e^{\perp}_{k_s}$ and $\zeta_{k_s-1}(z) = 1/\langle e_{k_s}, w_{k_p}\rangle$. Hence, we wish to estimate the norm of the vector

$$u = u_{p,q} := \sum_{p<s<q} 2^{-m_{k_s-1}}(-1)^{s-1}\zeta_{k_s-1}(z)\varphi'_{k_s}(z) = \sum_{p<s<q} 2^{-m_{k_p}}(-1)^{s-1}\frac{e^{\perp}_{k_s}}{\langle e_{k_s}, w_{k_p}\rangle}. \tag{2.31}$$

Since $|\langle u^{\perp}, w_{k_p}\rangle| = |\sum_{p<s<q}(-1)^{s-1}2^{-m_{k_p}}| \le 2^{-m_{k_p}} \le 2^{-n}$, we will establish this by estimating $|\langle u^{\perp}, w^{\perp}_{k_p}\rangle|$. For this, we switch from e_{k_s} to $\beta_s(t)$; recall that by Claim 2.1, $e_{k_s} \in \pm I_1$, $w_{k_p} \in \pm I_2$, $\gamma'(t) \in I \subset I_1$, therefore $|\langle e_{k_s}, w_{k_p}\rangle| \ge 1/2$, $|\langle \beta_s(t), w_{k_p}\rangle| = |\mathbb{E}(\langle \gamma', w_{k_p}\rangle|\Sigma_s)| \ge 1/2$, $\|\beta_s(t)\| \ge 1/2$, and $\beta_s(t)/\|\beta_s(t)\| \in I_1$. We also have $|\langle \beta_s(t), e^{\perp}_{k_s}\rangle| \le 2^{-s}$ since $s > p \ge m_{k_p} \ge n$ and so $t \notin D_s$, and $k_s(\gamma(t)) < \infty$. Hence,

$$\left| \frac{\langle \beta_s(t), w^{\perp}_{k_p}\rangle}{\langle \beta_s(t), w_{k_p}\rangle} - \frac{\langle e_{k_s}, w^{\perp}_{k_p}\rangle}{\langle e_{k_s}, w_{k_p}\rangle} \right| = \left| \frac{\langle \beta_s(t), e^{\perp}_{k_s}\rangle}{\langle e_{k_s}, w_{k_p}\rangle\langle \beta_s(t), w_{k_p}\rangle} \right| \le 2^{-s+2}, \tag{2.32}$$

and we see from $t \notin B$ that

$$\begin{aligned} |\langle u^{\perp}, w^{\perp}_{k_p}\rangle| &\le 2^{-m_{k_p}}\left(\left| \sum_{p<s<q}(-1)^{s-1}\frac{\langle \beta_s(t), w^{\perp}_{k_p}\rangle}{\langle \beta_s(t), w_{k_p}\rangle} \right| + \sum_{p<s<q} 2^{-s+2} \right) \\ &\le 2^{-m_{k_p}}\left(2(n+2) + 2^{-p+2}\right) \\ &\le 2^{-n}(2n+6). \end{aligned} \tag{2.33}$$

Consequently,

$$\|u_{p,q}\| \le 2^{-n}(2n+7). \tag{2.34}$$

After this digression, we are ready to finish the argument. Since $m_0 = 0$, $m_{j+1} \le m_j + 1$, and $\sup_j m_j \ge n+2$, there are indices j_0 and j_1 such that $m_{j_0-1} = n$, $m_j = n+1$, for $j_0 \le j < j_1$, and $m_{j_1} = n+2$. Let r_0 and r_1 be the least indices such that $k_{r_0} \ge j_0$ and $k_{r_1} \ge j_1$. We note that $k_{r_0} \ge m_{k_{r_0}} \ge m_{j_0-1} = n$. Hence, Claim 2.1 implies that there is $r_2 \ge r_0$ so that $w_k(z) = w_{k_{r_0}}(z)$ for $k_{r_0} \le k < k_{r_2}$, and $w_k(z) = w_{k_{r_2}}(z)$ for $k \ge k_{r_2}$. Let $r_3 = \min(r_1, r_2)$. It follows that (2.34) can be used with $p = r_0$ and $q = r_3$ as well as with $p = r_3$ and $q = r_1$, and we get

$$\|h'_{j_1}(z) - h'_{j_0-1}(z)\| \le 2^{-n+3} + \|u_{r_0,r_3}\| + 2^{-n+1} + \|u_{r_3,r_1}\| \le 2^{-n}(4n+24). \tag{2.35}$$

Since $m_{j_0-1} = n$, $\|h'_{j_0-1}(z)\| \le C_n$ and so

$$\|h'_{j_1}(z)\| \le C_n + 2^{-n}(4n+24) \le C_{n+1} = C_{m_{j_1-1}}. \tag{2.36}$$

But this means that $n+2 = m_{j_1} = m_{j_1-1} = n+1$, which is the contradiction we desired to prove (2.27), finishing the proof of the theorem.

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