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Research Article

Positive Solutions for Boundary Value Problem of Nonlinear Fractional Differential Equation

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We are concerned with the existence and nonexistence of positive solutions for the nonlinear fractional boundary value problem: $D_{0+}^{\alpha}u(t)+\lambda a(t)f(u(t))=0$, $0<t<1$, $u(0)=u^{'}(0)=u^{'}(1)=0$, where $2<\alpha<3$ is a real number and D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Our analysis relies on Krasnoselskiis fixed point theorem of cone preserving operators. An example is also given to illustrate the main results.

1. Introduction

One of the most frequently used tools for proving the existence of positive solutions to the integral equations and boundary value problems is Krasnoselskii theorem on cone expansion and compression and its norm-type version due to Guo [1]. In 1994, Wang [2] applied Krasnoselskii's work to eigenvalue problems to establish intervals of the parameter for which there is at least one positive solution. Since this pioneering work, a lot of research has been done in this area. Differential equations of fractional order, or fractional differential equations, in which an unknown function is contained under the operation of a derivative of fractional order, have been of great interest recently. Many papers and books on fractional calculus and fractional differential equations have appeared recently [3–8]. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations in terms of special functions. Recently, there are some papers which deal with the existence and multiplicity of solution (or positive solution) of nonlinear fractional differential equation by the use of techniques of nonlinear analysis. Bai and Lü [3] studied the existence and multiplicity of positive solutions of nonlinear fractional differential equation boundary value problem:

$$\begin{gathered} D_{0+}^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative. Zhang [7] considered the existence of solution of nonlinear fractional boundary value problems involving Caputo's derivative

$$\begin{gathered} D_t^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2, \\ u(0) = \nu \neq 0, \qquad u(1) = \rho \neq 0. \end{gathered} \tag{1.2}$$

In another paper, by using fixed point theorem on cones, Zhang [8] studied the existence and multiplicity of positive solutions of nonlinear fractional boundary value problem

$$\begin{gathered} D_t^{\alpha}u(t) = f(t,u(t)), \quad 0 < t < 1,\ 1 < \alpha \le 2, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(1) = 0, \end{gathered} \tag{1.3}$$

where D_t^{α} is the Caputo's fractional derivative.

The purpose of this paper is to establish the existence and nonexistence of positive solutions to nonlinear fractional boundary value problem

$$D_{0+}^{\alpha}u(t) + \lambda a(t) f(u(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha < 3, \tag{1.4}$$

$$u(0) = u'(0) = u'(1) = 0, \tag{1.5}$$

where λ is a positive parameter, $a : (0,1) \to [0,\infty)$ is continuous with $\int_0^1 a(t),\, dt > 0$, and $f : [0,\infty) \to [0,\infty)$ is continuous. Here, by a positive solution of the boundary value problem we mean a function which is positive on $(0,1)$ and satisfies differential equation (1.4) and the boundary condition (1.5). The paper has been organized as follows. In Section 2, we give basic definitions and provide some properties of certain Green's functions which are needed later. We also state Krasnoselskii's fixed point theorem for cone preserving operators. In Section 3, we establish some results for the existence and nonexistence of positive solutions to problem (1.4) and (1.5). In the end of this section, an example is also given to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we present here some notations and lemmas that will be used in the proof of our main results.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called **cone** of E if it satisfies the following conditions:

(1) $x \in K$, $\sigma \ge 0$ implies $\sigma x \in K$;

(2) $x \in K$, $-x \in K$ implies $x = 0$.

Definition 2.2. An operator is called **completely continuous** if it is continuous and maps bounded sets into precompact sets.

All results are based on the following fixed point theorem of cone expansion-compression type due to Krasnoselskii's. See, for example, [1, 9, 10].

THEOREM 2.3. *Let E be a Banach space and let $K \subset E$ be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be completely continuous operator. In addition, suppose that either*

(H1) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_2$ *or*

(H2) $\|Tu\| \leq \|u\|$, $\forall u \in K \cap \partial\Omega_2$ *and* $\|Tu\| \geq \|u\|$, $\forall u \in K \cap \partial\Omega_1$.

holds. Then T has a fixed pint in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Definition 2.4. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : [0,\infty) \to R$ is defined to be [4, 6]

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \quad n = [\alpha] + 1. \tag{2.1}$$

LEMMA 2.5 (see [3]). *Let $\alpha > 0$. If $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D^{\alpha}_{0+} u(t) = 0 \tag{2.2}$$

has solutions $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{N-1}$, $c_i \in \mathbb{R}$, $i = 0,1,\ldots,N$.

LEMMA 2.6 (see [3]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{N-1} \tag{2.3}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,\ldots,N$.

LEMMA 2.7. *Let $y \in C[0,1]$, then the boundary value problem*

$$D^{\alpha}_{0+} u(t) + y(t) = 0, \quad 0 < t < 1,\ 2 < \alpha < 3, \tag{2.4}$$

$$u(0) = u'(0) = u'(1) = 0 \tag{2.5}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) y(s) ds, \tag{2.6}$$

where

$$G(t,s) = \begin{cases} \dfrac{(1-s)^{\alpha-2} t^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq t \leq s \leq 1, \\[2ex] \dfrac{(1-s)^{\alpha-2} t^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leq s \leq t \leq 1. \end{cases} \tag{2.7}$$

Proof. We can reduce (2.4) to an equivalent integral equation

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{2.8}$$

By (2.5), the unique solution of problem (2.4), (2.5) is

$$\begin{aligned} u(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} y(s)ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned} \tag{2.9}$$

The proof is complete. □

It is obvious that

$$G(t,s) \geq 0, \quad G(1,s) \geq G(t,s), \quad 0 \leq t,\ s \leq 1. \tag{2.10}$$

Lemma 2.8. *$G(t,s) \geq q(t)G(1,s)$ for $0 \leq t,\ s \leq 1$, where $q(t) = t^{\alpha-1}$.*

Proof. If $t \geq s$, then

$$\begin{aligned} \frac{G(t,s)}{G(1,s)} &= \frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\ &= \frac{t(t-ts)^{\alpha-2} - (t-s)(t-s)^{\alpha-2}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\ &\geq \frac{t(t-ts)^{\alpha-2} - (t-s)(t-ts)^{\alpha-2}}{(1-s)^{\alpha-2} - (1-s)^{\alpha-1}} \\ &\geq t^{\alpha-2} \geq t^{\alpha-1}. \end{aligned} \tag{2.11}$$

If $t \leq s$, then

$$\frac{G(t,s)}{G(1,s)} = t^{\alpha-1}. \tag{2.12}$$

The proof is complete. □

3. Main results

In this section, we will apply Krasnoeselskii's fixed point theorem to the eigenvalue problem (1.4), (1.5). We note that $u(t)$ is a solution of (1.4), (1.5) if and only if

$$u(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))ds, \quad 0 \leq t \leq 1. \tag{3.1}$$

For our constructions, we shall consider the Banach space $X = C[0,1]$ equipped with standard norm $\|u\| = \max_{0\leq t\leq 1} |u(t)|$, $u \in X$. We define a cone P by

$$P = \{u \in X : u(t) \geq q(t)\|u\|,\ t \in [0,1]\}. \tag{3.2}$$

It is easy to see that if $u \in P$, then $\|u\| = u(1)$. Define an integral operator $T : P \to X$ by

$$Tu(t) = \lambda \int_0^1 G(t,s)a(s)f(u(s))\,ds, \quad 0 \le t \le 1,\ u \in P. \tag{3.3}$$

Notice from (2.10) and Lemma 2.8 that, for $u \in P$, $Tu(t) \ge 0$ on $[0,1]$ and

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &\ge \lambda q(t) \int_0^1 G(1,s)a(s)f(u(s))\,ds \\ &\ge \lambda q(t) \max_{0\le t\le 1} \int_0^1 G(t,s)a(s)f(u(s))\,ds \\ &= q(t)\|Tu(t)\|, \quad \forall t,s \in [0,1]. \end{aligned} \tag{3.4}$$

Thus, $T(P) \subset P$. In addition, standard arguments show that T is completely continuous. We define some important constants [11]:

$$\begin{aligned} A &= \int_0^1 G(1,s)a(s)q(s)\,ds, & B &= \int_0^1 G(1,s)a(s)\,ds, \\ F_0 &= \lim_{u\to 0^+} \sup \frac{f(u)}{u}, & f_0 &= \lim_{u\to 0^+} \inf \frac{f(u)}{u}, \\ F_\infty &= \lim_{u\to +\infty} \sup \frac{f(u)}{u}, & f_\infty &= \lim_{u\to +\infty} \inf \frac{f(u)}{u}. \end{aligned} \tag{3.5}$$

Here we assume that $1/Af_\infty = 0$ if $f_\infty \to \infty$, $1/BF_0 = \infty$ if $F_0 \to 0$, $1/Af_0 = 0$ if $f_0 \to \infty$, and $1/BF_\infty = \infty$ if $F_\infty \to 0$.

Theorem 3.1. *Suppose that $Af_\infty > BF_0$, then for each $\lambda \in (1/Af_\infty,\ 1/BF_0)$, the problem (1.4) and (1.5) has at least one positive solution.*

Proof. We choose $\epsilon > 0$ sufficiently small such that $(F_0 + \epsilon)\lambda B \le 1$. By the definition of F_0, we can see that there exists an $l_1 > 0$, such that $f(u) \le (F_0 + \epsilon)u$ for $0 < u \le l_1$. If $u \in P$ with $\|u\| = l_1$, we have

$$\begin{aligned} \|Tu(t)\| = Tu(1) &= \lambda \int_0^1 G(1,s)a(s)f(u(s))\,ds \\ &\le \lambda \int_0^1 G(1,s)a(s)(F_0+\epsilon)u(s)\,ds \\ &\le \lambda(F_0+\epsilon)\|u\| \int_0^1 G(1,s)a(s)\,ds \\ &\le \lambda B(F_0+\epsilon)\|u\| \le \|u\|. \end{aligned} \tag{3.6}$$

Then we have $\|Tu\| \leq \|u\|$. Thus if we let $\Omega_1 = \{u \in X : \|u\| < l_1\}$, then $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$. We choose $\delta > 0$ and $c \in (0, 1/4)$, such that

$$\lambda\left(\left(f_\infty - \delta\right) \int_c^1 G(1,s)a(s)q(s)ds\right) \geq 1. \tag{3.7}$$

There exists $l_3 > 0$, such that $f(u) \geq (f_\infty - \delta)u$ for $u > l_3$. Therefore, for each $u \in P$ with $\|u\| = l_2$, we have

$$\begin{aligned} ||Tu(t)|| = (Tu)(1) &= \lambda \int_0^1 G(1,s)a(s)f(u(s))ds \\ &\geq \lambda \int_c^1 G(1,s)a(s)\left(f_\infty - \epsilon\right)u(s)ds \\ &\geq \lambda\left(f_\infty - \epsilon\right)\|u\| \int_c^1 G(1,s)a(s)q(s)ds \geq \|u\|. \end{aligned} \tag{3.8}$$

Thus if we let $\Omega_2 = \{u \in E : \|u\| < l_2\}$, then $\Omega_1 \subset \overline{\Omega}_2$ and $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$. Condition (H1) of Krasnoesel'skii's fixed point theorem is satisfied. So there exists a fixed point of T in P. This completes the proof. □

Theorem 3.2. *Suppose that $Af_0 > BF_\infty$, then for each $\lambda \in (1/Af_0,\ 1/BF_\infty)$ the problem (1.4) and (1.5) has at least one positive solution.*

Proof. Choose $\epsilon > 0$ sufficiently small such that $(f_0 - \epsilon)\lambda A \geq 1$. From the definition of f_0, we see that there exists an $l_1 > 0$, such that $f(u) \geq (f_0 - \epsilon)u$ for $0 < u \leq l_1$. If $u \in P$ with $\|u\| = l_1$, we have

$$\begin{aligned} ||Tu(t)|| = (Tu)(1) &= \lambda \int_0^1 G(0,s)a(s)f(u(s))ds \\ &\geq \lambda\left(f_0 - \epsilon\right)\|u\|A \geq \|u\|. \end{aligned} \tag{3.9}$$

Then we have $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$. By the same method, we can see that if $u \in P$ with $\|u\| = l_2$, then we have $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$. Condition (H2) of Krasnoesel'skii's fixed-point theorem is satisfied. So there exists a fixed point of T in P. This completes the proof. □

Theorem 3.3. *Suppose that $\lambda Bf(u) < u$ for $u \in (0,\infty)$. Then the problem (1.4) and (1.5) has no positive solution.*

Proof. Assume to the contrary that u is a positive solution of (1.4) and (1.5). Then

$$\begin{aligned} u(1) = \lambda \int_0^1 G(1,s)a(s)f(u(s))ds &< \frac{1}{B}\int_0^1 G(1,s)a(s)u(s)ds \\ &\leq \frac{1}{B}u(1)\int_0^1 G(1,s)a(s)ds = u(1). \end{aligned} \tag{3.10}$$

This is a contradiction and completes the proof. □

Theorem 3.4. *Suppose that $\lambda A f(u) > u$ for $u \in (0,\infty)$. Then the problem (1.4) and (1.5) has no positive solution.*

Proof. Assume to the contrary that u is a positive solution of (1.4) and (1.5). Then

$$\begin{aligned} u(1) &= \lambda \int_0^1 G(1,s)a(s)f(u(s))ds > \frac{1}{A}\int_0^1 G(1,s)a(s)u(s)ds \\ &\geq \frac{u(1)}{A}\int_0^1 G(1,s)a(s)q(s)ds \geq u(1). \end{aligned} \tag{3.11}$$

This is a contradiction and completes the proof. □

Finally, we give an example to illustrate the results obtained in this paper.

Example 3.5. Consider the equation

$$\begin{gathered} D_{0+}^{(2.7)}u(t) + \lambda(2t+3)\frac{8u^2+u}{u+1}(4+\sin u) = 0, \quad 0 \leq t \leq 1, \\ u(0) = u'(0) = u'(1) = 0. \end{gathered} \tag{3.12}$$

Then $F_0 = f_0 = 4$, $F_\infty = 40$, $f_\infty = 24$, and $4u < f(u) < 40u$. By direct calculations, we obtain that $A = 0.240408$ and $B = 0.575602$. From Theorem 3.2, we see that if $\lambda \in (0.173316, 0.434328)$, then the problem (3.12) has a positive solution. From Theorem 3.3, we have that if $\lambda < 0.043433$, then the problem (3.12) has a positive solution. By Theorem 3.4, if $\lambda > 1.0399$, then the problem (3.12) has a positive solution.

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