

THE CAUCHY PROBLEM AND DECAY RATES FOR STRONG SOLUTIONS OF A BOUSSINESQ SYSTEM

FRANCISCO GUILLÉN GONZÁLEZ, MÁRCIO SANTOS DA ROCHA,
AND MARKO ROJAS MEDAR

Received 10 June 2004

The Boussinesq equations describe the motion of an incompressible viscous fluid subject to convective heat transfer. Decay rates of derivatives of solutions of the three-dimensional Cauchy problem for a Boussinesq system are studied in this work.

1. Introduction

In this work we show some theoretical results about decay rates of strong solutions of the three-dimensional Cauchy problem for Boussinesq equations, described by the following partial differential equation problem (see [6]):

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \pi &= \theta \mathbf{f} && \text{in } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } (0, T) \times \mathbb{R}^3, \\ \frac{\partial \theta}{\partial t} - \chi \Delta \theta + \mathbf{u} \cdot \nabla \theta &= 0 && \text{in } (0, T) \times \mathbb{R}^3, \\ \mathbf{u}(0, x) &= \mathbf{a}(x) && \text{in } \mathbb{R}^3, \\ \theta(0, x) &= b(x) && \text{in } \mathbb{R}^3, \end{aligned} \tag{1.1}$$

where the unknown are \mathbf{u} , θ , π which denote, respectively, the velocity field, the scalar temperature and the scalar pressure. Data are the positive constants ν , χ , respectively, the viscosity and the thermal conductivity coefficients and the function \mathbf{f} the external force field, and $\mathbf{a}(x)$, $b(x)$, respectively, represent the initial velocity and initial temperature.

The main objective of this work is to obtain a decay rate of derivatives for the strong solutions to the Cauchy problem (1.1). For this, we will consider the usual Lebesgue spaces $L^p(\mathbb{R}^3)$ with the usual norms $|\cdot|_p$. We will denote $L^p_\sigma(\mathbb{R}^3)$ the closure of $C^\infty_{0,\sigma}(\mathbb{R}^3) = \{\mathbf{v} \in C^\infty_0; \operatorname{div} \mathbf{v} = 0\}$ in $L^p(\mathbb{R}^3)$. We will denote too by $L^p(0, T; L^q(\mathbb{R}^3))$ the Banach space, classes of functions defined a.e. in $[0, T]$ on $L^q(\mathbb{R}^3)$, that are L^p -integrable in the sense of Bochner. For more details see [1, 3].

We observe that this model of fluids includes as a particular case the classical Navier-Stokes equations, which has been thoroughly studied (see, e.g., [7, 8]). Rojas Medar and

Lorca obtained results of uniqueness and existence of the local solutions and regularity of solutions for Boussinesq equations [9, 10].

Results of decay rates of strong solution were obtained by Cheng He and Ling Hsião [4]. In this paper, we are interested to get similar results for Boussinesq equations.

2. Results of decay rates

The main objective of this work is to establish the decay rates of derivatives about time variable and spaces variables for the strong solutions to the Cauchy problem of the Boussinesq equations (1.1). For this, we will consider a sequence of Cauchy problems for the linearized Boussinesq equations

$$\begin{aligned} \frac{\partial \mathbf{u}^k}{\partial t} - \nu \Delta \mathbf{u}^k + (\mathbf{u}^{k-1} \cdot \nabla) \mathbf{u}^{k-1} + \nabla \pi^k &= \theta^{k-1} \mathbf{f} \quad \text{on } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{u}^k &= 0 \quad \text{on } (0, T) \times \mathbb{R}^3, \\ \frac{\partial \theta^k}{\partial t} - \chi \Delta \theta^k + \mathbf{u}^{k-1} \cdot \nabla \theta^{k-1} &= 0 \quad \text{on } (0, T) \times \mathbb{R}^3, \\ \mathbf{u}^k(0, x) &= \mathbf{a}^k(x) \quad \text{on } \mathbb{R}^3, \\ \theta^k(0, x) &= b^k(x) \quad \text{on } \mathbb{R}^3, \end{aligned} \tag{2.1}$$

for $k \geq 1$, where $\mathbf{a}^k \in C_{0,\sigma}^\infty(\mathbb{R}^3)$ and $b^k \in C_0^\infty(\mathbb{R}^3)$ such that

$$\begin{aligned} \mathbf{a}^k &\longrightarrow \mathbf{a} \quad \text{in } L_\sigma^3(\mathbb{R}^3) \text{ strongly,} \\ b^k &\longrightarrow b \quad \text{in } L^3(\mathbb{R}^3) \text{ strongly,} \end{aligned} \tag{2.2}$$

with $|\mathbf{a}^k|_3 \leq |\mathbf{a}|_3$ and $|b^k|_3 \leq |b|_3$. The first term, $(\mathbf{u}^0, \pi^0, \theta^0)$, of this sequence is solution of the trivial Cauchy problem:

$$\begin{aligned} \frac{\partial \mathbf{u}^0}{\partial t} - \nu \Delta \mathbf{u}^0 + \nabla \pi^0 &= \mathbf{0} \quad \text{on } (0, T) \times \mathbb{R}^3, \\ \operatorname{div} \mathbf{u}^0 &= 0 \quad \text{on } (0, T) \times \mathbb{R}^3, \\ \frac{\partial \theta^0}{\partial t} - \chi \Delta \theta^0 &= 0 \quad \text{on } (0, T) \times \mathbb{R}^3, \\ \mathbf{u}^0(0, x) &= \mathbf{a}^0(x) \quad \text{on } \mathbb{R}^3, \\ \theta^0(0, x) &= b^0(x) \quad \text{on } \mathbb{R}^3. \end{aligned} \tag{2.3}$$

Let $\Gamma_\nu(t, x; s, y) = (4\nu\pi t)^{-3/2} \exp(-|x|^2/4\nu t)$ be a fundamental solution of the heat equation in \mathbb{R}^3 (with viscosity coefficient ν). Then, the solution of the linearized Boussinesq

system (2.1) can be written as follows:

$$\begin{aligned}
 u_i^k(t, x) &= \int_{\mathbb{R}^3} \Gamma_\nu(t, x; 0, y) a_i^k(y) dy \\
 &\quad - \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial u_i^{k-1}}{\partial x_j}(s, y) dy ds \\
 &\quad - \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \frac{\partial \pi^k}{\partial x_i}(s, y) dy ds \\
 &\quad + \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \theta^{k-1}(s, y) f_i(s, y) dy ds,
 \end{aligned} \tag{2.4}$$

$$\begin{aligned}
 \theta^k(t, x) &= \int_{\mathbb{R}^3} \Gamma_\chi(t, x; 0, y) b^k(y) dy \\
 &\quad - \int_0^t \int_{\mathbb{R}^3} \Gamma_\chi(t, x; s, y) \sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial \theta^{k-1}}{\partial x_j}(s, y) dy ds.
 \end{aligned} \tag{2.5}$$

The convergence for the above method can be seen in [2, 7].

Definition 2.1. A couple (\mathbf{u}, θ) is called strong solution for the system (1.1), if

$$\begin{aligned}
 \mathbf{u} &\in L^p(0, \infty; L^q_\sigma(\mathbb{R}^3)) \cap L^\infty(0, \infty; L^3_\sigma(\mathbb{R}^3)), \\
 \theta &\in L^p(0, \infty; L^q(\mathbb{R}^3)) \cap L^\infty(0, \infty; L^3(\mathbb{R}^3))
 \end{aligned} \tag{2.6}$$

for some $p > 2$ and $q > 3$, and satisfying

$$\begin{aligned}
 &\int_0^\infty \int_{\mathbb{R}^3} \left(\mathbf{u} \cdot \frac{d\varphi}{dt} + \mathbf{u} \cdot \Delta \varphi + (\mathbf{u} \cdot \nabla \varphi) \mathbf{u} \right) dx dt \\
 &= - \int_{\mathbb{R}^3} \mathbf{a} \cdot \varphi(0, x) dx + \int_0^\infty \int_{\mathbb{R}^3} \theta \mathbf{f} \cdot \varphi dx dt
 \end{aligned} \tag{2.7}$$

for all $\varphi \in C_0^\infty(0, \infty; C_{0,\sigma}^\infty(\mathbb{R}^3))$ and

$$\int_0^\infty \int_{\mathbb{R}^3} \left(\theta \frac{d\psi}{dt} + \theta \Delta \psi + (\mathbf{u} \cdot \nabla \psi) \theta \right) dx dt = - \int_{\mathbb{R}^3} b \psi(0, x) dx \tag{2.8}$$

for all $\psi \in C_0^\infty(0, \infty; C_0^\infty(\mathbb{R}^3))$.

LEMMA 2.2. For the pressure π^k the following estimate holds:

$$|\nabla \pi^k|_r \leq C |(\mathbf{u}^{k-1} \cdot \nabla) \mathbf{u}^{k-1} + \theta^{k-1} \mathbf{f}|_r \tag{2.9}$$

for $1 < r < \infty, k > 0$.

LEMMA 2.3. Let $\mathbf{a} \in L^3_\sigma(\mathbb{R}^3)$, $b \in L^3(\mathbb{R}^3)$, $|\mathbf{f}(t)|_q \leq C_0 t^{-1+3/2q}(|\mathbf{a}|_3 + |b|_3)$, $|\nabla \mathbf{f}(t)|_q \leq C_0 t^{-3/2+3/2q}(|\mathbf{a}|_3 + |b|_3)$, where the constant C_0 is independent of t and q . If $C^* C_0(|\mathbf{a}|_3 + |b|_3) \leq 1$ for some constant C^* , then

$$\begin{aligned} t^{1/2-3/2q} \left(|\mathbf{u}^k(t)|_q + |\theta^k(t)|_q \right) &\leq CC_0(|\mathbf{a}|_3 + |b|_3), \\ t^{1-3/2q} \left(|\nabla \mathbf{u}^k(t)|_q + |\nabla \theta^k(t)|_q \right) &\leq CC_0(|\mathbf{a}|_3 + |b|_3) \end{aligned} \tag{2.10}$$

for $3 \leq q \leq \infty$, $t \geq 0$ and $k \geq 0$.

Proof. We put

$$\begin{aligned} I_q^k &= t^{1/2-3/2q} \left(|\mathbf{u}^k(t)|_q + |\theta^k(t)|_q \right) \\ J_q^k &= t^{1-3/2q} \left(|\nabla \mathbf{u}^k(t)|_q + |\nabla \theta^k(t)|_q \right). \end{aligned} \tag{2.11}$$

We will assume by inductive hypotheses that the estimates (2.10) are true for $k - 1$. By the Young inequality for convolution, we can estimate the terms of (2.4) as follows:

$$\begin{aligned} \left| \int_{\mathbb{R}^3} \Gamma_\nu(t, \mathbf{x}; 0, y) a_i^k(y) dy \right|_q &\leq (4\nu\pi t)^{-3/2} \left(\int_{\mathbb{R}^3} e^{-|x-y|^2/4\nu t} dy \right)^{1/p} |a_i^k|_3 \\ &\leq C t^{-1/2+3/2q} |a_i^k|_3, \end{aligned} \tag{2.12}$$

where $1/p + 1/3 = 1 + 1/q$. Again, by the Young inequality we obtain

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, \mathbf{x}; s, y) \sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial u_i^{k-1}}{\partial x_j}(s, y) dy ds \right|_q \\ \leq C \int_0^t (t-s)^{-3/2(1/2-1/q)} |\mathbf{u}^{k-1}|_4 |\nabla \mathbf{u}^{k-1}|_4 ds \\ \leq CC_0^2(|\mathbf{a}|_3 + |b|_3)^2 t^{-1/2+3/2q}, \end{aligned} \tag{2.13}$$

where $1/p + 1/2 = 1 + 1/q$. Now, using (2.9) we have

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, \mathbf{x}; s, y) \frac{\partial \pi^k}{\partial x_i}(s, y) dy ds \right|_q &\leq CC_0^2(|\mathbf{a}|_3 + |b|_3)^2 t^{-1/2+3/2q}, \\ \left| \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, \mathbf{x}; s, y) \theta^{k-1}(s, y) f_i(s, y) dy ds \right|_q &\leq CC_0^2 t^{-1+3/2q} (|\mathbf{a}|_3 + |b|_3)^2. \end{aligned} \tag{2.14}$$

Moreover

$$|\mathbf{u}^k(t)|_q \leq C t^{-1/2+3/2q} \left(C_0(|\mathbf{a}|_3 + |b|_3) + C_0^2(|\mathbf{a}|_3 + |b|_3)^2 \right). \tag{2.15}$$

Analogously, we can obtain the estimate for $|\theta^k(t)|_q$. Now, differentiating (2.4) and using the fact

$$\left| \frac{\partial}{\partial x_i} \Gamma_\nu(t, \mathbf{x}; s, y) \right| \leq C(t-s)^{-2} e^{-\lambda|x-y|^2/4\nu(t-s)} \tag{2.16}$$

for $i = 1, 2, 3$ and some constant $\lambda > 0$, follows

$$\left| \frac{\partial}{\partial x_i} \int_{\mathbb{R}^3} \Gamma_\nu(t, x; 0, y) a_i^k(y) dy \right|_q \leq C t^{-1+3/2q} |\mathbf{a}|_3. \tag{2.17}$$

Analogously, we obtain

$$\left| \frac{\partial}{\partial x_i} \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial u_i^{k-1}}{\partial x_j}(s, y) dy ds \right|_q \tag{2.18}$$

$$\leq CC_0^2 (|\mathbf{a}|_3 + |\mathbf{b}|_3)^2 t^{-1+3/2q},$$

$$\left| \frac{\partial}{\partial x_i} \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \frac{\partial \pi^k}{\partial x_i}(s, y) dy ds \right|_q \tag{2.19}$$

$$\leq CC_0^2 (|\mathbf{a}|_3 + |\mathbf{b}|_3)^2 t^{-1+(3/2q)} \int_0^1 (1-w)^{-1/2-3/2r} w^{-3/2+3/2(1/r+1/q)} dw,$$

where $r > 3$ and $1/r + 1/q > 1/3$, to obtain the convergence of the last integral in (2.19). Finally, we obtain

$$\left| \frac{\partial}{\partial x_i} \int_0^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \theta^{k-1} f_i dy ds \right|_q \leq C \int_0^t (t-s)^{-1/2+3/2(1/q-1/l)} |\theta^{k-1} f_i|_l ds. \tag{2.20}$$

Without difficulty we can obtain for the equation of the temperature

$$J_q^k \leq C_0 (|\mathbf{a}|_3 + |\mathbf{b}|_3) + CC_0^2 (|\mathbf{a}|_3 + |\mathbf{b}|_3)^2 \leq 2C_0 (|\mathbf{a}|_3 + |\mathbf{b}|_3). \tag{2.21}$$

For $q = \infty$ we can obtain analogously as before

$$\begin{aligned} I_\infty^k &\leq C (|\mathbf{a}|_3 + |\mathbf{b}|_3) + C t^{1/2} \int_0^t (t-s)^{-3/4} |\mathbf{u}^{k-1}|_4 (|\nabla \mathbf{u}^{k-1}|_4 + |\nabla \theta^{k-1}|_4) ds \\ &\quad + C t^{1/2} \int_0^t (t-s)^{-3/4} |\theta^{k-1}|_4 |\mathbf{f}|_4 ds \leq C (|\mathbf{a}|_3 + |\mathbf{b}|_3) + CC_0^2 (|\mathbf{a}|_3 + |\mathbf{b}|_3)^2. \end{aligned} \tag{2.22}$$

Similarly, we obtain the estimative for J_∞^k . □

LEMMA 2.4. *Let $\mathbf{a} \in L_\sigma^3(\mathbb{R}^3)$, $\mathbf{b} \in L^3(\mathbb{R}^3)$ and $|\mathbf{f}|_q \leq C_0 t^{-1+3/2q} (|\mathbf{a}|_3 + |\mathbf{b}|_3)$, $|\nabla \mathbf{f}|_q \leq C_0 t^{-3/2+3/2q} (|\mathbf{a}|_3 + |\mathbf{b}|_3)$. If $C^* C_0 (|\mathbf{a}|_3 + |\mathbf{b}|_3) \leq 1$ for some constant C^* , then for $3 \leq q \leq \infty$, the following estimate is true uniformly in k :*

$$t^{3/2-3/2q} \sum_{l,j=1}^3 \left(\left| \frac{\partial^2 \mathbf{u}^k}{\partial x_l \partial x_j} \right|_q + \left| \frac{\partial^2 \theta^k}{\partial x_l \partial x_j} \right|_q \right) \leq 2C_0 (|\mathbf{a}|_3 + |\mathbf{b}|_3). \tag{2.23}$$

Proof. The identity (2.4) can be written as follows:

$$\begin{aligned}
 u_i^k(t, x) &= \int_{\mathbb{R}^3} \Gamma_\nu(t, x; 0, y) a_i^k(y) dy - \int_0^{t/2} \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \\
 &\quad \times \left(\sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial u_i^{k-1}}{\partial x_j}(s, y) + \frac{\partial \pi^k}{\partial x_i}(s, y) - \theta^{k-1}(s, y) f_i(s, y) \right) dy ds \\
 &\quad - \int_{t/2}^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \left(\sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial u_i^{k-1}}{\partial x_j}(s, y) \right. \\
 &\quad \quad \left. + \frac{\partial \pi^k}{\partial x_i}(s, y) - \theta^{k-1}(s, y) f_i(s, y) \right) dy ds
 \end{aligned} \tag{2.24}$$

analogously for temperature.

We will make the case $l = j$ (for the case $l \neq j$ the argument is analogous). By the Young inequality we obtain

$$\begin{aligned}
 &\left| \frac{\partial^2}{\partial x_l^2} \int_{\mathbb{R}^3} \Gamma_\nu(t, x; 0, y) a_i^k(y) dy \right|_q \\
 &\leq C \frac{1}{t} \left| \int_{\mathbb{R}^3} \left(\Gamma_\nu(t, x; 0, y) a_i^k(y) + \frac{(x_l - y_l)^2}{t^2} \Gamma_\nu(t, x; 0, y) a_i^k(y) \right) dy \right|_q \\
 &\leq C t^{-3/2+3/2q} |\mathbf{a}|_3.
 \end{aligned} \tag{2.25}$$

By analogous computations, we have

$$\begin{aligned}
 &\left| \frac{\partial^2}{\partial x_l^2} \int_0^{t/2} \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial u_i^{k-1}}{\partial x_j}(s, y) dy \right|_q \\
 &\leq CC_0^2 (|\mathbf{a}|_3 + |\mathbf{b}|_3)^2 t^{-3/2+3/2q} \int_0^{1/2} (1-w)^{-7/4+3/2q} w^{-3/4} dw.
 \end{aligned} \tag{2.26}$$

The estimation for the terms that involve $\int_0^{t/2}$ are obtained analogously. By other side

$$\begin{aligned}
 &\left| \frac{\partial^2}{\partial x_l^2} \int_{t/2}^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \sum_{j=1}^3 u_j^{k-1}(s, y) \frac{\partial u_i^{k-1}}{\partial x_j}(s, y) dy ds \right|_q \\
 &\leq CC_0^2 (|\mathbf{a}|_3 + |\mathbf{b}|_3)^2 t^{-3/2+3/2q} \int_{1/2}^1 (1-w)^{-1/2} w^{-2+3/2q} dw \\
 &\quad + CC_0 (|\mathbf{a}|_3 + |\mathbf{b}|_3) \int_{1/2}^1 (1-w)^{-1/2} w^{-2+3/2q} dw \\
 &\quad \times \sup_{s \in [t/2, t]} \left| s^{-3/2+3/2q} \frac{\partial^2 \mathbf{u}^{k-1}}{\partial x_l^2}(s) \right|_q.
 \end{aligned} \tag{2.27}$$

Analogously

$$\begin{aligned} & \left| \frac{\partial^2}{\partial x_i^2} \int_{t/2}^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \frac{\partial \pi}{\partial x_i} dy ds \right|_q \\ & \leq C \int_{t/2}^t (t-s)^{-2} \left(\left| \frac{\partial}{\partial x_i} ((\mathbf{u}^{k-1} \cdot \nabla) \mathbf{u}^{k-1}) \right|_q + \left| \frac{\partial \theta^{k-1} \mathbf{f}}{\partial x_i} \right|_q \right) ds \end{aligned} \tag{2.28}$$

and, finally,

$$\begin{aligned} & \left| \frac{\partial^2}{\partial x_i^2} \int_{t/2}^t \int_{\mathbb{R}^3} \Gamma_\nu(t, x; s, y) \theta^{k-1} f_i dy ds \right|_q \\ & \leq CC_0^2 (|\mathbf{a}|_3 + |\mathbf{b}|_3)^2 t^{-3/2+3/2q} \int_{1/2}^1 (1-w)^{-1/2} w^{-2+3/2q} dw. \end{aligned} \tag{2.29}$$

The proof of Lemma 2.4 is a consequence from the above estimative. □

LEMMA 2.5. *Let $\mathbf{a} \in L^3_\sigma(\mathbb{R}^3)$, $b \in L^3(\mathbb{R}^3)$ and $|\mathbf{f}|_q \leq C_0 t^{-1+3/2q} (|\mathbf{a}|_3 + |b|_3)$, $|\nabla \mathbf{f}|_q \leq C_0 t^{-3/2+3/2q} (|\mathbf{a}|_3 + |b|_3)$. If $M(|\mathbf{a}|_3 + |b|_3) \leq 1$ for some constant M , then for $3 \leq q \leq \infty$, the following estimative are verified uniformly in k :*

$$t^{3/2-3/2q} |\nabla \pi^k|_q \leq C (|\mathbf{a}|_3 + |b|_3)^2, \tag{2.30}$$

$$t^{3/2-3/2q} \left(\left| \frac{\partial \mathbf{u}^k}{\partial t} \right|_q + \left| \frac{\partial \theta^k}{\partial t} \right|_q \right) \leq C (|\mathbf{a}|_3 + |b|_3) + C (|\mathbf{a}|_3 + |b|_3)^2. \tag{2.31}$$

Proof. The proof is a consequence of Lemmas 2.2, 2.3, and 2.4, and the following facts

$$\begin{aligned} \frac{\partial \mathbf{u}^k}{\partial t} &= \nu \Delta \mathbf{u}^k - (\mathbf{u}^{k-1} \cdot \nabla) \mathbf{u}^{k-1} - \nabla \pi^k + \theta^{k-1} \mathbf{f}, \\ \frac{\partial \theta^k}{\partial t} &= \chi \Delta \theta^k - \mathbf{u}^{k-1} \cdot \nabla \theta^{k-1}. \end{aligned} \tag{2.32}$$

□

The main result in this paper is the following.

THEOREM 2.6. *Let $\mathbf{a} \in L^3_\sigma(\mathbb{R}^3)$, $b \in L^3(\mathbb{R}^3)$ and $|\mathbf{f}|_q \leq C_0 t^{-1+3/2q} (|\mathbf{a}|_3 + |b|_3)$, $|\nabla \mathbf{f}|_q \leq C_0 t^{-3/2+3/2q} (|\mathbf{a}|_3 + |b|_3)$. Then, there exists a positive constant ε such that, if $(|\mathbf{a}|_3 + |b|_3) \leq \varepsilon$, there exists a unique solution (\mathbf{u}, θ) for (1.1), which satisfy:*

$$\begin{aligned} t^{1/2-3/2q} \mathbf{u} &\in BC([0, \infty); L^q(\mathbb{R}^3)), \\ t^{1/2-3/2q} \theta &\in BC([0, \infty); L^q(\mathbb{R}^3)), \\ t^{1-3/2q} |\nabla \mathbf{u}| &\in BC([0, \infty); L^q(\mathbb{R}^3)), \\ t^{1/2-3/2q} |\nabla \theta| &\in BC([0, \infty); L^q(\mathbb{R}^3)), \end{aligned} \tag{2.33}$$

for $3 \leq q \leq \infty$ and moreover

$$\begin{aligned}
 t^{3/2-3/2q} \sum_{i,j=1}^3 \left| \frac{\partial^2 \mathbf{u}}{\partial x_j \partial x_i} \right| &\in BC([0, \infty); L^q(\mathbb{R}^3)), \\
 t^{3/2-3/2q} \sum_{i,j=1}^3 \left| \frac{\partial^2 \theta}{\partial x_j \partial x_i} \right| &\in BC([0, \infty); L^q(\mathbb{R}^3)), \\
 t^{3/2-3/2q} |\nabla \pi| &\in BC([0, \infty); L^q(\mathbb{R}^3)), \\
 t^{3/2-3/2q} \frac{\partial \mathbf{u}}{\partial t} &\in BC([0, \infty); L^q(\mathbb{R}^3)),
 \end{aligned} \tag{2.34}$$

for $3 \leq q \leq \infty$.

Proof. Using Lemma 2.3 with $q = 3$, we obtain

$$\begin{aligned}
 |\mathbf{u}^k|_3 + |\theta^k|_3 &\leq C(|\mathbf{a}|_3 + |b|_3), \\
 |\nabla \mathbf{u}^k|_3 + |\nabla \theta^k|_3 &\leq Ct^{-1/2}(|\mathbf{a}|_3 + |b|_3)
 \end{aligned} \tag{2.35}$$

then, for $1 < p < 2$ it is easy to show

$$\begin{aligned}
 \mathbf{u}^k &\in L^\infty(0, \infty; L^3(\mathbb{R}^3)), \\
 \theta^k &\in L^\infty(0, \infty; L^3(\mathbb{R}^3)), \\
 \nabla \mathbf{u}^k &\in L^p_{loc}(0, \infty; L^3(\mathbb{R}^3)), \\
 \nabla \theta^k &\in L^p_{loc}(0, \infty; L^3(\mathbb{R}^3)).
 \end{aligned} \tag{2.36}$$

Let q and q^* such that $1/q + 1/q^* = 1$ and we consider the following estimative:

$$\begin{aligned}
 \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle \Delta \mathbf{u}^k, \mathbf{v} \rangle| &\leq \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\nabla \mathbf{u}^k|_q |\nabla \mathbf{v}|_{q^*} \leq |\nabla \mathbf{u}^k|_q, \\
 \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle (\mathbf{u}^{k-1} \cdot \nabla) \mathbf{u}^{k-1}, \mathbf{v} \rangle| &\leq \left| |\mathbf{u}^{k-1}|^2 \right|_q, \\
 \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle \nabla \pi^k, \mathbf{v} \rangle| &\leq |\pi^k|_q.
 \end{aligned} \tag{2.37}$$

Now, using Hölder and Sobolev inequalities, we have

$$\begin{aligned}
 \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\langle \theta^{k-1} \mathbf{f}, \mathbf{v} \rangle| &\leq C \sup_{|\mathbf{v}|_{W_0^{1,q^*}}=1} |\theta^{k-1} \mathbf{f}|_{3q/(q+3)} |\nabla \mathbf{v}|_{q^*} \\
 &= C |\theta^{k-1} \mathbf{f}|_{3q/(q+3)}.
 \end{aligned} \tag{2.38}$$

Thus, by using Lemma 2.4 together with Sobolev and Hölder inequalities, we obtain

$$\left| \frac{\partial \mathbf{u}^k}{\partial t} \right|_{W^{-1,q}} \leq |\nabla \mathbf{u}^k|_q + C |\mathbf{u}^{k-1}|_3 |\nabla \mathbf{u}^{k-1}|_q + C |\theta^{k-1}|_3 |\mathbf{f}|_q, \tag{2.39}$$

consequently,

$$\left| \frac{\partial \mathbf{u}^k}{\partial t} \right|_{W^{-1,q}} \leq C t^{-1+3/2q}. \quad (2.40)$$

Analogously the following inequality can be proved for the temperature θ^k

$$\left| \frac{\partial \theta^k}{\partial t} \right|_{W^{-1,q}(\mathbb{R}^3)} \leq C t^{-1+3/2q}. \quad (2.41)$$

Therefore, for $1 < r < 2q/(2q - 3)$, we have

$$\begin{aligned} \frac{\partial \mathbf{u}^k}{\partial t} &\in L^r_{\text{loc}}(0, \infty; W^{-1,q}(\mathbb{R}^3)), \\ \frac{\partial \theta^k}{\partial t} &\in L^r_{\text{loc}}(0, \infty; W^{-1,q}(\mathbb{R}^3)). \end{aligned} \quad (2.42)$$

By using the compact embedding of $W^{1,3}(\mathbb{R}^3)$ on $L^3_{\text{loc}}(\mathbb{R}^3)$ and the Compactness Theorem in [11, Cap. 3], we obtain that there exists (\mathbf{u}, θ) such that

$$\begin{aligned} \mathbf{u}^k &\longrightarrow \mathbf{u} \quad \text{in } L^2_{\text{loc}}(0, \infty; L^3_{\text{loc}}(\mathbb{R}^3)) \text{ strongly,} \\ \theta^k &\longrightarrow \theta \quad \text{in } L^2_{\text{loc}}(0, \infty; L^3_{\text{loc}}(\mathbb{R}^3)) \text{ strongly.} \end{aligned} \quad (2.43)$$

Now, using the standard arguments, it is easily to show that (\mathbf{u}, θ) is a unique solution of (1.1) (see [5]). \square

Acknowledgments

The authors has been partially supported by D.G.E.S. and M.C. y T. (Spain), Project BFM2003-06446. M.A. Rojas-Medar is partially supported by CNPq-Brazil, Grant 301354/03-0.

References

- [1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, New York, 1975.
- [2] J. R. Cannon and E. DiBenedetto, *The initial value problem for the Boussinesq equations with data in L^p* , Approximation Methods for Navier-Stokes Problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979), Lecture Notes in Math., vol. 771, Springer, Berlin, 1980, pp. 129–144.
- [3] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier–Stokes system*, J. Differential Equations **62** (1986), no. 2, 186–212.
- [4] C. He and L. Hsiao, *The decay rates of strong solutions for Navier–Stokes equations*, J. Math. Anal. Appl. **268** (2002), no. 2, 417–425.
- [5] T. Hishida, *Existence and regularizing properties of solutions for the nonstationary convection problem*, Funkcial. Ekvac. **34** (1991), no. 3, 449–474.
- [6] D. D. Joseph, *Stability of Fluid Motions. Vol. I*, Springer Tracts in Natural Philosophy, vol. 27, Springer, Berlin, 1976.
- [7] ———, *Stability of Fluid Motions. Vol. II*, Springer Tracts in Natural Philosophy, vol. 28, Springer, Berlin, 1976.

- [8] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Mathematics and Its Applications, vol. 2, Gordon and Breach, New York, 1969.
- [9] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris, 1969.
- [10] M. A. Rojas-Medar and S. A. Lorca, *The equations of a viscous incompressible chemical active fluid. I. Uniqueness and existence of the local solutions*, Rev. Mat. Apl. **16** (1995), no. 2, 57–80.
- [11] ———, *The equations of a viscous incompressible chemical active fluid. II. Regularity of solutions*, Rev. Mat. Apl. **16** (1995), no. 2, 81–95.
- [12] R. Temam, *Navier-Stokes Equations*, Revised ed., Studies in Mathematics and Its Applications, vol. 2, North-Holland, Amsterdam, 1979.

Francisco Guillén González: Universidad de Sevilla, Facultad de Matemáticas, 41012 Sevilla, Spain
E-mail address: guillen@us.es

Márcio Santos da Rocha: DM-CCE-UDEL, Londrina-PR, Brazil
E-mail address: marcio@uel.br

Marko Rojas Medar: DMA-IMECC-UNICAMP, CP 6065, 13083-970, Campinas-SP, Brazil
E-mail address: marko@ime.unicamp.br