THEOREM ON THE UNION OF TWO TOPOLOGICALLY FLAT CELLS OF CODIMENSION 1 IN $\mathbb{R}^n$

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In this paper we give a complete and improved proof of the "Theorem on the union of two $(n-1)$-cells." First time it was proved by the author in the form of reduction to the earlier author's technique. Then the same reduction by the same method was carried out by Kirby. The proof presented here gives a more clear reduction. We also present here the exposition of this technique in application to the given task. Besides, we use a modification of the method, connected with cyclic ramified coverings, that allows us to bypass referring to the engulfing lemma as well as to other multidimensional results, and so the theorem is proved also for spaces of any dimension. Thus, our exposition is complete and does not require references to other works for the needed technique.

1. Notations and statement of the result

Specify the standard coordinate system $Ox^1\cdots x^n$ with the origin O and coordinate axes Ox^i in the space $\mathbb{R}^n$. The coordinate planes $Ox^1\cdots x^i$ will be identified with $\mathbb{R}^i$. The unit disc in $\mathbb{R}^i$ is denoted by B^i. The semispaces $x^{n-1}\geq 0$ and $x^{n-1}\leq 0$ are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, respectively, while semiplanes $\mathbb{R}^n_+\cap\mathbb{R}^{n-1}$ and $\mathbb{R}^n_-\cap\mathbb{R}^{n-1}$ as $\mathbb{R}^{n-1}_+$ and $\mathbb{R}^{n-1}_-$, respectively. Semidiscs $B^{n-1}\cap\mathbb{R}^{n-1}_+$ and $B^{n-1}\cap\mathbb{R}^{n-1}_-$ are denoted by B^{n-1}_+ and B^{n-1}_-, respectively.

We will say that an embedding $q: B^i\to M^n$ of an i-disc in a topological n-manifold without boundary is *topologically flat* if one can extend it to an embedding in M^n of its neighborhood in $\mathbb{R}^n$. It is known that a topologically flat embedding of a disc into $\mathbb{R}^n$ is extendable to a homeomorphism of $\mathbb{R}^n$ onto itself. An embedding of a submanifold is *locally flat* if every point has a neighborhood in it that is homeomorphic to a disc and the embedding on this disc is topologically flat. Any locally flat embedding of a disc is topologically flat (see, e.g., [8]).

Theorem 1.1. *Let an embedding $q: B^{n-1}\to\mathbb{R}^n$ be given, whose restrictions to both semidiscs B^{n-1}_+ and B^{n-1}_- are topologically flat. Then q is topologically flat.*

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We will denote the restriction of q onto $B^{n-1}_{\pm}$ as $q_{\pm}$, respectively.

Notice an important corollary to this theorem (first time mentioned by Cantrell [1]) that in the case $n \geq 4$ for embedding of an $(n-1)$-manifold into an n-manifold there are no isolated points where the condition of locally flatness is destroyed. If $n = 3$, it is not the case. The reason for this difference is the fact that for $n \geq 4$ an isolated singularity cannot exist on the boundary of an $(n-1)$-submanifold, and this is derived from the fact that in the former dimensions the arcs with only one singularity do not exist, (see [5, 6]) while it is well known that in the dimension 3 they do exist.

The proof of this theorem is based on a series of lemmas using the constructions of some elementary homeomorphisms described in Section 2. Here we introduce some notations.

Denote by Π_α the semiplane, bounded by subspace R^{n-2} and having the angle of α radians with $\mathbb{R}^{n-1}_+ = \Pi_0$. ($\Pi_{-\pi} = \Pi_\pi = \mathbb{R}^{n-1}_-$.) $Q[\alpha,\beta]$, $\alpha < \beta$ will denote a closed domain between Π_α and Π_β ($Q[\alpha,\beta] = \cup_{\alpha\leq\gamma\leq\beta}\Pi_\gamma$), $Q(\alpha,\beta)$ denotes the interior of $Q[\alpha,\beta]$.

For a point $z \in \mathbb{R}^n$ we denote by x_z its projection onto $\mathbb{R}^{n-2}$ and by y_z its projection onto $\mathbb{R}^{n-1}$.

Consider the system of 2-planes P_x, $x \in \mathbb{R}^{n-2}$ orthogonal to $\mathbb{R}^{n-2}$ at the corresponding points x. Consider also in every plane P_x an orthonormal coordinate system with the origin x and axes xs and xt, the former is parallel and codirected with the axe Ox^{n-1}, and the latter is parallel and codirected with the axe Ox^n, s and t have the meaning of coordinate parameters. For a point $z \in \mathbb{R}^n$ we denote by s_z and t_z its coordinates in the plane P_{x_z}. At last, $C_x(r)$ will denote a circle with the radius r in the plane P_x centred at x. For a point $z \in \mathbb{R}^n$ we denote by r_z its distance from $\mathbb{R}^{n-2}$, that is, the radius r of a circle $C_{x_z}(r)$ passing through z.

2. Preliminary statements

The following two statements will help us to construct some elementary homeomorphisms of $\mathbb{R}^n$ that send every circle $C_x(r)$ onto itself piecewise linearly.

Statement 2.1. Let for some α a closed subset $M \subset \mathbb{R}^n$ be given such that in some neighborhood of B^{n-2} it does not intersect $\Pi_\alpha \setminus \mathbb{R}^{n-2}$ and lies on one side of Π_α (i.e., in $Q(\alpha,\alpha+\pi)$ or in $Q(\alpha-\pi,\alpha)$).

Then there exists a function $\varepsilon(z) > 0$, $z \in \Pi_\alpha \setminus \mathbb{R}^{n-2}$ (possibly in a smaller neighborhood of B^{n-2}), that is continuous, tending to zero as z is tending to a point in $\mathbb{R}^{n-2}$, and such that for any circle $C_x(r)$ in this neighborhood its arc with the length $\varepsilon(z)$, having an end in $z \in \Pi_\alpha$ and lying on one side of Π_α as M, does not intersect M.

The construction of $\varepsilon(z)$ is standard and evident, so that it may be omitted.

For $z \in \Pi_\alpha$ consider arcs of the circles $C_{x_z}(r_z)$ having one end at z and the length $\varepsilon(z)$, where this function is chosen according to Statement 2.1 for some set M, the arcs are taken on one side of Π_α, as M. The surface, described by the second ends of these arcs (i.e., not on Π_α) will be called the *fence* separating M from Π_α.

Note that every circle C_x (sufficiently close to B^{n-2}) intersects every Π_α and every fence exactly one time.

Statement 2.2. Let four sets A, B, C, D in a neighborhood of B^{n-2} be given so that each of them is either a Π_α or a fence, and the points of intersections of B and C with any circle $C_x(r)$ in this neighborhood are located between points of intersection of this circle with the sets A and D. Then there exists a homeomorphism of $\mathbb{R}^n$, identical outside a neighborhood of B^{n-2} and outside the domain between A and D (containing B and C), that sends B into C in a smaller neighborhood.

For the proof it is sufficient to construct a homeomorphism on every circle $C_x(r)$ in a small neighborhood of B^{n-2} that maps linearly the arc between A and B into the arc between A and C and simultaneously the arc between B and D into the arc between C and D, such that it is identical on the second arc between A and D. In some larger neighborhood one can continuously reduce this homeomorphism to the identity.

The homeomorphisms constructed as in this proof will be called *arcwise*. Note that arcwise homeomorphisms are naturally isotopic to the identity.

Before turning to our lemmas, let us introduce the following definitions.

Defintions 2.3. An embedding $\gamma : \Pi_\alpha \to \mathbb{R}^n$, being identity on $\mathbb{R}^{n-2}$, touches Π_β at points of B^{n-2} if for every $\varepsilon > 0$ one can find $\delta > 0$ so that $\gamma\Pi_\alpha \cap O_\delta(B^{n-2}) \subset Q[\beta - \varepsilon, \beta + \varepsilon]$.

Analogously, a sequence of points $z_n \in \mathbb{R}^n$ touches Π_α at a point $x \in B^{n-2}$ if for every $\varepsilon > 0$ there exists n_ε such that $z_n \in Q[\beta - \varepsilon, \beta + \varepsilon] \cap O_\varepsilon(x)$ for all $n > n_\varepsilon$.

3. Lemmas

Lemma 3.1. *Let an embedding $p_1 : B^n \setminus (B_-^{n-1} \setminus B^{n-2}) \to \mathbb{R}^n$ be identical on B^{n-2}, and for every $\alpha \in [-\pi + \pi/4, \pi - \pi/4]$ the set $p_1(\Pi_\alpha)$ touches Π_α. Let also $p_1 B_+^{n-1} \subset \mathbb{R}_+^n$.*

Then the cell $B_-^{n-1} \cup p_1 B_+^{n-1}$ is embedded topologically flat, that is, there is a homeomorphism $\bar{p}_1$ of $\mathbb{R}^n$ that maps B^{n-1} onto $B_-^{n-1} \cup p_1 B_+^{n-1}$. (The tangency of Π_α for $\alpha \in Q(-\pi/2, +\pi/2)$ is not essential and has only a technical role.)

Proof. First we will construct a mapping $w : \mathbb{R}^n \to \mathbb{R}^n$, that orthogonally projects B_-^{n-1} onto B^{n-2}, is homeomorphic outside B_-^{n-1}, and is identical on $\mathbb{R}_+^n$. Under the given conditions it is clear that the composition $w^{-1}\pi w$ coincides with p_1 on B_+^{n-1}. At the same time it occurs that this composition can be extended identically on B_-^{n-1}. The obtained extension is the homeomorphism $\bar{p}_1$ we are looking for.

The beginning of this construction of w is determined by the requirements that $w = 1$ on $\mathbb{R}_+^n$ and $w(y) = x_y$ for $y \in B_-^{n-1}$. Extend w identically to the points $y \in \mathbb{R}_-^{n-1}$ whose projections x_y onto $\mathbb{R}^{n-2}$ are situated outside B^{n-2}. If $x_y \in B^{n-2}$ and $y \in \mathbb{R}_-^{n-1} \setminus B_-^{n-1}$, we take as $w(y)$ the point that is obtained from y by the shift along the direction of the axe Ox^{n-1} in the distance equal to the intersection segment of B_-^{n-1} with the axe $x_y s$ in P_{x_y}. Thus we have constructed w on the space $\mathbb{R}^{n-1}$.

For every point $y \in \mathbb{R}_-^{n-1}$, we denote by L_y the straight line going through y and being parallel to the axe Ox^n. If x_y lies outside B^{n-2}, set $w = 1$ on L_y.

Let $y \in B^{n-1}$. Define that w sends L_y isometrically into the union of two rays in P_{x_y} starting at the point $x_y \in B^{n-2}$ with the angle $\alpha = \pm(\pi/2 - \pi/4 \cdot s_y)$ with respect to the axe $x_y s$ ($s_y < 0$ is the coordinate of y in P_{x_y}).

Notice that $\alpha \to \pi/2$, when $s_y \to 0$, that is, $y \to x_y$.

If $x_y \in B^{n-2}$ and $y \in \mathbb{R}^{n-1}_- \setminus B^{n-1}$, then w sends L_y isomorphically to the pair of rays in P_{x_y} starting at the point $w(y)$ with the angles $\alpha = \pm(\pi/2 - \pi/4 \cdot s_{y'})$, where y' is an intersection of the half-axe $x_y s \cap \mathbb{R}^n_-$ with the boundary of B^{n-1}.

Now w is well posed on the entire $\mathbb{R}^n$; it is continuous and identical on $\mathbb{R}^n_+$ and outside $\cup_{x \in B^{n-2}} P_x$. Also w retracts B^{n-1}_- onto B^{n-2} by the orthogonal projection and it is homeomorphic outside B^{n-1}_-.

It remains to note that a sequence of points z_n tends to a point $y \in B^{n-1}_-$ if and only if $w(y_n)$ tends to x_y, touching $\Pi_\alpha \cup \Pi_{-\alpha}$, where α is chosen according to the point y as above, that is, $\alpha_y = \pm(\pi/2 - \pi/4 \cdot s_y)$.

Indeed, take a spherical neighborhood V_ε with radius $\varepsilon > 0$ of a point x_y in the plane $x^{n-1} = 0$ and consider the set W_ε of points $z \in \mathbb{R}^n_-$ that are projected to V_ε. Let $U_\varepsilon(y)$ be the intersection of W_ε with the domain between two planes, being parallel to $x^{n-1} = 0$ and located on different sides of y in the distance ε. Let $U'_\varepsilon(x_y)$ be the intersection of W_ε with $Q(\alpha_y - \pi/2 \cdot \varepsilon, \alpha_y + \pi/2 \cdot \varepsilon) \cup Q(-\alpha_y - \pi/2 \cdot \varepsilon, -\alpha_y + \pi/2 \cdot \varepsilon)$. Then for every $\varepsilon' > 0$ one can find a $\varepsilon > 0$ such that $w(U_\varepsilon(y)) \subset U'_{\varepsilon'}$, and, conversely, for every $\varepsilon > 0$ one can find a ε' such that $w(U_\varepsilon(y)) \supset U'_{\varepsilon'}(x_y)$. Hence the sequence of points $z_n \in \mathbb{R}^n$ tends to $y \in Int\, B^{n-1}_-$ if and only if $w(z_n)$ tends to x_y and touches $\Pi_{\alpha_y} \cup \Pi_{-\alpha_y}$.

A sequence z_n tends to $y \in \partial B^{n-1}_- \setminus B^{n-2}$ if and only if $wz_n \to x_y$ and for every $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ all z_n are located outside $Q(-\pi/2 - \pi/4 + \varepsilon, +\pi/2 + \pi/4 - \varepsilon)$. It is clear that the same property is fulfilled for the sequence $hw(z_n)$.

This proves that the homeomorphism $\widetilde{p}_1 = w^{-1} p_1 w$ is extended identically to B^{n-1}_-, as what was in demand. The constructed homeomorphism $\widetilde{p}_1$ coincides with the given p_1 on B^{n-1}_+ and is identical on B^{n-1}_+. Thus, the union of cells $B^{n-1}_- \cup p_1 B^{n-1}_+ = \bar{p}_1 B^n$ is embedded locally flat at least at the points of $B^{n-1} \setminus \partial B^{n-2}$. But then one can easily construct a homeomorphism of the whole space that sends $B^{n-1}_- \cup p_1 B^{n-1}_+$ into B^{n-1}_-. It is a standard construction (see [3]), which we leave as an exercise. So, the embedding of $B^{n-1}_- \cup p_1 B^{n-1}_+$ is topologically flat. □

Lemma 3.2. *The assertion of Lemma 3.1 is true for the embedding $p_2 : Q[-\pi/2, \pi/2] \to Q[-\pi/2, \pi/2]$, for which $p_2 \Pi_\alpha$ touches Π_α with $\alpha \in (-\pi/2, \pi/2)$, $p_2 B^{n-1}_+ \subset Q(-\pi/4, \pi/4)$.*

Proof. Construct the arc homeomorphism $\rho : Q[-\pi, \pi] \to Q[-\pi/2, \pi/2]$, identical on $Q[-\pi/4, \pi/4]$, that sends linearly the arc of each circle $C_x(r)$ between the points of its intersections with Π_π and $\Pi_{\pi/4}$ to the arc between its intersections with $\Pi_{\pi/2}$, $\Pi_{\pi/4}$, and, analogously, sends the arc between $\Pi_{-\pi}$ and $\Pi_{-\pi/4}$ to the arc between $\Pi_{\pi/2}$ and $\Pi_{-\pi/4}$. It is clear that touching Π_α is transformed into touching $\rho \Pi_\alpha$. Then, the hypothesis of Lemma 3.1 is satisfied for $\rho^{-1} p_2 \rho$ that coincides with p_2 on B^{n-2}_+. Thus, the embedding of the cell $B^{n-1}_- \cup p_2 B^{n-2}_+$ is topologically flat. □

Lemma 3.3. *Let an embedding $p_3 : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be given in some neighborhood of B^{n-2}, where the images of $\Pi_{-\pi/2}$, $\Pi_{\pi/2}$, and two more semiplanes Π_α, $-\pi/2 < \alpha < \pi/2$ (let them be for definiteness $\Pi_{-\pi/4}$ and Π_0) touch their preimages: $\Pi_{-\pi/2}, \Pi_{\pi/2}, \Pi_{-\pi/4}, \Pi_0$ at the points of B^{n-2}.*

Then there exists an isotopy $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$, identical outside $p_3 \mathbb{R}^n_+$ and outside a given neighborhood of B^{n-2}, such that $\phi_0 = 1$ and $\bar{p}_3 \Pi_\alpha = \phi_1 p_3 \Pi_\alpha$ touches Π_α at the points of B^{n-2} for all $\alpha \in [-\pi/2, \pi/2]$. In particular, the condition of Lemma 3.2 for p_2 is fulfilled for $\bar{p}_3$.

Proof. At first note that if the hypothesis of touching is fulfilled for some dense set of $\alpha \in [-\pi/2, \pi/2]$, it will be fulfilled for all α from this interval. So, we will try to obtain its fulfillment for a countable dense set of values, namely for the set $\{\alpha = \pi d\}$, where d is a dyadic rational and $|d| \le 1/2$.

Enumerate these numbers into a sequence α_k (starting with the four given values of α), and begin constructing a countable sequence of ε_k-isotopies $\phi^{(k)}$, each of which is identical on the ε_k-neighborhood of B^{n-2} and achieves the touching condition for the homeomorphism $\phi^{(k)} \cdots \phi^{(1)} p_3$ at the points of B^{n-2} for every next α_k without loosing this property for the preceding α_i.

In this construction, independently from the preceding steps, one may take the numbers ε_k arbitrarily small. Hence the sequence of isotopies $\phi_t^{(k)} \cdots \phi_t^{(1)}$ will tend to an isotopy, and in the limit the touching condition will be fulfilled for all $\alpha \in [-\pi/2, \pi/2]$.

It is sufficient to describe one step of the construction, say, for values of α given in the lemma. Next steps are absolutely analogous.

Let us show how to obtain this touching condition for $\alpha = \pi/4$. The construction of the isotopy in demand takes several steps. □

Step 1. Note that by applying the arcwise homeomorphisms one can get that the images of Π_α touch Π_α on one side; for example, that $p_3\Pi_0$ touches Π_0 and lies in $Q(0, \pi/2)$, and $p_3\Pi_{-\pi/4}$ touches $\Pi_{-\pi/4}$ and lies in $Q(-\pi/4, 0)$. (Certainly, it is sufficient for each touching condition to be fullfilled in a small neighborhood of B^{n-2}.) Let us show this for $p_3\Pi_0$.

Note that $\Pi_{3\pi/8}$ and $\Pi_{\pi/4}$ lie in a small neighborhood of B^{n-2} between $p_3\Pi_{\pi/2}$ and $p_3\Pi_0$ and do not intersect them except for $\mathbb{R}^{n-2}$.

Construct an arcwise homeomorphism τ', identical outside $Q(0, 3\pi/8)$, that sends $\Pi_{\pi/4}$ to the fence S_0 touching Π_0. Replace p_3 by $\widetilde{p}_3 = \tau' p_3$. All the hypotheses of the lemma remain true, but now $\widetilde{p}_3\Pi_0$ lies between the fence S_0 and $\Pi_{-\pi/16}$ (in a small neighborhood of B^{n-2}), since $\widetilde{p}_3\Pi_0$ touches Π_0.

Construct now an arcwise homeomorphism τ'', identical outside the domain in $\mathbb{R}^n_+$ bounded by S_0 and $\Pi_{-\pi/8}$, that sends $\Pi_{\pi/16}$ to Π_0. The lemma's hypotheses remain true after replacing $\widetilde{p}_3$ by $p_3^\circ = \tau''\widetilde{p}_3$, but then $p_3^\circ\Pi_0$ lies between Π_0 and S_0, that is, on one side of Π_0.

Thus from the very beginning we may suppose that $p_3\Pi_0$ lies in $Q(0, \pi/4)$ and touches Π_0 as well as, analogously, that $p_3\Pi_{-\pi/4} \subset Q(-\pi/4, 0)$ and touches $\Pi_{-\pi/2}$. Moreover, $p_3\Pi_{\pi/2} \subset Q(\pi/4, \pi/2)$ and touches $\Pi_{\pi/2}$ as well as $p_3\Pi_{-\pi/2} \subset Q(-\pi/2, -\pi/4)$ and touches $\Pi_{-\pi/2}$.

Step 2. Construct an arcwise homeomorphism τ_1, identical outside $Q(0, \pi/2)$, that sends $\Pi_{\pi/4}$ to the fence S_1 touching Π_0, closely to Π_0 so that $p_3S_1 \subset Q(0, \pi/4)$. Let $t_1 = p_3\tau_1 p_3^{-1}$. Then $t_1 = 1$ outside $p_3Q(0, \pi/2)$ and $t_1 p_3\Pi_{\pi/4} \subset Q(0, \pi/4)$. Let $p_3' = t_1 p_3$.

Step 3. Construct now a fence S_2 touching $\Pi_{-\pi/4}$ closely so that $p_3'S_2$ lies in $Q(-\pi/4, 0)$.

Let τ_2 be an arcwise homeomorphism identical outside $Q(-\pi/4, \pi/4)$ that sends Π_0 to S_2. Let $t_2 = p_3'\tau_2 p_3'^{-1}$. Then $t_2 = 1$ outside $p_3'Q(-\pi/4, \pi/4)$, $t_2 p_3'\Pi_0 \subset Q(-\pi/4, 0)$, and $t_2 p_3'\Pi_{\pi/4} \subset Q(0, \pi/4)$. Let $p_3'' = t_2 p_3'$.

Step 4. Construct a fence S_2 that touches Π_0 and separates Π_0 from $p_3''\Pi_{\pi/4}$. Construct also another fence S_2' touching $\Pi_{\pi/4}$ and lying in $Q(0,\pi/4)$. Let τ_3 be an arcwise homeomorphism, identical outside of $Q(0,\pi/4)$, that sends S_2 to S_2'. Then $p_3''' = \tau_3 p_3''\Pi_{\pi/4}$ lies in the domain between $\Pi_{\pi/4}$ and S_2', that is, it touches $\Pi_{\pi/4}$ on one side.

Step 5. Consider now the homeomorphism $\bar{p}_3 = t_2^{-1}\pi_3''' = t_2^{-1}\tau_3 t_2 t_1 p_3$. It is clear that $\bar{p}_3 = p_3$ on $\Pi_{-\pi/2} \cup \Pi_{\pi/2} \cup \Pi_{-\pi/4} \cup \Pi_0$ and that $\bar{p}_3\Pi_{\pi/4}$ touches $\Pi_{\pi/4}$ on one side. Since $\bar{p}_3 \cdot p_3^{-1}$ is identical on $\Pi_{-\pi/2} \cup \Pi_{\pi/2}$, there is an isotopy ϕ_t, identical outside $Q(-\pi/2,\pi/2)$ and such that $\phi_0 = 1$, $\phi_1 = \bar{p}_3 p_3^{-1}$, that is, $\bar{p}_3 = \phi_1 p_3$.

It is clear that one can make this isotopy ε-small and identical outside the ε-neighborhood of B^{n-2} for any given ε. All conditions of the lemma are satisfied.

Lemma 3.4. *Let a homeomorphism $p_4 : \mathbb{R}^n \to \mathbb{R}^n$ be identical on $\mathbb{R}^{n-2}$ such that $p_4 Q_k \subset Q_k$, $0 \le k \le 3$ where $Q_0 = Q[-\pi/8,\pi/8]$ and Q_i, $1 \le i \le 3$ are obtained from Q_0 by consecutive turns by 90° counter-clockwise (from x^{n-1} to x^n).*

Then for every $\varepsilon > 0$ there exists a ε-isotopy $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$, identical on $\mathbb{R}^{n-2}$ and outside a given neighborhood of B^{n-2}, and such that in a smaller neighborhood of B^{n-2} the homeomorphism $\bar{p}_4 = \psi_1 p_4$, restricted to $\mathbb{R}^n_+$, fulfills the conditions of Lemma 3.3 for p_3.

The proof of this lemma follows form an evident construction with arcwise homeomorphisms.

First of all, one may assume, as in the proof of the preceding lemma, that in a neighborhood of B^{n-2} the images of Π_{α_k} for $\alpha_k = k\pi/2$, $0 \le k \le 3$ touch Π_{α_k} at points of B^{n-2} and on the wishful side of Π_{α_k}.

Indeed, let $\Pi_{\pm\pi/4}$ and $\Pi_{\pm 3\pi/4}$ remain immovable. Move the semiplanes $\Pi_{+\pi/8}$ and $\Pi_{-\pi/8}$ by an arcwise homeomorphism σ_0, that is identical outside $Q[-\pi/4,\pi/4]$, to Π_0 and to a fence S_0 that touches Π_0, respectively.

By the same way, let an arcwise homeomorphism σ_1, identical outside $Q[\pi/4,3\pi/4]$, move the semiplane $\Pi_{\pi/2+\pi/8}$ to $\Pi_{\pi/2}$ and $\Pi_{\pi/2-\pi/8}$ to a fence S_1 touching $\Pi_{\pi/2}$. Let σ_2 be an arcwise homeomorphism, identical outside $Q[3\pi/4,5\pi/4]$, that sends $\Pi_{-\pi-\pi/8}$ to $\Pi_{-\pi}$ and $\Pi_{-\pi+\pi/8}$ to a fence S_2 touching $\Pi_{-\pi}$.

At last, let σ_3 be an arcwise homeomorphism that is identical outside $Q[-3\pi/4,-\pi/4]$ and sends $\Pi_{-\pi/2-\pi/8}$ and $\Pi_{-\pi/2+\pi/8}$ to $\Pi_{-\pi/2}$ and to a fence S_3 touching $\Pi_{-\pi/2}$, respectively.

Let σ be a homeomorphism that in each fourth-space, limited by semiplanes $\Pi_{\pm\pi/4}$ and $\Pi_{\pm 3\pi/4}$, coincides with the corresponding σ_k, $0 \le k \le 3$. Construct an arcwise homeomorphism $\tau : \mathbb{R}^n \to \mathbb{R}^n$ that sends the domain between Π_π and S_2 to the domain between $\Pi_{-\pi/2}$ and S_3 as well as the domain between $\Pi_{-\pi/2}$ and S_3 to the domain between $\Pi_{-\pi/4}$ and a fence touching $\Pi_{-\pi/4}$. Also, let $\tau = 1$ in $Q[-\pi/8,\pi-\pi/8]$ and on $\Pi_{\pm\pi/4} \cup \Pi_{\pm 3\pi/4}$.

It is clear that $\bar{p}_4 = \tau\sigma p_4$ satisfies the hypothesis of Lemma 3.3 for p_3. Namely, $\bar{p}_4\mathbb{R}^n_+ \subset \mathbb{R}^n_+$, $\bar{p}_4\Pi_{\pm\pi/2}$ touches $\Pi_{\pm\pi/2}$, $\bar{p}_4\Pi_0$ touches Π_0, and $\bar{p}_4\Pi_{-\pi/4}$ touches $\Pi_{-\pi/4}$; also, $\bar{p}_4$ is isotopic to p_4 by a small isotopy, since an arcwise homeomorphism is isotopic to the identity and its mesh does not supersede diameters of circles $C_x(r)$ and of their images on which it is not identical.

It should be pointed out that the homeomorphism τ, constructed above, is identical on $\Pi_{\pi/2+\pi/4}$.

Lemma 3.5. *Assume that there exists a homeomorphism* $p_5 : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $p_5 Q[-\pi/8, \pi/8] \subset Q(-\pi/8, \pi/8)$.

There is an isotopy $\chi_t : \mathbb{R}^n \to \mathbb{R}^n$, *identical on* $\mathbb{R}^{n-2}$ *and outside a neighborhood of* B^{n-2}, *such that* $\chi_0 = 1$ *and* $\chi_1 p_5 = \bar{p}_5$ *satisfies the hypothesis of Lemma 3.4 for* p_4 *and coincides with* p_5 *on* $B_-^{n-1} \cup p_5 B_+^{n-1}$.

Proof. Consider the 4-sheeted covering $\nu : \mathbb{R}_2^n \to \mathbb{R}_1^n$, branched over the subspace $\mathbb{R}_1^{n-2} \subset \mathbb{R}_1^n$. (It is convenient to indicate distinction between the same objects in the domain and in the image spaces of the covering ν by means of lower indices.) Denote by $j : \mathbb{R}_2^n \to \mathbb{R}_1^n$ the natural identification of both spaces. Let us concretize the covering by identifying every plane $P_{kx} \subset \mathbb{R}_i^n$, where $x \in \mathbb{R}_k^{n-2}$, $k = 1, 2$, with the complex line $\mathbb{C}^1$ (xs is the real and xt is the imaginary axes), and by representing ν as the function $z \mapsto e^{\mathbf{i}\cdot 3\varphi_z} z$, where $z = \rho_z e^{\mathbf{i}\varphi_z}$. Here $j = \nu$ on $\mathbb{R}_{2+}^{n-1}$.

According to the hypothesis of Lemma 3.4 the homeomorphism $p_5 : \mathbb{R}_1^n \to \mathbb{R}_1^n$ is given. Consider the homeomorphism $\widetilde{p}_5 : \mathbb{R}_2^n \to \mathbb{R}_2^n$, covering p_5 ($\nu\widetilde{p}_5 = p_5\nu$). We have $\nu\widetilde{p}_5 = p_5 j$ on $\mathbb{R}_{2+}^{n-1}$.

Construct now a homeomorphism $\beta : \mathbb{R}_2^n \to \mathbb{R}_2^n$, patching up the covering p so that $\nu\beta = j$ on $Q_2[-\pi/8, \pi/8]$. Namely, β is an arcwise homeomorphism, identical on $Q_2[-\pi/4, \pi/4]$, that sends $Q_2[-\pi/8, \pi/8]$ into $Q_2[-\pi/32, \pi/32]$. (One may analogously redefine β on other three quadrants, separated by planes $x^n = \pm x^{n-1}$, so that the mapping would remain a covering, but it is not important for us.)

As a result, $\nu\beta^{-1} = j$ on $Q_2[-\pi/8, \pi/8]$. The homeomorphism $\bar{p}_5 : \mathbb{R}_1^n \to \mathbb{R}_1^n$, defined by equality $\bar{p}_5 = j\beta^{-1}\widetilde{p}_5\beta j^{-1}$, coincides on $Q_1[-\pi/8, \pi/8]$ with $p_5 = \nu\beta\beta^{-1}\nu^{-1}p_5\nu\beta\beta^{-1}\nu^{-1}$. Moreover,

$$\bar{p}_5 Q_1[-k\pi/8, k\pi/8] \subset Q_1[-k\pi/8, k\pi/8], \quad 0 \le k \le 3. \tag{3.1}$$

So, $\bar{p}_5$ satisfies the hypothesis of Lemma 3.4 for the homeomorphism p_4, as what is required.

Besides, $\bar{p}_5$ is isotopic to the homeomorphism p_5 under isotopy that is identical on $\mathbb{R}_1^{n-2}$, since $\bar{p}_5 p_5^{-1}$ is identical on $Q_1[-\pi/8, \pi/8]$. □

4. Proof of the theorem

Since the embedding q_- is topologically flat, it can be extended to a homeomorphism of $\mathbb{R}^n$ and so we can assume that q is identical on B_-^{n-1}. Construct two fences $S_{-\pi}$ and S_π on two different sides of B_-^{n-1}, that are touching $\mathbb{R}^{n-1}$ from above and from below and separating B_-^{n-1} from qB_+^{n-1}. Then move them by an arcwise homeomorphism τ, identical on B_-^{n-1}, onto $\Pi_{-\pi/8}$ and onto $\Pi_{\pi/8}$, respectively, and replace q with $\widetilde{q} = \tau q$. We obtain $\widetilde{q}B_+^{n-1} \subset Q(-\pi/8, \pi/8)$. Suppose that this is valid for q from the very beginning.

Since q_+ is topologically flat, it can be extended to a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ ($h|_{B_+^{n-1}} = q|_{B_+^{n-1}} \subset Q(-\pi/8, +\pi/8) \subset \mathbb{R}_+^n$). As $hB_+^{n-1} \cap B_-^{n-1} = B^{n-2}$, applying as well as above the arcwise homeomorphisms, identical on B_-^{n-1}, one can wangle $hQ[-\pi/2, \pi/2] \subset Q(-\pi/2, \pi/2)$.

Thus the assertion of our theorem now takes the following form.

Theorem 4.1. *Suppose that there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$, for which $hB_+^{n-1} \subset Q(-\pi/8, \pi/8)$. Then the cell $B_-^{n-1} \cup hB_+^{n-1}$ is topologically flat. More precisely: there exists a homeomorphism $\bar{h} : \mathbb{R}^n \to \mathbb{R}^n$, identical on B_-^{n-1}, that coincides with h on B_+^{n-1}.*

The proof of this statement follows from the above sequence of lemmas as follows:

First, having constructed an arcwise homeomorphism γ, identical on $\mathbb{R}_-^n$, that sends $\Pi_{-\pi/8}$ and $\Pi_{\pi/8}$, respectively, to fences S_- and S_+, touching Π_0 from below and from above and separating Π_0 from $h^{-1}\Pi_{-\pi/8}$ and $h^{-1}\Pi_{\pi/8}$, we may replace h by a homeomorphism $h\gamma$ that coincides with h on B_+^{n-1} and moves $Q[-\pi/8, \pi/8]$ into $Q(-\pi/8, \pi/8)$. Then we obtain a homeomorphism satisfying the hypotheses of Lemma 3.5 for p_5.

By Lemma 3.5 we obtain a homeomorphism that coincides with the given h on B_+^{n-1} and satisfies the hypotheses of Lemma 3.4 for the homeomorphism p_4. By Lemma 3.4 we can construct a homeomorphism $\bar{p}_4$ that satisfies the condition for p_3 from Lemma 3.3, is isotopic to p_4, and is identical on $\Pi_{3\pi/4}$ by its construction.

Denote by D the semiball in $\Pi_{3\pi/4}$, bounded by B^{n-2}, and by γ the arcwise homeomorphism, constructed in Lemma 3.4, that is identical on D. Evidently, the cell $B_-^{n-1} \cup hB_+^{n-1}$ is topologically flat if and only if the same is true for the cell $D \cup hB_+^{n-1}$, if and only if it is so for $D \cup \gamma hB_+^{n-1}$, and if and only if it is so for $B_-^{n-1} \cup \gamma hB_+^{n-1}$, because these cells are obtained one from another by the application of some (arcwise) homeomorphisms of the space.

So, it is sufficient to prove that the cell $B_-^{n-1} \cup p_3B_+^{n-1}$ is topologically flat, where p_3 is the embedding given in Lemma 3.3.

According to Lemma 3.3 we can replace p_3 by an embedding, isotopic to p_3 under the isotopy, identical on B_-^{n-1}, that satisfies the conditions of Lemma 3.2. This isotopy sends the cell $B_-^{n-1} \cup p_3B_+^{n-1}$ to the cell $B_-^{n-1} \cup \bar{p}_3B_+^{n-1} = B_-^{n-1} \cup p_2B_+^{n-1}$ and we have to prove that the latter is locally flat. But this is the assertion of Lemma 3.2.

The theorem follows.

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