A DEGREE THEORY FOR COMPACT PERTURBATIONS OF PROPER C^1 FREDHOLM MAPPINGS OF INDEX 0

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Received 14 June 2004

We construct a degree for mappings of the form $F+K$ between Banach spaces, where F is C^1 Fredholm of index 0 and K is compact. This degree generalizes both the Leray-Schauder degree when $F=I$ and the degree for C^1 Fredholm mappings of index 0 when $K=0$. To exemplify the use of this degree, we prove the "invariance-of-domain" property when $F+K$ is one-to-one and a generalization of Rabinowitz's global bifurcation theorem for equations $F(\lambda,x)+K(\lambda,x)=0$.

1. Introduction

We generalize the Leray-Schauder degree to mappings $F+K$ between real Banach spaces X and Y, where F is C^1 Fredholm of index 0 and K is compact. Recall that the Leray-Schauder theory addresses the case when $X=Y$ and $F(x)=x$. Throughout this paper, a (nonlinear) compact operator is a continuous operator mapping bounded subsets to relatively compact ones.

Under rather restrictive additional assumptions, it is sometimes possible to reduce a problem involving an operator $F+K$ as above to one having the desired structure in the Leray-Schauder theory. For instance, if F is a homeomorphism of X onto Y and F^{-1} maps bounded subsets to bounded subsets, the equation $F(x)+K(x)=z$ becomes $y+K\circ F^{-1}(y)=z$ and $K\circ F^{-1}$ is compact. More generally, this reduction is possible if the above assumptions are made about $F+N$, where $N\in C^1(X,Y)$ is compact (just write $F+K=(F+N)+(K-N)$). However, the existence of such an operator N is not guaranteed when F is nonlinear.

The possible nonexistence of an equivalent Leray-Schauder formulation is already an issue when $K=0$ and the question is to define a degree for C^1 Fredholm mappings F of index 0. The first investigations by Caccioppoli [5, 6] in this direction, resulting in a mod 2 degree, go back to 1936, only two years after the work of Leray and Schauder. In the C^2 case, the mod 2 degree was subsequently rediscovered and more rigorously justified by Smale [26].

The first instance of a $\mathbb{Z}$-valued degree for C^2 Fredholm mappings of index 0 can be found in [8, 9] by Elworthy and Tromba. Although its definition via Fredholm structures

Abstract and Applied Analysis 2005:7 (2005) 707–731
DOI: 10.1155/AAA.2005.707

made this degree of limited practical value, this work revealed that an integer-valued degree theory for Fredholm mappings of index 0 *cannot* comply with the homotopy invariance property and, more specifically, that the degree *must* be allowed to change sign under some Fredholm homotopies. An attempt to combine the Elworthy-Tromba approach with the ideas of Caccioppoli to handle the C^1 case can be found in the work of Borisovich et al. [4], where compact perturbations are also discussed. A theory for the nonnegative index case, in which the degree is defined as a framed cobordism, was further developed in [28] by Zvyagin and [29] by Zvyagin and Ratiner.

The construction of a "user-friendly" degree was described by Fitzpatrick et al. [11], based on the concept of parity of a path of linear Fredholm operators of index 0 (see Section 2), which also makes it possible to assess whether a sign change occurs during homotopy. The technical difficulties to extend this construction to the C^1 case were resolved in [17, 18] by Pejsachowicz and Rabier. An approach technically simpler in places and in a setting where homotopy invariance holds was proposed by Benevieri and Furi [1, 2]. Several applications to existence or bifurcation questions in ODEs or PDEs on unbounded domains (which, typically, give rise to operators beyond the Leray-Schauder theory) have been described in [14, 15, 20, 21, 25], among others. Further comments on the development of the degree theory for Fredholm mappings and additional references, notably to the work of the Russian school, can be found in [12].

The starting point of this paper is the degree described in [18] for C^1 Fredholm mappings of index 0. A concise but complete review of its definition and properties is given in Section 2. A special feature is that the numerical value of the degree depends upon the choice of a *base-point* of the mapping F of interest, which is simply a point $p \in X$ such that $DF(p)$ is invertible (in these introductory comments, there is no need to consider the case when no base-point exists). Then, if $\Omega \subset X$ is an open subset such that F is proper on $\overline{\Omega}$, the base-point degree $d_p(F,\Omega,y) \in \mathbb{Z}$ is defined for all $y \notin F(\partial\Omega)$. Different choices of the base-point lead to the same degree up to sign and the possible sign change is characterized as a parity.

If now F above is replaced by $F+K$ with $K \in C^0(X,Y)$ compact and Ω is bounded, the general idea is to replace K by a suitable C^1 approximation K_ε, so that $F+K_\varepsilon$ is C^1 Fredholm of index 0 and proper on $\overline{\Omega}$. As a result, a base-point degree is available for $F+K_\varepsilon$ which, in principle, can be used to define the desired degree for $F+K$. However, the justification of this natural approach is faced with various nontrivial difficulties.

The first problem is of course the very *existence* of such "suitable" approximations K_ε of K of class C^1, which turns out to be a delicate technical issue. When the dual space X^* is separable and hence X is separable and has an equivalent norm of class C^1 away from the origin (Restrepo [23]), the problem can be greatly simplified thanks to an approximation theorem of Bonic and Frampton [3]. Details can be found in the dissertation [24]. Our strategy when no C^1 norm is available consists in first considering the case when K is *finite dimensional*. The C^1 approximation remains delicate and cannot be established on all of $\overline{\Omega}$, but only on well-chosen compact subsets.

The approximation of K by C^1 mappings K_ε is not the only difficulty, even when K is finite dimensional: if the base-point degree for $F+K_\varepsilon$ is to be used to define the degree for $F+K$, it must naturally be shown that it is independent of the approximation K_ε. But $d_p(F+K_\varepsilon,\Omega,y)$ makes sense only if p is a base-point of $F+K_\varepsilon$, and p need not remain a base-point after K_ε is changed. This makes it impossible to compare the degrees $d_p(F+K_\varepsilon,\Omega,y)$ for all the possible approximations K_ε. A proof of the independence of $d_p(F+K_\varepsilon,\Omega,y)$ with p depending upon K_ε is even more hopeless since, even when K_ε is fixed, different choices of p may yield degrees of opposite signs.

The way to overcome this ambiguity is to choose p above to be a base-point of F in the first place and to confine attention to approximations K_ε "based at p", that is, satisfying $DK_\varepsilon(p)=0$. Thus, p is a base-point of $F+K_\varepsilon$ for all the approximations K_ε of interest and it now makes sense to ask whether $d_p(F+K_\varepsilon,\Omega,y)$ is independent of the approximation K_ε based at p. This happens to be true, along with the existence of such approximations.

This yields a reasonable definition for a degree $d_p(F+K,\Omega,y)$ depending upon the base-point p of F (not of $F+K$, since this makes no sense if K is not differentiable). The next obvious question is whether this degree is independent of the decomposition $F+K$: if $T=F+K=G+L$ with F and G Fredholm of index 0 and K and L compact, and if p is a base-point of F *and* of G, does the above construction provide the same value for $d_p(T,\Omega,y)$ when $T=F+K$ or $T=G+L$? (This is not an issue with the Leray-Schauder degree since $F=G=I$ and $K=L$ in that theory.) The answer is *negative*, but this does not induce any new complication, as we now explain.

First, this negative answer shows that the degree for $F+K$ constructed above depends not only upon the base-point p of F but also upon the decomposition, that is, upon F. As a result, $d_p(F+K,\Omega,y)$ is not a correct notation for the degree and $d_{F,p}(F+K,\Omega,y)$ should be used instead. At a first sight, this opens up the possibility that a plethora of integers might represent the degree of the same mapping, depending upon which decomposition is chosen. But the reality is much simpler: two decompositions $T=F+K=G+L$ and two base-points p and q of F and G, respectively, can only lead to the same degree ($d_{F,p}(T,\Omega,y)=d_{G,q}(T,\Omega,y)$) or to two degrees of opposite signs ($d_{F,p}(T,\Omega,y)=-d_{G,q}(T,\Omega,y)$). Furthermore, the possible change of sign can once again be characterized as a parity.

The degree $d_{F,p}(F+K,\Omega,y)$ satisfies all the properties expected of a topological degree, except (just as when $K=0$) that it may change its sign under homotopies. However, special homotopies, notably those affecting only K, leave the degree invariant. When F is a *local diffeomorphism*, $d_{F,p}(F+K,\Omega,y)$ is actually independent of $p\in X$ and can then be denoted by $d_F(F+K,\Omega,y)$. In particular, when $X=Y$ and $F=I$, we obtain a degree $d_I(I+K,\Omega,y)$ for the compact perturbations of the identity. Not surprisingly, this degree coincides with the Leray-Schauder degree, which thus appears as a genuine special case of $d_{F,p}(F+K,\Omega,y)$.

To demonstrate the fact that, in its general form, the degree of this paper is as versatile as the Leray-Schauder degree, we use it in Section 8 to prove the "invariance-of-domain" property and in Section 9 to generalize the well-known Rabinowitz global bifurcation theorem to problems of the form $F(\lambda,x)+K(\lambda,x)=0$.

Throughout the paper, $\mathscr{L}(X,Y)$ is the space of bounded linear operators from X to Y, $GL(X,Y)$ the subset of linear isomorphisms, $\Phi_n(X,Y)$ the subset of Fredholm operators of index $n\in\mathbb{Z}$, and $\mathscr{K}(X,Y)$ the subspace of compact operators. For $m\in\mathbb{N}$ and $n\in\mathbb{Z}$, we let $\Phi_n C^m(X,Y)$ denote the set of C^m Fredholm mappings of index n from X to Y. As is customary, when $X = Y$, we use $\mathcal{L}(X)$, $GL(X)$, $\Phi_n(X)$, $\mathcal{K}(X)$, and $\Phi_n C^m(X)$, respectively.

2. Background

We briefly review the definition and properties of the base-point degree for proper C^1 Fredholm mappings of index 0. Details can be found in [18] and the references therein.

A basic concept is that of *parity* of a continuous path of linear Fredholm operators of index 0. Let X and Y be real Banach spaces and, given a compact interval $[a,b] \subset \mathbb{R}$, let $A \in C^0([a,b],\Phi_0(X,Y))$ be a path of linear Fredholm operators of index 0 such that $A(a), A(b) \in GL(X,Y)$. As shown in [10, Theorem 1.3.6] or [27, Theorem 2.10], there is $P \in C^0([a,b],GL(Y,X))$ (parametrix) such that $PA = I - C$ where $C \in C^0([a,b],\mathcal{K}(X))$. Then, $I - C(a) \in GL(X)$ and $I - C(b) \in GL(X)$ have well-defined indices $(= (-1)^\ell$ where ℓ is the number of negative eigenvalues, counted with multiplicity) $i(I - C(a)) \in \{-1,1\}$ and $i(I - C(b)) \in \{-1,1\}$ and the parity $\sigma(A)$ of A (or $\sigma(A,[a,b])$ if displaying the interval is important) is defined by

$$\sigma(A) := i(I - C(a))\,i(I - C(b)) \in \{-1,1\}. \tag{2.1}$$

Of course, it can be shown that this formula is independent of the parametrix P.

To understand the meaning of $\sigma(A)$, it is helpful to consider the case $X = Y = \mathbb{R}^N$ when, as is easily seen, $\sigma(A) = \operatorname{sgn}\det A(a)\operatorname{sgn}\det A(b)$. In the infinite-dimensional setting, $\sigma(A)$ may be thought of as a generalization of this formula in the absence of any determinant function. However, in contrast to the finite-dimensional case, $\sigma(A)$ need *not* depend only upon the endpoints a and b (see [10]). Alternatively, $\sigma(A)$ may be viewed (generically) as the mod 2 count of the number of times $A(t)$ crosses the subset of noninvertible linear Fredholm operators of index 0 as t runs over $[a,b]$.

The parity has a number of interesting properties, including homotopy invariance (provided that the endpoints remain invertible during the homotopy), multiplicativity with respect to consecutive intervals (i.e., $\sigma(A,[a,c]) = \sigma(A,[a,b])\sigma(A,[b,c])$ if $A(a)$, $A(b)$, and $A(c)$ are invertible), and multiplicativity with respect to composition. Also important in practice, the parity is unchanged by reparametrizations and the parity of a path of linear isomorphisms is always 1.

Let now $F \in \Phi_0 C^1(X,Y)$ be given and let $\Omega \subset X$ be an open subset. Assume that F is proper on $\overline{\Omega}$ and let $y \notin F(\partial\Omega)$ be a regular value of $F_{|\Omega}$, that is, $DF(x) \in GL(X,Y)$ whenever $x \in \Omega$ and $F(x) = y$. Then, by properness, the set $F^{-1}(y) \cap \Omega = F^{-1}(y) \cap \overline{\Omega}$ is finite, say

$$F^{-1}(y) \cap \Omega = \{x_1,\dots,x_k\}, \tag{2.2}$$

for some integer $k \geq 0$.

Given $p \in X$ such that $DF(p) \in GL(X,Y)$ (a *base-point* of F), the degree $d_p(F,\Omega,y)$ is defined by the sum of parities:

$$d_p(F,\Omega,y) = \sum_{i=1}^{k} \sigma(DF \circ \gamma_i), \tag{2.3}$$

where γ_i is any continuous curve in X joining p to x_i, $1 \leq i \leq k$. If $k = 0$ (so that $F^{-1}(y) = \emptyset$), we set $d_p(F,\Omega,y) = 0$. The homotopy invariance of the parity ensures that the above definition of $d_p(F,\Omega,y)$ is independent of γ_i. It does, however, depend upon p, but passing from a base-point to another can only leave the degree unchanged or change it into its negative (see Corollary 2.4). Thus, the "absolute" degree $|d|$ defined by $|d|(F,\Omega, y) = |d_p(F,\Omega,y)|$ is independent of p. This makes it possible to define $|d|$ even when no base point exists, by setting $|d|(F,\Omega,y) = 0$ in this case.

Remark 2.1. It follows from the above definition that when F is a linear isomorphism, then $d_p(F,\Omega,y) = 1$ regardless of $p \in X$ whenever $y \in F(\Omega)$. This is often useful in practical calculations, together with homotopy arguments (see below). That $d_p(F,\Omega,y)$ can *never* be -1 when F is linear points to differences—not contradictions or incompatibilities—between the base-point degree and the Leray-Schauder degree.

The most technical step consists in defining $d_p(F,\Omega,y)$ when $y \notin F(\partial\Omega)$ is not necessarily a regular value of $F_{|\Omega}$. When F is C^2, this is done by approximating y by regular values (see [11]). When F is only C^1, this approach fails and must be replaced by approximating F rather than y (see [17, 18]).

The properties of the base-point degree, listed below, are almost the expected ones, the notable exception being that it is only invariant up to sign under homotopies. However, the sign change (or lack thereof) can be fully monitored by the parity, as indicated in Theorem 2.2.

Theorem 2.2. *Let $h \in \Phi_1 C^1([0,1] \times X, Y)$ be proper on $[0,1] \times \overline{\Omega}$ and let $y \notin h([0,1] \times \partial\Omega)$ be given. The following properties hold.*

(i) *If $p_0 \in X$ and $p_1 \in X$ are base-points of $h(0,\cdot)$ and $h(1,\cdot)$, respectively, then*

$$d_{p_0}(h(0,\cdot),\Omega,y) = \nu d_{p_1}(h(1,\cdot),\Omega,y), \tag{2.4}$$

where $\nu := \sigma(D_x h \circ \Gamma) \in \{-1,1\}$ and Γ is any continuous curve in $[0,1] \times X$ joining $(0,p_0)$ to $(1,p_1)$.

(ii) *If $p_0 \in X$ is a base-point of $h(0,\cdot)$ and if $h(1,\cdot)^{-1}(y) = \emptyset$, then*

$$d_{p_0}(h(0,\cdot),\Omega,y) = 0. \tag{2.5}$$

(iii) *If $p_0 \in X$ is a base-point of $h(0,\cdot)$ and if $h(1,\cdot)$ has no base-point, then*

$$d_{p_0}(h(0,\cdot),\Omega,y) = 0. \tag{2.6}$$

The following corollary gives a simple but useful case when homotopy invariance holds.

Corollary 2.3. *Let $h \in \Phi_1 C^1([0,1] \times X, Y)$ be proper on $[0,1] \times \overline{\Omega}$ and let $y \notin h([0,1] \times \partial\Omega)$ be given. If $p \in X$ is a base-point of $h(t,\cdot)$ for all $t \in [0,1]$ (i.e., $D_x h(t,p) \in GL(X,Y)$ for all $t \in [0,1]$), then $d_p(h(0,\cdot),\Omega,y) = d_p(h(1,\cdot),\Omega,y)$.*

Corollary 2.4. *Let $F \in \Phi_0 C^1(X,Y)$ be proper on $\overline{\Omega}$ and let $p,q \in X$ be base-points of F. If $y \notin F(\partial\Omega)$, then $d_q(F,\Omega,y) = \nu d_p(F,\Omega,y)$ where $\nu := \sigma(DF \circ \gamma) \in \{-1,1\}$ and γ is any continuous curve in X joining p to q.*

Corollary 2.5 (local constancy). *Let $F \in \Phi_0 C^1(X,Y)$ be proper on $\overline{\Omega}$ and let $p \in X$ be a base-point of F. The degree $d_p(F,\Omega,y)$ depends only upon the connected component of $Y \backslash F(\partial\Omega)$ containing y.*

Corollary 2.6 (normalization). *Let $F \in \Phi_0 C^1(X,Y)$ be proper on $\overline{\Omega}$ and let $p \in X$ be a base-point of F. If $y \notin F(\partial\Omega)$ and $d_p(F,\Omega,y) \neq 0$, then $F^{-1}(y) \cap \Omega \neq \emptyset$.*

The following two properties of the degree are also important for calculations.

Theorem 2.7 (excision). *Let $F \in \Phi_0 C^1(X,Y)$ be proper on $\overline{\Omega}$ and let Σ be a closed subset of Ω. Then, F is proper on $\overline{\Omega \backslash \Sigma}$. Furthermore, if $y \notin F(\Sigma \cup \partial\Omega)$, then $y \notin F(\partial(\Omega \backslash \Sigma))$ and if $p \in X$ is a base-point of F, $d_p(F,\Omega \backslash \Sigma,y) = d_p(F,\Omega,y)$.*

Theorem 2.8 (additivity on domain). *Suppose that $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint open subsets of X and let $F \in \Phi_0 C^1(X,Y)$ be proper on $\overline{\Omega}$. Then, F is proper on $\overline{\Omega}_\alpha$, $\alpha = 1,2$. Furthermore, if $y \notin F(\partial\Omega)$, then $y \notin F(\partial\Omega_\alpha)$, $\alpha = 1,2$, and if $p \in X$ is a base-point of F, $d_p(F,\Omega,y) = d_p(F,\Omega_1,y) + d_p(F,\Omega_2,y)$.*

The absolute degree $|d|$ is *homotopy invariant*, which is consistent with Theorem 2.2 when base-points exist, but is true in general. Corollaries 2.5 and 2.6 as well as Theorem 2.7 also hold for $|d|$.

All of the above can be repeated if F is only defined on a connected and simply connected open subset $\mathbb{O}$ of X with $\Omega \subset \mathbb{O}$ (it even suffices that the first cohomology group $H^1(\mathbb{O})$ vanishes; see [12]) and closures are understood relative $\mathbb{O}$. It will be obvious from the proofs that the results of this paper can also be extended verbatim to this setting.

3. Degree for finite-dimensional perturbations: definition

In this section, $F \in \Phi_0 C^1(X,Y)$ and the compact *finite dimensional* mapping $K \in C^0(X,Y)$ are given and Ω denotes a *bounded* open subset of X. We define a degree for $F+K$ that generalizes the base-point degree of Section 2 when $K=0$. After this degree is constructed and its main properties established, it will be a simple matter to drop the assumption that K is finite dimensional (Section 7).

We will further assume that F is proper on $\overline{\Omega}$ and that

$$F(\overline{\Omega}) \text{ is bounded}, \tag{3.1}$$

although (3.1) will be dropped in Section 7.

Let Y_0 be a finite-dimensional subspace of Y such that $K(X) \subset Y_0$. Since $F(\overline{\Omega})$ is closed (by the properness of F), it follows from (3.1) that $F(\overline{\Omega}) \cap Y_0$ is a compact subset of Y

and hence that

$$S(F,\overline{\Omega},Y_0) := F^{-1}(F(\overline{\Omega}) \cap Y_0) \cap \overline{\Omega} \tag{3.2}$$

is a compact subset of X. For example, if $X = Y$ and $F = I$, then $S(I,\overline{\Omega},Y_0) = \overline{\Omega} \cap Y_0$. Note that $(F+K)^{-1}(0) \cap \overline{\Omega} \subset S(F,\overline{\Omega},Y_0)$ since $K(X) \subset Y_0$.

Definition 3.1. Given $\varepsilon > 0$, a mapping $K_\varepsilon \in C^1(X,Y_0)$ is called a regular ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ (regular finite-dimensional ε-approximation, for short) if

$$\sup_{x \in S(F,\overline{\Omega},Y_0)} ||K_\varepsilon(x) - K(x)|| \leq \varepsilon. \tag{3.3}$$

If also $DK_\varepsilon(p) = 0$ for some $p \in X$, K_ε is said to be based at p.

The (delicate) question about the existence of compact and regular approximations based at p is settled by Theorem 3.2 below. For clarity, its proof is postponed until the next section.

Theorem 3.2. *For every finite-dimensional subspace Y_0 containing $K(X)$, every $\varepsilon > 0$, and every $p \in X$, there is a compact and regular ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ based at p. More generally, there is a compact and regular ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ based at all the points of any given finite sequence $p_1,\ldots,p_r \in X$.*

For the time being, it will suffice to define the degree at the value $0 \notin (F+K)(\partial\Omega)$. To justify the definition given in (3.6) below, it must be checked that $d_p(F+K_\varepsilon,\Omega,0)$ exists and is independent of the choice of the regular ε-approximation K_ϵ of K on $S(F,\overline{\Omega},Y_0)$ based at p *and* of the subspace Y_0. This is done in the next lemma, where we implicitly use the fact that the properness of F on $\overline{\Omega}$ and the compactness of K imply that $F+K$ is proper on $\overline{\Omega}$, so that $(F+K)(\partial\Omega)$ is closed.

Lemma 3.3. *Suppose that $0 \notin (F+K)(\partial\Omega)$, that $0 < \varepsilon < \operatorname{dist}(0,(F+K)(\partial\Omega))$, and that $p \in X$ is a base-point of F.*

(i) *If $K_\varepsilon \in C^1(X,Y_0)$ is a compact and regular ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ based at p, then $F+K_\varepsilon \in \Phi_0C^1(X,Y)$, $F+K_\varepsilon$ is proper on $\overline{\Omega}$, $0 \notin (F+K_\varepsilon)(\partial\Omega)$, and p is a base-point of $F+K_\varepsilon$. In particular, $d_p(F+K_\varepsilon,\Omega,0)$ is defined.*

(ii) *With K_ε as in* (i), *let Y_0' be another finite-dimensional subspace of Y such that $K(X) \subset Y_0'$ and let $K_\varepsilon' \in C^1(X,Y_0')$ be a compact and regular ε-approximation of K on $S(F,\overline{\Omega},Y_0')$ based at p. Then, $d_p(F+K_\varepsilon,\Omega,0) = d_p(F+K_\varepsilon',\Omega,0)$.*

Proof. (i) That $F+K_\varepsilon \in \Phi_0C^1(X,Y)$ follows from $F \in \Phi_0C^1(X,Y)$ and K_ε compact (whence $DK_\varepsilon(x) \in \mathcal{K}(X,Y)$ for all $x \in X$; see, e.g., [7, page 56]). The properness of $F+K_\varepsilon$ on $\overline{\Omega}$ is due to the properness of F, the compactness of K_ε, and the boundedness of Ω.

Suppose now that $x \in \overline{\Omega}$ and that $F(x)+K_\varepsilon(x) = 0$. Then, $F(x) = -K_\varepsilon(x) \in Y_0$ and hence $x \in S(F,\overline{\Omega},Y_0)$. As a result, $\|(F+K)(x)\| = \|(F+K)(x)-(F+K_\varepsilon)(x)\| = \|K(x)-K_\varepsilon(x)\| \leq \varepsilon < \operatorname{dist}(0,(F+K)(\partial\Omega))$, so that $x \notin \partial\Omega$. This means that $0 \notin (F+K_\varepsilon)(\partial\Omega)$. That p is a base-point of $F+K_\varepsilon$ is obvious.

(ii) After replacing Y_0' by $Y_0 + Y_0'$ and K_ε' by a compact and regular finite-dimensional ε-approximation of K on $S(F,\overline{\Omega}, Y_0 + Y_0')$ (whose existence follows from Theorem 3.2), we may assume with no loss of generality that $Y_0 \subset Y_0'$.

(ii-a) Assume first that $Y_0' = Y_0$. Then, $K_{\varepsilon t} := (1-t)K_\varepsilon + tK_\varepsilon'$ is also a compact and regular ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ based at p for every $t \in [0,1]$ and $d_p(F + K_\varepsilon,\Omega,0) = d_p(F + K_\varepsilon',\Omega,0)$ follows readily from Corollary 2.3.

(ii-b) Consider now the general case when $Y_0 \subset Y_0'$. By (ii-a) and Theorem 3.2, it suffices to compare $d_p(F + K_\varepsilon,\Omega,0)$ and $d_p(F + K_\varepsilon',\Omega,0)$ when K_ε' is also an ε'-approximation for some $0 < \varepsilon' \le \varepsilon$, say $K_\varepsilon' = K_{\varepsilon'}'$ for consistency.

Let $P_0 \in \mathcal{L}(Y)$ project onto Y_0 and set $\varepsilon' := (\varepsilon/\|P_0\|) \le \varepsilon$. Since $P_0K = K$ and $K_{\varepsilon'}'$ is an ε'-approximation of K on $S(F,\overline{\Omega},Y_0')$, we have

$$\sup_{x\in S(F,\overline{\Omega},Y_0')} \|P_0K_{\varepsilon'}'(x) - K(x)\| \le \|P_0\|\varepsilon' \le \varepsilon. \tag{3.4}$$

Since $S(F,\overline{\Omega},Y_0) \subset S(F,\overline{\Omega},Y_0')$, $P_0K_{\varepsilon'}'$ is a compact and regualr ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ based at p, so that $d_p(F + K_\varepsilon,\Omega,0) = d_p(F + P_0K_{\varepsilon'}',\Omega,0)$ by (ii-a). It remains to show that

$$d_p(F + P_0K_{\varepsilon'}',\Omega,0) = d_p(F + K_{\varepsilon'}',\Omega,0), \tag{3.5}$$

which follows from Corollary 2.3 after checking that $F + (1-t)P_0K_{\varepsilon'}' + tK_{\varepsilon'}'$ does not vanish on $\partial\Omega$ for $t \in [0,1]$. But if $x \in \overline{\Omega}$ and $F(x) + (1-t)P_0K_{\varepsilon'}'(x) + tK_{\varepsilon'}'(x) = 0$, then $x \in S(F,\overline{\Omega},Y_0')$ and $F(x) + K(x) = (1-t)(K(x) - P_0K_{\varepsilon'}'(x)) + t(K(x) - K_{\varepsilon'}'(x))$. Thus, by (3.4), $\|F(x) + K(x)\| \le (1-t)\varepsilon + t\varepsilon' \le \varepsilon < \operatorname{dist}(0,(F + K)(\partial\Omega))$, so that $x \notin \partial\Omega$. □

From Lemma 3.3, if $0 \notin (F + K)(\partial\Omega)$ and if $p \in X$ is a base-point of F, the definition

$$d_{F,p}(F + K,\Omega,0) := d_p(F + K_\varepsilon,\Omega,0) \tag{3.6}$$

makes sense whenever $0 < \varepsilon < \operatorname{dist}(0,(F + K)(\partial\Omega))$ and K_ε is a regular finite dimensional ε-approximation of K based at p.

Remark 3.4. In particular, if $K = 0$, we may choose $K_\varepsilon = 0$ and (3.6) reads as $d_{F,p}(F,\Omega, y) = d_p(F,\Omega, y)$. Thus, for $F \in \Phi_0C^1(X,Y)$ proper on $\overline{\Omega}$, there is no difference between $d_p(F,\Omega, y)$ defined in Section 2 and $d_{F,p}(F,\Omega, y)$ defined above.

4. Existence of compact regular approximations

This section is devoted to a generalization of Theorem 3.2 when F is Fredholm of any index, which will be useful when dealing with homotopies. The finite-dimensional subspace Y_0 of Y and the mappings $F \in \Phi_nC^1(X,Y)$ for some $n \in \mathbb{Z}$ and $K \in C^0(X,Y_0)$ as well as the bounded open subset Ω of X are given once and for all. It is also assumed throughout that F is proper on $\overline{\Omega}$, that $F(\overline{\Omega})$ is bounded, and that K is compact (i.e., bounded on bounded subsets since K is finite dimensional). Given $\varepsilon > 0$ and a base-point $p \in X$ of F, our goal is to find $K_\varepsilon \in C^1(X,Y_0)$ such that $DK_\varepsilon(p) = 0$, K_ε is compact, and $\sup_{x\in S(F,\overline{\Omega},Y_0)} \|K_\varepsilon(x) - K(x)\| \le \varepsilon$, where $S(F,\overline{\Omega},Y_0)$ is given by (3.2). The compactness of $S(F,\overline{\Omega},Y_0)$ or the definition of a regular finite-dimensional ε-approximation is not affected by the fact that the index of F is not necessarily 0.

Definition 4.1. The finite-dimensional subspace Y_1 of Y is said to be $S(F,\overline{\Omega},Y_0)$-regular if $Y_0 \subset Y_1$ and there is $P_1 \in \mathscr{L}(Y)$ projecting onto Y_1 such that $Q_1F : X \to rge Q_1$ is a submersion on $S(F,\overline{\Omega},Y_0)$, where $Q_1 := I - P_1$.

If Y_1 is an $S(F,\overline{\Omega},Y_0)$-regular subspace of Y, there is an open neighborhood U of $S(F,\overline{\Omega},Y_0)$ in X such that

$$M(Y_1) := (Q_1F)^{-1}(0) \cap U \tag{4.1}$$

is a finite-dimensional C^1 submanifold of X containing $S(F,\overline{\Omega},Y_0)$. The existence of regular subspaces is settled by the following.

LEMMA 4.2. *There is an $S(F,\overline{\Omega},Y_0)$-regular subspace of Y.*

Proof. Given $x \in X$, it is well known that a finite-dimensional direct complement Y^x of $rge\,DF(x)$ remains a complement (though not necessarily direct) of $rge\,DF(\xi)$ for ξ in an open neighborhood U^x of x. Since $S(F,\overline{\Omega},Y_0)$ is compact, it may be covered by finitely many neighborhoods $U^{x_1},\dots,U^{x_n}$. Set $Y_1 := Y^{x_1} + \cdots + Y^{x_n} + Y_0$, a finite-dimensional subspace of Y containing Y_0, and let Z_1 be any closed direct complement of Y_1.

Denote by P_1 and $Q_1 = I - P_1$ the projections onto the spaces Y_1 and Z_1, respectively. If $x \in S(F,\overline{\Omega},Y_0)$, then $rge\,DF(x) + Y_1 = Y$, so that, given $z_1 \in Z_1$, there are $w \in X$ and $y_1 \in Y_1$ such that $DF(x)w + y_1 = z_1$. Thus, $Q_1DF(x)w = z_1$, which shows that $Q_1F : X \to Z_1$ is a submersion on $S(F,\overline{\Omega},Y_0)$ and hence that Y_1 is $S(F,\overline{\Omega},Y_0)$-regular. □

If Y_1 is an $S(F,\overline{\Omega},Y_0)$-regular subspace of Y, the function $K_{|M(Y_1)}$ can be uniformly approximated by C^1 functions on every compact subset of $M(Y_1)$ and hence on $S(F,\overline{\Omega},Y_0)$. The tool needed to extend C^1 functions defined on C^1 finite-dimensional submanifolds is a variant of the Whitney embedding theorem, proved in [17, Theorem 7.1] and reproduced in Lemma 4.3 below, showing that a compact subset of such a submanifold can always be "flattened" by a diffeomorphism of the *whole* space.

LEMMA 4.3. *Let X be a real Banach space, $M \subset X$ a finite-dimensional C^1 submanifold, and $N \subset M$ a compact subset. There is a finite-dimensional subspace X_1 of X and a C^1 diffeomorphism Φ of X onto itself such that $\Phi(N) \subset X_1$.*

THEOREM 4.4. *For every $\varepsilon > 0$ and every finite sequence $p_1,\dots,p_r \in X$, there is a compact and regular ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ with values in Y_0 and based at $p_1,\dots,p_r$.*

Proof. Let Y_1 be an $S(F,\overline{\Omega},Y_0)$-regular subspace of Y (Lemma 4.2) and choose $M = M(Y_1)$, $N = S(F,\overline{\Omega},Y_0)$ in Lemma 4.3. The corresponding diffeomorphism Φ maps $S(F,\overline{\Omega},Y_0)$ onto a compact subset $Q := \Phi(S(F,\overline{\Omega},Y_0))$ of the finite-dimensional subspace X_1 of X. By the Dugundji extension theorem (see, e.g., [7]), $(K \circ \Phi^{-1})_{|Q} \in C^0(Q,Y_0)$ can be extended to a mapping $\widetilde{K} \in C^0(X_1,Y_0)$ with values in the (compact) convex hull $\operatorname{conv}(K(S(F,\overline{\Omega},Y_0))) \subset Y_0$.

Let $\pi_1 \in \mathcal{L}(X)$ project onto X_1 and let $p_{1i} := \pi_1\Phi(p_i) \in X_1$, $1 \le i \le r$. We claim that, given $\varepsilon > 0$, there is $\widetilde{K}_\varepsilon \in C^1(X_1, Y_0)$ such that

$$\sup_{x_1 \in X_1} \|\widetilde{K}_\varepsilon(x_1) - \widetilde{K}(x_1)\| \le \varepsilon \tag{4.2}$$

and $D\widetilde{K}_\varepsilon(p_{1i}) = 0$, $1 \le i \le r$. Since $\dim X_1 < \infty$, this is clear without the requirement that $D\widetilde{K}_\varepsilon(p_{1i}) = 0$. But it is also clear that such an approximation exists which is constant in some neighborhood of p_{1i} and hence satisfies $D\widetilde{K}_\varepsilon(p_{1i}) = 0$ for all indices i.

Now, define $\widehat{K}_\varepsilon := \widetilde{K}_\varepsilon \circ \pi_1 \in C^1(X, Y_0)$ and $\widehat{K} := \widetilde{K} \circ \pi_1$. Then,

$$\begin{aligned} &\sup_{x_1 \in Q} \|\widehat{K}_\varepsilon(x_1) - \widehat{K}(x_1)\| \\ &\qquad = \sup_{x_1 \in Q} \|\widetilde{K}_\varepsilon(x_1) - \widetilde{K}(x_1)\| \le \sup_{x_1 \in X_1} \|\widetilde{K}_\varepsilon(x_1) - \widetilde{K}(x_1)\| \le \varepsilon \end{aligned} \tag{4.3}$$

by (4.2) and $D\widehat{K}_\varepsilon(\Phi(p_i)) = D\widetilde{K}_\varepsilon(p_{1i}) = 0$, $1 \le i \le r$. Therefore, $K_\varepsilon := \widehat{K}_\varepsilon \circ \Phi \in C^1(X, Y_0)$ satisfies $DK_\varepsilon(p_i) = 0$, $1 \le i \le r$, and

$$\begin{aligned} &\sup_{x \in S(F, \overline{\Omega}, Y_0)} \|K_\varepsilon(x) - K(x)\| \\ &\qquad = \sup_{x_1 \in Q} \|K_\varepsilon \circ \Phi^{-1}(x_1) - K \circ \Phi^{-1}(x_1)\| = \sup_{x_1 \in Q} \|\widehat{K}_\varepsilon(x_1) - \widehat{K}(x_1)\| \le \varepsilon. \end{aligned} \tag{4.4}$$

To complete the proof, it remains to show that K_ε is compact. But this follows at once from the remark that, by (4.2), $K_\varepsilon(X)$ is a bounded subset of Y_0 since $\widetilde{K}$ has values in the compact subset $\operatorname{conv}(K(S(F, \overline{\Omega}, Y_0)))$. □

5. Degree for finite-dimensional perturbations: homotopy variance

We begin with a convenient definition.

Definition 5.1. The homotopy $h \in C^0([0,1] \times X, Y)$ will be called Ω-admissible if it can be written in the form $h = h_\Phi + h_\kappa$ with $h_\Phi \in \Phi_1 C^1([0,1] \times X, Y)$, h_Φ proper on $[0,1] \times \overline{\Omega}$, and $h_\kappa \in C^0([0,1] \times X, Y)$ compact.

If $h = h_\Phi + h_\kappa \in C^0([0,1] \times X, Y)$ is an Ω-admissible homotopy with h_κ *finite dimensional* and if $y \notin h([0,1] \times \partial\Omega)$, then the degree $d_{h_\Phi(t,\cdot),p_t}(h(t,\cdot), \Omega, y)$ is defined whenever p_t is a base-point of $h_\Phi(t,\cdot)$ (see (3.2)). The next theorem explains how the degrees $d_{h_\Phi(0,\cdot),p_0}(h(0,\cdot), \Omega, y)$ and $d_{h_\Phi(1,\cdot),p_1}(h(1,\cdot), \Omega, y)$ are related.

Theorem 5.2. *Let $h = h_\Phi + h_\kappa \in C^0([0,1] \times X, Y)$ be an Ω-admissible homotopy with h_κ finite dimensional and $h_\Phi([0,1] \times \overline{\Omega})$ bounded and suppose that $0 \notin h([0,1] \times \partial\Omega)$. If $p_0 \in X$ and $p_1 \in X$ are base-points of $h_\Phi(0,\cdot)$ and $h_\Phi(1,\cdot)$, respectively, then*

$$d_{h_\Phi(0,\cdot),p_0}(h(0,\cdot), \Omega, 0) = \nu d_{h_\Phi(1,\cdot),p_1}(h(1,\cdot), \Omega, 0), \tag{5.1}$$

where $\nu := \sigma(D_x h_\Phi \circ \Gamma) \in \{-1, 1\}$ and Γ is any continuous curve in $[0,1] \times X$ joining $(0, p_0)$ to $(1, p_1)$.

Proof. By the arguments of the proof of Lemma 3.3(i), the set $h([0,1]\times\partial\Omega)$ is closed in Y, so that $\mathrm{dist}(0,h([0,1]\times\partial\Omega))>0$. Let $0<\varepsilon<\mathrm{dist}(0,h([0,1]\times\partial\Omega))$ be chosen once and for all. In order to use Theorem 4.4 with F replaced by h_Φ and K replaced by h_κ (and Ω replaced by $(0,1)\times\Omega$) so as to obtain regular approximations of h_κ, it is necessary to extend both h_Φ and h_κ to the whole space $\mathbb{R}\times X$. This issue is straightforward for h_κ, but it is immediately realized that extending h_Φ to a Fredholm mapping is not such an easy matter. However, a notable exception to this statement occurs when $D_t h_\Phi(0,\cdot)=D_t h_\Phi(1,\cdot)=0$, for then the extension $\widetilde{h}_\Phi$ of h_Φ defined by $\widetilde{h}_\Phi(t,x):=h_\Phi(0,x)$ for $t\le 0$ and $\widetilde{h}_\Phi(t,x):=h_\Phi(1,x)$ for $t\ge 1$ is in $\Phi_1C^1(\mathbb{R}\times X,Y)$ and, evidently, $\widetilde{h}_{\Phi|_{[0,1]\times\overline{\Omega}}}=h_{\Phi|_{[0,1]\times\overline{\Omega}}}$ is proper.

At this stage, the pertinent remark is that neither the assumptions of Theorem 5.2 nor its conclusion is affected by changing $h(t,x)$ into $h(\varphi(t),x)$ where φ is any C^1 homeomorphism of $[0,1]$ onto itself such that $\varphi(0)=0$ and $\varphi(1)=1$. In particular, this change does *not* modify the set $h([0,1]\times\partial\Omega)$ or the mappings h_Φ and h_κ when $t=0$ or $t=1$. The only slightly less obvious point is that $\nu:=\sigma(D_xh_\Phi\circ\Gamma)$ is unchanged. But this follows from the homotopy invariance of the parity since, as is readily checked, φ and the identity of $[0,1]$ are homotopic. Since φ can be chosen so that $(d\varphi/dt)(0)=(d\varphi/dt)(1)=0$, it follows that, for the purpose of proving Theorem 5.2, we may and will assume with no loss of generality that $D_th_\Phi(0,\cdot)=D_th_\Phi(1,\cdot)=0$.

Let then $\widetilde{h}_\Phi$ be the extension of h_Φ introduced above and let $\widetilde{h}_\kappa$ extend h_κ in the same way, so that $\widetilde{h}_\kappa\in C^0(\mathbb{R}\times X,Y)$ is compact and finite dimensional. It follows from Theorem 4.4 with X replaced by $\mathbb{R}\times X$ and Ω replaced by $(0,1)\times\Omega$ that given any finite-dimensional subspace Y_0 of Y such that $h_\kappa(\mathbb{R}\times X)\subset Y_0$, there is a compact and regular finite-dimensional ε-approximation $\widetilde{h}_{\kappa,\varepsilon}\in C^1(\mathbb{R}\times X,Y_0)$ of $\widetilde{h}_\kappa$ on $S(h_\Phi,[0,1]\times\overline{\Omega},Y_0):=\widetilde{h}_\Phi^{-1}(\widetilde{h}_\Phi([0,1]\times\overline{\Omega})\cap Y_0)\cap[0,1]\times\overline{\Omega}=h_\Phi^{-1}(h_\Phi([0,1]\times\overline{\Omega})\cap Y_0)\cap[0,1]\times\overline{\Omega}$ (thus independent of the extension) based at both $(0,p_0)$ and $(1,p_1)$. In particular,

$$D_x\widetilde{h}_{\kappa,\varepsilon}(0,p_0)=D_x\widetilde{h}_{\kappa,\varepsilon}(1,p_1)=0. \tag{5.2}$$

In particular, $\widetilde{h}_{\kappa,\varepsilon}(0,\cdot)$ is a compact and regular ε-approximation of $\widetilde{h}_\kappa(0,\cdot)=h_\kappa(0,\cdot)$ on $S(h_\Phi(0,\cdot),\overline{\Omega},Y_0)=h_\Phi(0,\cdot)^{-1}(h_\Phi(\{0\}\times\overline{\Omega})\cap Y_0)\cap\overline{\Omega}$ based at p_0 and $\widetilde{h}_{\kappa,\varepsilon}(1,\cdot)$ is a compact and regular ε-approximation of $\widetilde{h}_\kappa(1,\cdot)=h_\kappa(1,\cdot)$ on $S(h_\Phi(1,\cdot),\overline{\Omega},Y_0)=h_\Phi(1,\cdot)^{-1}$ $(h_\Phi(\{1\}\times\overline{\Omega})\cap Y_0)\cap\overline{\Omega}$ based at p_1. Also, $h_\Phi(\{0\}\times\overline{\Omega})$ and $h_\Phi(\{1\}\times\overline{\Omega})$ are bounded (as required by (3.1)) since $h_\Phi([0,1]\times\overline{\Omega})$ is bounded by hypothesis. Therefore, by the definition (3.6),

$$d_{h_\Phi(0,\cdot),p_0}(h(0,\cdot),\Omega,0)=d_{p_0}(h_\Phi(0,\cdot)+\widetilde{h}_{\kappa,\varepsilon}(0,\cdot),\Omega,0), \tag{5.3}$$

$$d_{h_\Phi(1,\cdot),p_1}(h(1,\cdot),\Omega,y)=d_{p_1}(h_\Phi(1,\cdot)+\widetilde{h}_{\kappa,\varepsilon}(1,\cdot),\Omega,y). \tag{5.4}$$

Now, $h_\Phi+\widetilde{h}_{\kappa,\varepsilon}\in\Phi_1C^1([0,1]\times X,Y)$ is proper on $[0,1]\times\overline{\Omega}$ (being a finite-dimensional perturbation of h_Φ) and $0\notin(h_\Phi+\widetilde{h}_{\kappa,\varepsilon})([0,1]\times\partial\Omega)$. By Theorem 2.2,

$$d_{p_1}(h_\Phi(1,\cdot)+\widetilde{h}_{\kappa,\varepsilon}(1,\cdot),\Omega,y)=\nu d_{p_0}(h_\Phi(0,\cdot)+\widetilde{h}_{\kappa,\varepsilon}(0,\cdot),\Omega,y), \tag{5.5}$$

where $\nu := \sigma(D_x h_\Phi \circ \Gamma + D_x \widetilde{h}_{\kappa,\varepsilon} \circ \Gamma) \in \{-1,1\}$ and Γ is any continuous curve in $[0,1] \times X$ joining $(0,p_0)$ to $(1,p_1)$. We claim that, more simply,

$$\nu = \sigma(D_x h_\Phi \circ \Gamma). \tag{5.6}$$

Indeed, consider the homotopy $H(s,t) := D_x h_\Phi(\Gamma(t)) + sD_x \widetilde{h}_{\kappa,\varepsilon}(\Gamma(t))$, so that $H(0,t) = D_x h_\Phi \circ \Gamma$ and $H(1,t) = D_x h_\Phi \circ \Gamma + D_x \widetilde{h}_{\kappa,\varepsilon} \circ \Gamma$. Then, $H(s,0) = D_x h_\Phi(0,p_0)$ and $H(s,1) = D_x h_\Phi(1,p_1)$ for all $s \in [0,1]$ since $D_x \widetilde{h}_{\kappa,\varepsilon}(0,p_0) = D_x \widetilde{h}_{\kappa,\varepsilon}(1,p_1) = 0$ (see (5.2)). This is to say that the endpoints remain invertible during the homotopy, whence $\sigma(H(0,\cdot)) = \sigma(H(1,\cdot))$. This proves (5.6) and thus the theorem by (5.3), (5.4), and (5.5). □

Generalizations and corollaries of Theorem 5.2 will be mentioned in Section 7. For the time being, we only clarify the F-dependence of the degree $d_{F,p}$.

With F and K satisfying the assumptions required in Section 3 to define $d_{F,p}(F + K,\Omega,0)$ by (3.6), suppose also that $F + K = G + L$ with $G \in \Phi_0 C^1(X,Y)$ and $L \in C^0(X,Y)$ compact and finite dimensional. Then, $G = F + K - L$ is proper on $\overline{\Omega}$ and $G(\overline{\Omega})$ is bounded since $F(\overline{\Omega})$ is bounded by hypothesis. Therefore, with $T := F + K = G + L$ and assuming that $0 \notin T(\partial\Omega)$, we have a degree $d_{F,p}(T,\Omega,0)$ and a degree $d_{G,q}(T,\Omega,0)$ whenever p and q are base-points of F and G, respectively. Up to sign, these two degrees coincide, as shown below.

Theorem 5.3. *Above,*

$$d_{F,p}(T,\Omega,0) = \nu d_{G,q}(T,\Omega,0), \tag{5.7}$$

where $\nu \in \{-1,1\}$ is the parity of any path (although perhaps not apparent, this is a path of Fredholm operators of index 0; see the proof) $\{(1-t)DF(\gamma(t)) + tDG(\gamma(t)) : t \in [0,1]\}$ with $\gamma \in C^0([0,1],X)$ being a curve joining p to q.

Proof. The formula (5.7) is the special case of Theorem 5.2 where $\Gamma(t) = (t,\gamma(t))$ and where $p_0 = p$, $p_1 = q$, $h_\Phi(t,x) = F(x) + t(K-L)(x)$, and $h_\kappa(t,x) = (1-t)K(x) + tL(x)$. Note that h_Φ is C^1 because $K - L = G - F$ is C^1 (although neither K nor L need be C^1) and Fredholm of index 0 since F is Fredholm of index 0 and $K - L$ is compact and C^1. Note also that $h_\Phi([0,1] \times \overline{\Omega})$ is bounded since $F(\overline{\Omega})$ is bounded and that $h_\Phi + h_\kappa = F + K = T$ is independent of t. □

Theorem 5.3 helps to clarify the question of extending the definition (3.6) in the case that F has no base-point. Indeed, given any $q \in X$, it is always possible to write $T = F + K$ in the form $T = G + L$ in such a way that q is a base-point of G. For instance, choose $G = F + A$ and $L = K - A$ where $A \in \mathscr{L}(X,Y)$ is a suitable operator with finite rank. Then, $d_{G,q}(T,\Omega,0)$ makes sense (if $0 \notin T(\partial\Omega)$) and can be used in place of the nonexisting $d_{F,p}(T,\Omega,0)$.

6. Degree for finite-dimensional perturbations: main properties

We now prove that the degree (3.6) possesses the main properties valid in the C^1 case: normalization, excision, and additivity on domain. We continue to assume that $\Omega \subset X$ is a bounded open subset and that $F(\overline{\Omega})$ is bounded.

Theorem 6.1. *Let $F \in \Phi_0 C^1(X,Y)$ be proper on $\overline{\Omega}$, let $K \in C^0(X,Y)$ be compact, and let $p \in X$ be a base-point of F. The following properties hold.*

(i) *If $0 \notin (F+K)(\partial\Omega)$ and $d_{F,p}(F+K,\Omega,0) \neq 0$, then $(F+K)^{-1}(0) \cap \Omega \neq \varnothing$.*

(ii) *If Σ is a closed subset of Ω and $0 \notin (F+K)(\Sigma \cup \partial\Omega)$, then $0 \notin (F+K)(\partial(\Omega\backslash\Sigma))$ and $d_{F,p}(F+K,\Omega\backslash\Sigma,0) = d_{F,p}(F+K,\Omega,0)$.*

(iii) *If $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are disjoint open subsets of Ω and $0 \notin (F+K)(\partial\Omega)$, then $0 \notin (F+K)(\partial\Omega_1) \cup (F+K)(\partial\Omega_2)$ and $d_{F,p}(F+K,\Omega,0) = d_{F,p}(F+K,\Omega_1,0) + d_{F,p}(F+K,\Omega_2,0)$.*

Proof. Throughout the proof, Y_0 denotes a finite-dimensional subspace of Y such that $K(X) \subset Y_0$ and, given $\varepsilon > 0$, K_ε is a compact and regular ε-approximation of K on $S(F,\overline{\Omega},Y_0)$ based at p. Recall that Theorem 3.2 ensures the existence of K_ε.

(i) It suffices to show that if $(F+K)^{-1}(0) \cap \overline{\Omega} = \varnothing$, then $d_{F,p}(F+K,\Omega,0) = 0$. To see this, note that $\operatorname{dist}(0,(F+K)(\overline{\Omega})) > 0$ by the properness of $F+K$ on $\overline{\Omega}$. Choose $0 < \varepsilon < \operatorname{dist}(y,(F+K)(\overline{\Omega}))$, so that $0 \notin (F+K_\varepsilon)(\overline{\Omega})$ by the arguments of the proof of Lemma 3.3(i), that is, $(F+K_\varepsilon)^{-1}(0) \cap \overline{\Omega} = \varnothing$. Thus, $d_p(F+K_\varepsilon,\Omega,0) = 0$ by Corollary 2.6, so that $d_{F,p}(F+K,\Omega,0) = 0$ by (3.6).

(ii) That $0 \notin (F+K)(\partial(\Omega\backslash\Sigma))$ follows from $\partial(\Omega\backslash\Sigma) \subset \Sigma \cup \partial\Omega$ (trivial using the closedness of Σ in Ω). Since $\Sigma \cup \partial\Omega$ is closed in $\overline{\Omega}$, the properness of $F+K$ yields that $(F+K)(\Sigma \cup \partial\Omega)$ is closed in Y. If $0 < \varepsilon < \operatorname{dist}(0,(F+K)(\Sigma \cup \partial\Omega))$, then $0 \notin (F+K_\varepsilon)(\Sigma \cup \partial\Omega)$, once again by the arguments of the proof of Lemma 3.3(i), so that $d_p(F+K_\varepsilon,\Omega,0) = d_p(F+K_\varepsilon,\Omega\backslash\Sigma,0)$ by Theorem 2.7. But since $\partial\Omega \subset \Sigma \cup \partial\Omega$ and $\partial(\Omega\backslash\Sigma) \subset \Sigma \cup \partial\Omega$, we have that

$$\varepsilon < \min\{\operatorname{dist}(0,(F+K)(\partial\Omega)), \operatorname{dist}(0,(F+K)(\partial(\Omega\backslash\Sigma)))\}, \tag{6.1}$$

whence, by (3.6), the left-hand side is $d_{F,p}(F+K,\Omega,0)$ and the right-hand side is $d_{F,p}(F+K,\Omega\backslash\Sigma,0)$. (Note that K_ε is also an ε-approximation of K on $S(F, \overline{\Omega\backslash\Sigma}, Y_0)$.)

(iii) First, $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$ since Ω_1 and Ω_2 are disjoint, so that $0 \notin (F+K)(\partial\Omega_1) \cup (F+K)(\partial\Omega_2)$. Also, if $0 < \varepsilon < \operatorname{dist}(0,(F+K)(\partial\Omega))$, then

$$0 < \varepsilon < \operatorname{dist}\left(0,(F+K)\left(\partial\Omega_\alpha\right)\right), \quad \alpha = 1,2. \tag{6.2}$$

As a result, K_ε is a compact and regular ε-approximation of K on $S(F,\overline{\Omega}_\alpha,Y_0)$ based at p for $\alpha = 1,2$. By (3.6), $d_{F,p}(F+K,\Omega,0) = d_p(F+K_\varepsilon,\Omega,0)$ and $d_{F,p}(F+K,\Omega_\alpha,0) = d_p(F+K_\varepsilon,\Omega_\alpha,0)$, $\alpha = 1,2$. The conclusion thus follows from Theorem 2.8. □

Theorem 6.2 (Borsuk's theorem). *In addition to the hypotheses of Theorem 6.1, assume that $\Omega = -\Omega$, that $0 \in \Omega$, and that $F+K$ is odd. If $0 \notin (F+K)(\partial\Omega)$, then $d_{F,p}(F+K,\Omega,0)$ is odd.*

Proof. In a first step, assume that both F and K are odd. Then, for every finite-dimensional subspace Y_0 of Y such that $K(X) \subset Y_0$ and every $\varepsilon > 0$, there is an odd compact and regular ε-approximation K_ε of K on $S(F,\overline{\Omega},Y_0)$ based at p: just start with any approximation K_ε based at both p and $-p$ given by Theorem 3.2 and replace $K_\varepsilon(x)$ by $(1/2)(K_\varepsilon(x) - K_\varepsilon(-x))$.

Then, $F + K_\varepsilon$ is odd and $0 \notin (F + K_\varepsilon)(\partial\Omega)$ if $0 < \varepsilon < \operatorname{dist}(0, (F + K)(\partial\Omega))$. As shown in [13], this implies (although [13] discusses the case of C^2 Fredholm mappings, everything carries over to the C^1 case) that $d_p(F + K_\varepsilon, \Omega, 0)$ is odd and hence, by (3.6), that $d_{F,p}(F + K, \Omega, 0)$ is odd.

Now, suppose only that $F + K$ is odd. We reduce the problem to the previous case. Set $G(x) = (1/2)(F(x) - F(-x))$ and $L(x) = (1/2)(K(x) - K(-x))$. Both G and L are odd, $F + K = G + L$ (since $F + K$ is odd), L is compact and finite dimensional, and G is Fredholm of index 0 with G proper on $\overline{\Omega}$. (For the last two properties, it suffices to observe that $G = F + K - L$ and that $K - L = G - F$ is compact and C^1.) If G has a base-point q, it follows from Theorem 5.3 that $d_{F,p}(F + K, \Omega, 0) = \mp d_{G,q}(G + L, \Omega, 0)$ and $d_{G,q}(G + L, \Omega, 0)$ is odd by the first step above, so that $d_{F,p}(F + K, \Omega, 0)$ is odd.

If G has no base-point, pick $q \in X$ and let $A \in \mathscr{L}(X, Y)$ be an operator of finite rank such that $DG(q) + A$ is invertible, so that $G + A$ is Fredholm of index $0, (G + A)$ is proper on $\overline{\Omega}$, and q is a base-point of $G + A$. Then, $G + A$ is odd, $L - A$ is compact, finite dimensional, and odd, and $F + K = (G + A) + (L - A)$. This reduces the problem to the case just discussed when G has a base-point. □

7. Degree for compact perturbations

We are now in a position to eliminate the assumption that K is a finite-dimensional mapping via uniform approximation on $\overline{\Omega}$ by compact finite-dimensional mappings. The existence of such approximations is standard and already used in the construction of the Leray-Schauder degree. Specifically, with $F \in \Phi_0 C^1(X, Y)$ proper on the closure $\overline{\Omega}$ of the bounded open subset Ω of X and with $K \in C^0(X, Y)$ compact, we define, assuming that $p \in X$ is a base-point of F, that $F(\overline{\Omega})$ is bounded (but see Theorem 7.6), and that $0 \notin (F + K)(\partial\Omega)$,

$$d_{F,p}(F + K, \Omega, 0) := d_{F,p}(F + K^f, \Omega, 0), \tag{7.1}$$

where $K^f \in C^0(X, Y)$ is compact, finite dimensional and $\sup_{x \in \overline{\Omega}} \|K(x) - K^f(x)\|$ is small enough. That this definition is the same for any two such choices K_0^f and K_1^f of K^f follows at once from Theorem 5.2 with $h_\Phi = F$, $h_\kappa(t, x) = (1 - t)K_0^f(x) + tK_1^f(x)$, and $p_0 = p_1 = p$ (so that $\nu = 1$). More generally, if $y \notin (F + K)(\partial\Omega)$, we set

$$d_{F,p}(F + K, \Omega, y) := d_{F,p}(F + K - y, \Omega, 0). \tag{7.2}$$

Most of the properties of the degree $d_{F,p}(F + K, \Omega, y)$ follow at once from the case when $y = 0$ and from the analogous properties when K is finite dimensional. For convenience, these properties are summarized in the next two theorems.

Theorem 7.1. *Let $h = h_\Phi + h_\kappa \in C^0([0,1] \times X, Y)$ be an Ω-admissible homotopy (see Definition 5.1) with $h_\Phi([0,1] \times \overline{\Omega})$ bounded and suppose that $y \notin h([0,1] \times \partial\Omega)$. If $p_0 \in X$*

and $p_1 \in X$ *are base-points of* $h_\Phi(0,\cdot)$ *and* $h_\Phi(1,\cdot)$*, respectively, then*

$$d_{h_\Phi(0,\cdot),p_0}(h(0,\cdot),\Omega,y) = \nu d_{h_\Phi(1,\cdot),p_1}(h(1,\cdot),\Omega,y), \tag{7.3}$$

where $\nu := \sigma(D_x h_\Phi \circ \Gamma) \in \{-1,1\}$ *and* Γ *is any continuous curve in* $[0,1]\times X$ *joining* $(0,p_0)$ *to* $(1,p_1)$.

Proof. Just replace h_κ by a close enough finite-dimensional approximation h_κ^f and use Theorem 5.2. □

If $h_\Phi = F$ is independent of t in Theorem 7.1 and hence the same base-point $p = p_0 = p_1$ can be chosen for $h_\Phi(0,\cdot) = h_\Phi(1,\cdot)$, the result is especially simple since we obtain homotopy *invariance.*

Corollary 7.2. *Let* $F \in \Phi_0 C^1(X,Y)$ *be proper on* $\overline{\Omega}$ *with* $F(\overline{\Omega})$ *bounded and let* $h_\kappa \in C^0([0,1]\times X,Y)$ *be compact. Let* $h = F + h_\kappa$ *and let* $p \in X$ *be a base-point of* F*. If* $y \notin h([0,1]\times\partial\Omega)$*, then*

$$d_{F,p}(F+h_\kappa(0,\cdot),\Omega,y) = d_{F,p}(F+h_\kappa(1,\cdot),\Omega,y). \tag{7.4}$$

Proof. In Theorem 7.1, let $\Gamma(t) := (t,p)$, so that ν is the parity of the constant path $DF(p)$ and hence $\nu = 1$. □

Theorem 7.3. *Let* $F \in \Phi_0 C^1(X,Y)$ *be proper on* $\overline{\Omega}$*, let* $K \in C^0(X,Y)$ *be compact, and let* $p \in X$ *be a base-point of* F*. If* $F(\overline{\Omega})$ *is bounded, the following properties hold.*

(i) *The degree* $d_{F,p}(F+K,\Omega,y)$ *depends only upon the connected component of* $Y\backslash(F+K)(\partial\Omega)$ *containing* y.

(ii) *If* $y \notin (F+K)(\partial\Omega)$ *and* $d_{F,p}(F+K,\Omega,y) \neq 0$*, then* $(F+K)^{-1}(y)\cap\Omega \neq \emptyset$.

(iii) *If* Σ *is a closed subset of* Ω *and* $y \notin (F+K)(\Sigma\cup\partial\Omega)$*, then* $y \notin (F+K)(\partial(\Omega\backslash\Sigma))$ *and* $d_{F,p}(F+K,\Omega\backslash\Sigma,y) = d_{F,p}(F+K,\Omega,y)$.

(iv) *If* $\Omega = \Omega_1 \cup \Omega_2$ *where* Ω_1 *and* Ω_2 *are disjoint open subsets of* Ω *and* $y \notin (F+K)(\partial\Omega)$*, then* $y \notin (F+K)(\partial\Omega_1)\cup(F+K)(\partial\Omega_2)$ *and* $d_{F,p}(F+K,\Omega,y) = d_{F,p}(F+K,\Omega_1,y) + d_{F,p}(F+K,\Omega_2,y)$.

(v) *If* $T := F+K = G+L$ *with* $G \in \Phi_0 C^1(X,Y)$ *and* $L \in C^0(X,Y)$ *compact and if* $y \notin T(\partial\Omega)$ *and* $q \in X$ *is a base-point of* G*, then*

$$d_{F,p}(T,\Omega,y) = \nu d_{G,q}(T,\Omega,y), \tag{7.5}$$

where $\nu \in \{-1,1\}$ *is the parity of any path (so that* $\nu = 1$ *or* $\nu = -1$ *irrespective of* y*)* $\{(1-t)DF(\gamma(t)) + tDG(\gamma(t)) : t \in [0,1]\}$ *with* $\gamma \in C^0([0,1],X)$ *being a curve joining* p *to* q.

(vi) *If* p *and* q *are two base-points of* F*, then*

$$d_{F,p}(F+K,\Omega,y) = \nu d_{F,q}(F+K,\Omega,y), \tag{7.6}$$

where $\nu \in \{-1,1\}$ *is the parity of any path* $DF\circ\gamma$ *with* $\gamma \in C^0([0,1],X)$ *being a curve joining* p *to* q.

(vii) *If in addition F is a local diffeomorphism, $d_{F,p}(F+K,\Omega,y)$ is independent of $p \in X$ (and hence can be denoted by $d_F(F+K,\Omega,y)$).*

(viii) *If in addition $\Omega = -\Omega$, $0 \in \Omega$, $F+K$ is odd, and $0 \notin (F+K)(\partial\Omega)$, then $d_{F,p}(F+K,\Omega,0)$ is odd.*

Proof. (i) It suffices to show that $d_{F,p}(F+K,\Omega,y)$ is locally constant. Given $y \notin (F+K)(\partial\Omega)$, let $B(y)$ be an open ball centered at y such that $B(y) \subset Y \backslash (F+K)(\partial\Omega)$. For $z \in B(y)$ and $t \in [0,1]$, set $h_\kappa(t,x) := K(x) + (1-t)y + tz$. Then, $d_{F,p}(F+K-y,\Omega,0) = d_{F,p}(F+K-z,\Omega,0)$ by Corollary 7.2, that is, $d_{F,p}(F+K,\Omega,y) = d_{F,p}(F+K,\Omega,z)$.

Parts (ii), (iii), and (iv) follow at once from their analog when K is finite dimensional.

(v) This follows from Theorem 7.1 in the same way that Theorem 5.3 follows from Theorem 5.2.

(vi) This the special case of (v) where $G = F$ and $L = K$.

(vii) Just use (vi) and the fact that the parity of a path of isomorphisms is 1.

(viii) As in the proof of Theorem 6.2, reduce the problem to the case when both F and K are odd. Then, replace K by an odd finite-dimensional approximation K^f and use Theorem 6.2. □

In (7.5), the sign change may occur irrespective of p and q and hence may be only due to using different representations (see Section 8). Also, since Theorem 7.3(v) shows that the two pairs (F,K) and (G,L) such that $F+K = G+L = T$ always provide the same degree up to sign, we can define the absolute degree (if $y \notin (F+K)(\partial\Omega)$)

$$|d|(T,\Omega,y) = \left| d_{F,p}(T,\Omega,y) \right|, \tag{7.7}$$

independent of the representation $T = F+K$ such that F has base-points (recall that such representations exist) and of the base-point p of F.

Theorem 7.4. *Under the same assumptions as in Theorem 7.1, $|d|(h(0,\cdot),\Omega,y) = |d|(h(1,\cdot),\Omega,y)$.*

Proof. This is obvious from (7.7) if $h_\Phi(0,\cdot)$ and $h_\Phi(1,\cdot)$ have base-points. Otherwise, let $A_0, A_1 \in \mathcal{K}(X,Y)$ be such that $h_\Phi(0,\cdot)+A_0$ and $h_\Phi(1,\cdot)+A_1$ have base-points. With $A(t) = (1-t)A_0 + tA_1$, replace $h_\Phi(t,x)$ by $h_\Phi(t,x)+A(t)x$ and $h_\kappa(t,x)$ by $h_\kappa(t,x) - A(t)x$ to reduce the problem to the case when base-points exist. □

It is straightforward to check that $|d|$ also satisfies Theorem 7.3(i), (ii), (iii), and (viii).

The final step consists in removing the assumption that $F(\overline{\Omega})$ is bounded to define the degree $d_{F,p}(F+K,\Omega,y)$ if $y \notin (F+K)(\partial\Omega)$. To do this, we first observe that since F is locally bounded (being continuous) and $F+K$ is proper on $\overline{\Omega}$, there is an open neighborhood $\omega \subset \Omega$ of $(F+K)^{-1}(y) \cap \Omega$ such that $F(\overline{\omega})$ is bounded. Evidently, $y \notin (F+K)(\partial\omega)$ and hence $d_{F,p}(F+K,\omega,y)$ is well defined.

Lemma 7.5. *$d_{F,p}(F+K,\omega,y)$ is independent of the open subset ω with the above properties.*

Proof. Let ω_1 and ω_2 denote two choices of ω, so that $(F+K)^{-1}(y) \cap \Omega \subset \omega_1 \cap \omega_2$. Therefore, if $\Sigma_1 := \omega_1 \backslash \omega_2$ (a closed subset of ω_1), then $y \notin (F+K)(\Sigma_1 \cup \partial\omega_1)$ and hence, by excision (Theorem 7.3(iii)), $d_{F,p}(F+K,\omega_1,y) = d_{F,p}(F+K,\omega_1\cap\omega_2,y)$. By similar arguments, $d_{F,p}(F+K,\omega_2,y) = d_{F,p}(F+K,\omega_1\cap\omega_2,y)$, whence

$$d_{F,p}(F+K,\omega_1,y) = d_{F,p}(F+K,\omega_2,y). \tag{7.8}$$

□

It follows from Lemma 7.5 that if $y \notin (F+K)(\partial\omega)$ and $p \in X$ is a base-point of F, we may define

$$d_{F,p}(F+K,\Omega,y) = d_{F,p}(F+K,\omega,y), \tag{7.9}$$

where $\omega \subset \Omega$ is any open neighborhood of $(F+K)^{-1}(y)\cap\Omega$ such that $F(\overline{\omega})$ is bounded. With this definition, it is routine to check the following.

Theorem 7.6. *Theorems 7.1, 7.3, and 7.4 and Corollary 7.2 remain true without assuming that $F(\overline{\Omega})$ is bounded or that $h_\Phi([0,1]\times\overline{\Omega})$ is bounded.*

8. Further remarks and complements

(1) A special case arises when the compact operator K is C^1 and hence the degree $d_q(F+K,\Omega,y)$ of Section 2 already exists if q is a base-point of $F+K$. If so, by letting $G = F+K$ and $L = 0$ and by using Remark 3.4, it follows from Theorem 7.3(v) that

$$d_{F,p}(F+K,\Omega,y) = \nu d_q(F+K,\Omega,y), \tag{8.1}$$

where $\nu \in \{-1,1\}$ is the parity of any path $\{DF(\gamma(t))+tDK(\gamma(t)) : t\in[0,1]\}$ and $\gamma \in C^0([0,1],X)$ is a curve joining p to q. In particular, if p is also a base-point of $F+K$, then we can choose $\gamma(t) = p$ for $t\in[0,1]$ to get

$$d_{F,p}(F+K,\Omega,y) = \nu d_p(F+K,\Omega,y), \tag{8.2}$$

where ν is the parity of the path $\{DF(p)+tDK(p) : t\in[0,1]\}$.

(2) As we will see in an example in the next section, the absolute degree offers a convenient way to deal with situations when no base-point exists, without having to modify F and K to reinstate the existence of base-points. However, some caution should be exercised: if F has no base-point, then $|d|(F,\Omega,y) = 0$ (see Section 2), but, in general, this does *not* imply that $|d|(F+K,\Omega,y) = 0$.

(3) When $Y = X$ and $F = I$, the degree $d_{I,p}(I+K,\Omega,y)$ is *exactly* the Leray-Schauder degree $d_{LS}(I+K,\Omega,y)$: by Theorem 7.3(vii),

$$d_{I,p}(I+K,\Omega,y) = d_I(I+K,\Omega,y) \tag{8.3}$$

is independent of $p\in X$. Furthermore, by Theorem 7.3, the degree d_I has all the properties characterizing the Leray-Schauder degree (see, e.g., [7, Theorem 8.1]). This includes

$d_I(I,\Omega,y) = d_p(I,\Omega,y) = 1$ for every $p \in X$ (by Remark 3.4 and a trivial calculation), which implies $d_I = d_{LS}$. In particular, if $X = Y = \mathbb{R}^N$, then $d_I(K,\Omega,y) = d_I(I+(K-I),\Omega,y)$ is the Brouwer degree of K.

(4) The relations $d_{I,p} = d_I = d_{LS}$ enable us to give a very simple example when $\nu = -1$ in (7.5) irrespective of base-points, so that the sign change is only due to using different representations. Let $X = Y$ and let $K \in \mathcal{K}(X)$ be such that $I+K \in GL(X)$ and that the sum of the algebraic multiplicities of the negative eigenvalues of $I+K$ is odd. If Ω is any nonempty bounded open subset of X and $y \in (I+K)(\Omega)$, then $d_{LS}(I+K,\Omega,y) = -1$ and hence $d_{I,p}(I+K,\Omega,y) = -1$ for all $p \in X$. On the other hand, since $I+K \in \Phi_0 C^1(X)$, we have $d_{I+K,q}(I+K,\Omega,y) = d_q(I+K,\Omega,y)$ by Remark 3.4 and $d_q(I+K,\Omega,y) = 1$ for every $q \in X$ by Remark 2.1 since $I+K \in GL(X)$. Thus, $d_{I+K,q}(I+K,\Omega,y) = 1$. This is to say that when $T = I+K$, $F = I$, $G = I+K$, and $L = 0$ in Theorem 7.3(v), then $d_{F,p}(T,\Omega,y) = -d_{G,q}(T,\Omega,y)$ for all $p,q \in X$.

(5) At the end of Section 2, it was pointed out that the degree theory of this paper can be repeated when F is only defined and is Fredholm of index 0 on some connected and simply connected open subset $\mathcal{O} \subset X$ containing Ω. This is sometimes useful. For instance, to prove the "*invariance-of-domain*" property that if $F \in \Phi_0 C^1(X,Y)$, $K \in C^0(X,Y)$ is compact, and $F+K$ is one-to-one, then $F+K$ is an open mapping. The problem is easily reduced to the case when $F(0) = K(0) = 0$ and to proving that $(F+K)(X)$ contains a neighborhood of 0. To see this, set $h := h_\Phi + h_\kappa$ where $h_\Phi(t,x) := F(x) - F(-tx)$ and $h_\kappa(t,x) := K(x) - K(-tx)$. While there is no guarantee that h_Φ is Fredholm on $[0,1]\times X$, it is readily checked that it is Fredholm of index 1 on $[0,1]\times B_\delta(0)$ if the ball $B_\delta(0)$ has small enough radius $\delta > 0$. Then, by the local properness of Fredholm mappings (see [26]), h_Φ is proper on $[0,1]\times \overline{B}_\delta(0)$ after shrinking δ if necessary. Since the injectivity of $F+K$ implies that $0 \notin h([0,1]\times \partial B_\delta(0))$, the conclusion follows from the homotopy invariance of $|d|$ and by Theorem 7.3(i), (ii), and (viii) for $|d|$.

(6) The following generalization of Theorem 7.1 when the open set Ω is varied will be used in the next section. When $E \subset \mathbb{R}\times X$ and $t \in \mathbb{R}$, we let $E_t \subset X$ denote the set $E_t := \{x \in X : (t,x) \in E\}$.

Theorem 8.1. *Let $\widetilde{\Omega}$ be a bounded open subset of $\mathbb{R}\times X$ and let $h = h_\Phi + h_\kappa$ where $h_\Phi \in \Phi_1 C^1(\mathbb{R}\times X, Y)$ is proper on $\overline{\widetilde{\Omega}}$ and $h_\kappa \in C^0(\mathbb{R}\times X,Y)$ is compact. On the other hand, let $[a,b] \subset \mathbb{R}$ be a compact interval and let $y \notin \cup_{t\in[a,b]} h(\{t\}\times(\partial\widetilde{\Omega})_t)$. If $p_a, p_b \in X$ are base-points of $h_\Phi(a,\cdot)$ and $h_\Phi(b,\cdot)$, respectively, then*

$$d_{h_\Phi(b,\cdot),p_b}(h(b,\cdot),\widetilde{\Omega}_b,y) = \nu d_{h_\Phi(a,\cdot),p_a}(h(a,\cdot),\widetilde{\Omega}_a,y), \tag{8.4}$$

where $\nu := \sigma(D_x h_\Phi \circ \Gamma) \in \{-1,1\}$ and Γ is any continuous curve in $\mathbb{R}\times X$ joining (a,p_a) to (b,p_b). Furthermore,

$$|d|(h(b,\cdot),\widetilde{\Omega}_b,y) = |d|(h(a,\cdot),\widetilde{\Omega}_a,y), \tag{8.5}$$

irrespective of the existence of base-points for $h_\Phi(a,\cdot)$ or $h_\Phi(b,\cdot)$.

Proof. Suppose first that $p_a = p_b = p$ and that p is a base-point of $h_\Phi(t,\cdot)$ for all $t \in [a,b]$. Since $\Gamma(t) := (t,p)$ is a curve joining (a,p) to (b,p) and the parity of a path of isomorphisms is 1, (8.4) amounts to saying that $d_{h_\Phi(t,\cdot),p}(h(t,\cdot),\widetilde{\Omega}_t,y)$ is independent of $t \in [a,b]$, that is, locally constant on $[a,b]$.

Since $\partial\widetilde{\Omega}_{t_0} \subset (\partial\widetilde{\Omega})_{t_0}$ and $y \notin h(\{t_0\}\times(\partial\widetilde{\Omega})_{t_0})$ by hypothesis, it follows that $h(t_0,\cdot)^{-1}(y)\cap \overline{\widetilde{\Omega}}_{t_0}$ is a compact subset of $\widetilde{\Omega}_{t_0}$ for every $t_0 \in [a,b]$. Hence, there is an open subset $\Omega \subset X$ with $\overline{\Omega} \subset \widetilde{\Omega}_{t_0}$ such that $h(t_0,\cdot)^{-1}(y) \cap \overline{\widetilde{\Omega}}_{t_0} \subset \Omega$. By a simple contradiction argument, there is $\varepsilon > 0$ such that $\overline{\Omega} \subset \widetilde{\Omega}_t$ and $h(t,\cdot)^{-1}(y) \cap \overline{\widetilde{\Omega}}_t \subset \Omega$ whenever $t \in [a,b]$ and $|t-t_0| < \varepsilon$. By excision (Theorems 7.6 and 7.3(iii)), $d_{h_\Phi(t,\cdot),p}(h(t,\cdot),\widetilde{\Omega}_t,y) = d_{h_\Phi(t,\cdot),p}(h(t,\cdot),\Omega,y)$ if $t \in [a,b]$ and $|t-t_0| < \varepsilon$ and the right-hand side is independent of any such t by Theorem 7.1 (using once again the fact that the parity of a path of isomorphisms is 1).

Returning to the general case, we now claim that, given any point $p \in X$, there is a finite-dimensional subspace Z of Y and there is $A \in C^1([a,b],\mathscr{L}(X,Z))$ such that p is a base-point of $\hat{h}_\Phi(t,x) := h_\Phi(t,x) + A(t)x$ for all $t \in [a,b]$. This will be justified later on in the proof. Since $h = \hat{h}_\Phi + \hat{h}_\kappa$ with $\hat{h}_\kappa(t,x) := h_\kappa(t,x) - A(t)x$, it follows from the first part of the proof that

$$d_{\hat{h}_\Phi(a,\cdot),p}(h(a,\cdot),\widetilde{\Omega}_a,y) = d_{\hat{h}_\Phi(b,\cdot),p}(h(b,\cdot),\widetilde{\Omega}_b,y). \tag{8.6}$$

Also, by Theorem 7.3(v),

$$\begin{aligned} d_{h_\Phi(a,\cdot),p_a}(h(a,\cdot),\widetilde{\Omega}_b,y) &= \nu_a d_{\hat{h}_\Phi(a,\cdot),p}(h(a,\cdot),\widetilde{\Omega}_b,y), \\ d_{\hat{h}_\Phi(b,\cdot),p}(h(b,\cdot),\widetilde{\Omega}_b,y) &= \nu_b d_{h_\Phi(b,\cdot),p_b}(h(b,\cdot),\widetilde{\Omega}_b,y), \end{aligned} \tag{8.7}$$

where $\nu_a \in \{-1,1\}$ is the parity of any path $\{D_x h_\Phi(a,\gamma_a(s)) + sA(a) : s \in [0,1]\}$ with $\gamma_a \in C^0([0,1],X)$ being a curve joining p_a to p and $\nu_b \in \{-1,1\}$ is the parity of any path $\{D_x h_\Phi(b,\gamma_b(s)) + sA(b) : s \in [0,1]\}$ with $\gamma_b \in C^0([0,1],X)$ being a curve joining p to p_b. Thus, from (8.6) and (8.7),

$$d_{h_\Phi(a,\cdot),p_a}(h(a,\cdot),\widetilde{\Omega}_a,y) = \nu_a\nu_b d_{h_\Phi(b,\cdot),p_b}(h(b,\cdot),\widetilde{\Omega}_b,y) \tag{8.8}$$

and it suffices to show that $\nu_a\nu_b = \nu$ as in (8.4). Since the parity is unaffected by reparametrization and since the parity of a path of isomorphisms is 1, it follows from the multiplicative property of the parity with respect to consecutive intervals that $\nu_a\nu_b$ is the parity of the path on $[a-1,b+1]$ defined by

$$\begin{gathered} D_x h_\Phi(a,\gamma_a(t-a+1)) + (t-a+1)A(a) \quad \text{if } a-1 \le t \le a, \\ D_x h_\Phi(t,p) + A(t) \quad \text{if } a \le t \le b, \\ D_x h_\Phi(b,\gamma_b(t-b)) + (b+1-t)A(b) \quad \text{if } b \le t \le b+1. \end{gathered} \tag{8.9}$$

This path is clearly homotopic to the path (with the same invertible endpoints $D_x h_\Phi(a, p_a)$ and $D_x h_\Phi(b, p_b)$) $D_x h_\Phi \circ \Gamma$ where

$$\Gamma(t) := \begin{cases} (a, \gamma_a(t-a+1)) & \text{if } a-1 \leq t \leq a, \\ (t, p) & \text{if } a \leq t \leq b, \\ (b, \gamma_b(b+1-t)) & \text{if } b \leq t \leq b+1, \end{cases} \tag{8.10}$$

joins (a, p_a) to (b, p_b). Thus, $\nu_a \nu_b = \sigma(D_x h_\Phi \circ \Gamma) = \nu$. The "furthermore" part follows from (8.4) and the definition of the absolute degree (if necessary, modify $h_\Phi(a, \cdot)$ and $h_\Phi(b, \cdot)$ so that base-points exist).

To complete the proof, we now establish the claimed existence of Z and A above. First, by the arguments of the proof of Lemma 4.2, there is a finite-dimensional subspace Z of Y such that $rge D_x h(t,p) + Z = Y$ for all $t \in [a,b]$. By setting $X_t := D_x h_\Phi(t,p)^{-1}(Z)$, we obtain a finite-dimensional C^0 vector bundle $(X_t)_{t\in[a,b]}$ with dimension $\dim Z$ (since $D_x h_\Phi(t,p)$ has index 0).

Let $P \in C^0([a,b], \mathscr{L}(X))$ be such that $P(t)$ projects onto X_t for all $t \in [a,b]$. The existence of P is standard (locally by trivialization and next globally with a partition of unity). Let also $B \in C^0([a,b], \mathscr{L}(Z,X))$ be a trivialization (every vector bundle over a contractible base is trivial; see [16]) of $(X_t)_{t\in[a,b]}$, that is, $rge B(t) = X_t$ for $t \in [a,b]$. Then, $B(t)^{-1}P(t) \in \mathscr{L}(X,Z)$ exists for all $t \in [a,b]$ and $B^{-1}P \in C^0([a,b], \mathscr{L}(X,Z))$. For the latter point, assume by contradiction that there is a sequence $(t_n) \subset [a,b]$ such that $t_n \to t$ but $\|B(t_n)^{-1}P(t_n) - B(t)^{-1}P(t)\| > \varepsilon > 0$ for all n. Then, $\|B(t_n)^{-1}P(t_n)x_n - B(t)^{-1}P(t)x_n\| \geq \varepsilon$ for some sequence $(x_n) \subset X$ with $\|x_n\| = 1$. It is readily checked that there is a constant $\alpha > 0$ such that $\|B(s)z\| \geq \alpha\|z\|$ for every $s \in [a,b]$ and every $z \in Z$, so that $\|B(s)^{-1}\|_{\mathscr{L}(X_s,Z)} \leq \alpha^{-1}$ for every $s \in [a,b]$. Hence,

$$||P(t_n)x_n - B(t_n)B(t)^{-1}P(t)x_n|| \geq \alpha\varepsilon, \tag{8.11}$$

for all n. Since $\dim X_t < \infty$, there is $\xi \in X_t$ and a subsequence (x_{n_k}) such that $P(t)x_{n_k} \to \xi$. Then, $P(t_{n_k})x_{n_k} = (P(t_{n_k}) - P(t))x_{n_k} + P(t)x_{n_k} \to \xi$ by the continuity of P and the boundedness of (x_n). Since also $B(t_{n_k}) \to B(t)$ in $\mathscr{L}(Z,X)$, a contradiction arises with (8.11) for k large enough.

The above shows that $B^{-1}P \in C^0([a,b], \mathscr{L}(X,Z))$. We also claim that $D_x h_\Phi(t,p) + \lambda B(t)^{-1}P(t) \in GL(X,Y)$ for all $t \in [a,b]$ if $\lambda > 0$ is large enough. Since $\lambda B(t)^{-1}P(t)$ is compact and $D_x h_\Phi(t,p)$ is Fredholm of index 0, it suffices to prove the injectivity. Let then $t \in [a,b]$ and $x \in X$ be such that $D_x h_\Phi(t,p)x + \lambda B(t)^{-1}P(t)x = 0$. This means that $D_x h_\Phi(t,p)P(t)x + \lambda B(t)^{-1}P(t)x = -D_x h_\Phi(t,p)Q(t)x$, where $Q(t) := I - P(t)$. Since the left-hand side is in Z by definition of X_t, this implies that $Q(t)x \in D_x h_\Phi(t,p)^{-1}(Z) = X_t$, so that $Q(t)x = 0$ since $Q(t)$ projects onto a complement of X_t. Thus, $D_x h_\Phi(t,p)P(t)x + \lambda B(t)^{-1}P(t)x = 0$, that is, $B(t)^{-1}P(t)x = -\lambda^{-1}D_x h_\Phi(t,p)P(t)x$ and hence $\|B(t)^{-1}P(t)x\| \leq \lambda^{-1}\|P(t)x\| \max_{s\in[a,b]} \|D_x h_\Phi(s,p)\|_{\mathscr{L}(X,Y)}$. Now, $\|B(t)^{-1}P(t)x\| \geq \beta\|P(t)x\|$ where $\beta := 1/\max_{s\in[a,b]} \|B(s)\|_{\mathscr{L}(Z,X)}$. If λ is large enough, then $\lambda^{-1} \max_{s\in[a,b]} \|D_x h_\Phi(s,p)\|_{\mathscr{L}(X,Y)} < \beta$. It follows that $P(t)x = 0$, whence $x = 0$.

Let now $\lambda > 0$ be chosen so that $D_x h_\Phi(t,p) + \lambda B(t)^{-1}P(t) \in GL(X,Y)$ for all $t \in [a,b]$. If $\varepsilon > 0$ is small enough, then $D_x h_\Phi(t,p) + A(t) \in GL(X,Y)$ for all $t \in [a,b]$ whenever $A \in C^0([a,b],\mathcal{L}(X,Z))$ and $\max_{s\in[a,b]} \|A(s) - \lambda B(s)^{-1}P(s)\|_{\mathcal{L}(X,Z)} < \varepsilon$. Any such choice with $A \in C^1([a,b],\mathcal{L}(X,Z))$ answers the question. $\square$

9. Application to global bifurcation

We consider a mapping $G : \mathbb{R} \times X \to Y$ of the form

$$G(\lambda,x) = F(\lambda,x) + K(\lambda,x), \tag{9.1}$$

with $F \in \Phi_1 C^1(\mathbb{R}\times X, Y)$ and $K \in C^0(\mathbb{R}\times X,Y)$ compact satisfying $F(\lambda,0) = K(\lambda,0) = 0$ for all $\lambda \in \mathbb{R}$. We also assume that $K(\lambda,\cdot)$ is Fréchet differentiable at 0 (but not necessarily elsewhere) and that

$$D_x K(\lambda,0) = 0 \quad \forall \lambda \in \mathbb{R}. \tag{9.2}$$

Theorem 9.1 below is a generalization of Rabinowitz's global bifurcation theorem when $F = I - \lambda L$ with $L \in \mathcal{K}(X)$ (see [22]). When $K = 0$, generalizations have already been given in [11, 18].

Theorem 9.1. *In addition to the above assumptions, suppose that there are $\lambda_- < \lambda_+$ such that $D_x F(\lambda_\mp,0) \in GL(X,Y)$ and that $\sigma(D_x F(\cdot,0),[\lambda_-,\lambda_+]) = -1$ (see Remark 9.3). Denote by S the closure in $\mathbb{R}\times X$ of $G^{-1}(0)\backslash(\mathbb{R}\times\{0\})$ and by C the connected component of $S \cup [\lambda_-,\lambda_+]\times\{0\}$ containing $[\lambda_-,\lambda_+]\times\{0\}$. Then, either C is noncompact or C contains a point $(\lambda^*,0)$ with $\lambda^* \notin [\lambda_-,\lambda_+]$.*

Proof. By contradiction, assume that C is compact and contains no point $(\lambda,0)$ with $\lambda \notin [\lambda_-,\lambda_+]$. Since Fredholm mappings are locally proper (see [26]), we can find a bounded open neighborhood $\tilde{\Omega}$ of C in $\mathbb{R}\times X$ such that F is proper on $\overline{\tilde{\Omega}}$. Then, by a variant of the Rabinowitz construction in [22], $\tilde{\Omega}$ may be shrunk so as to "isolate" C from the remainder of $S \cup [\lambda_-,\lambda_+]\times\{0\}$. In other words, $\tilde{\Omega}$ may be assumed to satisfy the condition that (F is proper on $\overline{\tilde{\Omega}}$ and)

$$\overline{\tilde{\Omega}} \cap (S \cup [\lambda_-,\lambda_+]\times\{0\}) = C. \tag{9.3}$$

In particular,

$$\Big\{(\lambda,x) \in \overline{\tilde{\Omega}}\backslash C, G(\lambda,x) = 0\Big\} \Longrightarrow \lambda \notin [\lambda_-,\lambda_+], \quad x = 0 \tag{9.4}$$

and since $\tilde{\Omega}$ is a neighborhood of $[\lambda_-,\lambda_+]\times\{0\}$, there is $\delta > 0$ such that

$$[\lambda_- - \delta, \lambda_+ + \delta]\times\{0\} \subset \tilde{\Omega}. \tag{9.5}$$

Furthermore, after shrinking $\delta > 0$, it follows from the hypothesis $D_x F(\lambda_\mp,0) \in GL(X,Y)$ that we may assume that $D_x F(\lambda,0) \in GL(X,Y)$ for all $\lambda \in [\lambda_- - \delta,\lambda_-] \cup [\lambda_+,\lambda_+ + \delta]$. This means that $p = 0$ is a base-point of $F(\lambda,\cdot)$ for all $\lambda \in [\lambda_- - \delta,\lambda_-] \cup [\lambda_+,\lambda_+ + \delta]$.

By (9.4) and (9.5), we have that

$$0 \notin G\Big(\{\lambda\} \times (\partial\widetilde{\Omega})_\lambda\Big) \quad \text{for } \lambda \in [\lambda_- - \delta, \lambda_+ + \delta]. \tag{9.6}$$

Indeed, if $G(\lambda,x) = 0$ with $(\lambda,x) \in \partial\widetilde{\Omega}$ and $\lambda \in [\lambda_- - \delta, \lambda_+ + \delta]$, then $(\lambda,x) \notin C \subset \widetilde{\Omega}$ and hence $x = 0$ by (9.4), which contradicts (9.5).

By (9.6) for $\lambda \in [\lambda_- - \delta, \lambda_-]$ and Theorem 8.1 with (t called λ and) $h_\Phi = F$, $h_\kappa = K$, and the remark above that $p = 0$ is a base-point of $F(\lambda,\cdot)$ for all $\lambda \in [\lambda_- - \delta, \lambda_-]$, we obtain

$$d_{F(\lambda_- - \delta,\cdot),0}\big(G(\lambda_- - \delta,\cdot),\widetilde{\Omega}_{\lambda_- - \delta},0\big) = d_{F(\lambda_-,\cdot),0}\big(G(\lambda_-,\cdot),\widetilde{\Omega}_{\lambda_-},0\big), \tag{9.7}$$

by simply choosing $\Gamma(\lambda) = (\lambda,0)$, $\lambda \in [\lambda_- - \delta, \lambda_-]$ in Theorem 8.1. If so, $D_xF \circ \Gamma = D_xF(\cdot,0)$ is a path of isomorphisms (thus with parity 1).

Now, we claim that $d_{F(\lambda_- - \delta,\cdot),0}(G(\lambda_- - \delta,\cdot),\widetilde{\Omega}_{\lambda_- - \delta},0) = 1$. Here, the relevant remark is that there is $\varepsilon_0 > 0$ such that if $G(\lambda,x) = 0$ with $\lambda \le \lambda_- - \delta$ and $(\lambda,x) \in \overline{\widetilde{\Omega}}$, then either $x = 0$ or $\|x\| \ge \varepsilon_0$. This follows from (9.4) and $\operatorname{dist}(C,(-\infty,\lambda_- - \delta] \times \{0\}) > 0$ (since C is compact and contains no point $(\lambda,0)$ with $\lambda \notin [\lambda_-,\lambda_+]$). As a result, if $0 < \varepsilon < \varepsilon_0$ and if $\widetilde{\Omega}^\varepsilon$ denotes the open set $\widetilde{\Omega} \backslash (\mathbb{R} \times \overline{B}_\varepsilon(0))$, we have

$$0 \notin G\Big(\{\lambda\} \times (\partial\widetilde{\Omega}^\varepsilon)_\lambda\Big) \quad \forall \lambda \le \lambda_- - \delta. \tag{9.8}$$

Indeed, $(\partial\widetilde{\Omega}^\varepsilon)_\lambda \subset \partial B_\varepsilon(0) \cup (\partial\widetilde{\Omega})_\lambda \backslash B_\varepsilon(0)$ and, from the above, $G(\lambda,x) \neq 0$ if $\lambda \le \lambda_- - \delta$ and $x \in \partial B_\varepsilon(0)$ while $G(\lambda,x) \neq 0$ if $(\lambda,x) \in \partial\widetilde{\Omega} \subset \overline{\widetilde{\Omega}} \backslash C$ and $\|x\| \ge \varepsilon$ by (9.4).

The boundedness of $\widetilde{\Omega}^\varepsilon$ shows that $\widetilde{\Omega}^\varepsilon_\lambda = \emptyset$ if $\lambda \ll \lambda_- - \delta$. It thus follows from (9.8) and Theorem 8.1 and from $\widetilde{\Omega}^\varepsilon_\lambda = \widetilde{\Omega}_\lambda \backslash \overline{B}_\varepsilon(0)$ that

$$|d|\big(G(\lambda,\cdot),\widetilde{\Omega}_\lambda \backslash \overline{B}_\varepsilon(0),0\big) = 0 \tag{9.9}$$

for all $\lambda \le \lambda_- - \delta$. (Here, the use of the absolute degree is required since it is not known whether $F(\lambda,\cdot)$ has base-points for $\lambda \ll \lambda_- - \delta$.) In particular, $d_{F(\lambda_- - \delta,\cdot),0}(G(\lambda_- - \delta,\cdot), \widetilde{\Omega}_{\lambda_- - \delta} \backslash \overline{B}_\varepsilon(0),0) = 0$ and so, by excision (since there is no solution $x \in \partial B_\varepsilon(0)$ of $G(\lambda_- - \delta,x) = 0$ and since $\overline{B}_\varepsilon(0) \subset \widetilde{\Omega}_{\lambda_- - \delta}$ if $\varepsilon > 0$ is small enough by (9.5)),

$$d_{F(\lambda_- - \delta,\cdot),0}\big(G(\lambda_- - \delta,\cdot),\widetilde{\Omega}_{\lambda_- - \delta},0\big) = d_{F(\lambda_- - \delta,\cdot),0}\big(G(\lambda_- - \delta,\cdot),B_\varepsilon(0),0\big), \tag{9.10}$$

provided that $\varepsilon > 0$ is small enough. Recalling that $D_xF(\lambda_- - \delta,0) \in GL(X,Y)$ and that $D_xK(\lambda_- - \delta,0) = 0$ (see (9.2)), $\varepsilon > 0$ can be chosen so small that $0 \notin h([0,1] \times \partial B_\varepsilon(0))$ where $h(t,x) := F(\lambda_- - \delta,x) + tK(\lambda_- - \delta,x)$. Thus, by Corollary 7.2 and Remark 3.4, we find

$$\begin{aligned} &d_{F(\lambda_- -\delta,\cdot),0}(G(\lambda_- - \delta,\cdot),B_\varepsilon(0),0) \\ &\quad = d_{F(\lambda_- -\delta,\cdot),0}(F(\lambda_- - \delta,\cdot),B_\varepsilon(0),0) = d_0(F(\lambda_- - \delta,\cdot),B_\varepsilon(0),0). \end{aligned} \tag{9.11}$$

Now, the assumption $D_xF(\lambda_- - \delta,0) \in GL(X,Y)$ implies that $F(\lambda_- - \delta,x) = 0$ has no nonzero solution in $\overline{B}_\varepsilon(0)$ if $\varepsilon > 0$ is small enough, whence $d_0(F(\lambda_- - \delta,\cdot),B_\varepsilon(0),0) = 1$ by definition of the degree at regular values (see Section 2) since the base-point $p = 0$ and the solution $x = 0$ coincide. By substitution into (9.10) and (9.11), we get

$$d_{F(\lambda_- -\delta,\cdot),0}(G(\lambda_- - \delta,\cdot),\widetilde{\Omega}_{\lambda_- -\delta},0) = 1, \tag{9.12}$$

as claimed earlier.

With this, it follows from (9.7) that $d_{F(\lambda_-,\cdot),0}(G(\lambda_-,\cdot),\widetilde{\Omega}_{\lambda_-},0) = 1$. Naturally,

$$d_{F(\lambda_+,\cdot),0}(G(\lambda_+,\cdot),\widetilde{\Omega}_{\lambda_+},0) = 1 \tag{9.13}$$

by similar arguments. Thus,

$$d_{F(\lambda_-,\cdot),0}(G(\lambda_-,\cdot),\widetilde{\Omega}_{\lambda_-},0) = d_{F(\lambda_+,\cdot),0}(G(\lambda_+,\cdot),\widetilde{\Omega}_{\lambda_+},0) = 1. \tag{9.14}$$

On the other hand, by (9.6) for $\lambda \in [\lambda_-,\lambda_+]$ and Theorem 8.1,

$$d_{F(\lambda_-,\cdot),0}(G(\lambda_-,\cdot),\widetilde{\Omega}_{\lambda_-},0) = \nu d_{F(\lambda_+,\cdot),0}(G(\lambda_+,\cdot),\widetilde{\Omega}_{\lambda_+},0), \tag{9.15}$$

where ν is the parity of any path $D_xF \circ \Gamma$ with Γ joining $(\lambda_-,0)$ to $(\lambda_+,0)$. One such path is $D_xF(\lambda,\cdot)$, $\lambda \in [\lambda_-,\lambda_+]$, with parity -1 by hypothesis. Clearly, (9.15) with $\nu = -1$ contradicts (9.14) and the proof is complete. □

Remark 9.2. If also F is proper on the closed bounded subsets of $\mathbb{R} \times X$, then "C is noncompact" in Theorem 9.1 is equivalent to "C is unbounded". In particular, when $F(\lambda,\cdot)$ is linear (as in the original theorem of Rabinowitz), Fredholmness implies the properness on closed bounded subsets.

Remark 9.3. In Theorem 9.1, the condition $\sigma(D_xF(\cdot,0),[\lambda_-,\lambda_+]) = -1$ generalizes the hypothesis that "λ crosses a characteristic value of L of odd algebraic multiplicity" when $F(\lambda,\cdot) = I - \lambda L$ and $L \in \mathscr{L}(X)$ is compact, and its generalizations when $D_xF(\lambda,\cdot)$ is nonlinear in λ (see [19]). Furthermore, unlike all the "odd algebraic multiplicity" crossing assumptions, $\sigma(D_xF(\cdot,0),[\lambda_-,\lambda_+]) = -1$ is *not* a local condition, insofar as it does not require $D_xF(\lambda,\cdot)$ to be singular only at isolated points. On the other hand, $D_xF(\lambda,\cdot)$ must be singular at *some* point of $[\lambda_-,\lambda_+]$, for otherwise $\sigma(D_xF(\cdot,0),[\lambda_-,\lambda_+]) = 1$.

Acknowledgment

Part of this work was done while the first author was visiting the Bernoulli Center, EPFL, Switzerland, whose hospitality and support are gratefully acknowledged.

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